The abstract approach to classifying C^*-algebras

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The Classification Theorem.

- A, B : [unital sep. simple nuclear] [$\mathcal{Z}$-stable] [UCT]
- $A \cong B \iff KT_u(A) \cong KT_u(B)$

$\wr$ $(K_0(A), K_1(A), [1_A]_0, T(A), r_B)$ — Aff $T(A)$

r_B: $T(A) \times K_0(B) \to \mathbb{R}$, $(\tau, [p]_0) \mapsto \tau(p)$

$\mathcal{Z}$-stability.

- $A \cong A \otimes \mathcal{Z} \overset{A \text{ sep. unital}}{\iff} \exists$ unital $*$-hom $\mathcal{Z}_{2,3} \longrightarrow A_\infty \cap A'$ (where $A_\infty = \ell^\infty(A)/c_0(A)$)

where $\mathcal{Z}_{2,3} \subset C([0,1], M_6)$ (endpoints: $M_2 \otimes 1$ at 0, $1 \otimes M_3$ at 1)

- Fact: $KT_u(A) \cong KT_u(A \otimes \mathcal{Z})$

UCT.

(a consequence of the UCT:) have a surjection

$$KK(A,B) \longrightarrow \mathrm{Hom}(K_0(A), K_0(B)) \oplus \mathrm{Hom}(K_1(A), K_1(B))$$

- "generalized $*$-hom's"
- formal differences of $*$-hom's: $KK(A, \mathbb{C})$
  (φ^+, φ^-), $\varphi^\pm : A \to B(\mathcal{H})$ s.t. $\varphi^+(a) - \varphi^-(a) \in \mathcal{K}$

Classifying alg's by classif. $*$-hom's.

Goal: produce an invariant $\underline{K}T_u$ ("total invariant") s.t.

(1) every suitable $\underline{K}T_u(A) \xrightarrow{\Phi} \underline{K}T_u(B)$ is induced by a suitable $*$-hom $A \xrightarrow{\varphi} B$ (existence)

(2) this φ is unique up suitable notion of equiv. (uniqueness)

$\wr$ approx. unitary equiv.
$\varphi \approx_u \psi$ if $\exists (u_n) \subset \mathcal{U}(B)$ s.t. $\|\varphi(a) - u_n^* \psi(a) u_n\| \to 0$, $a \in A$

Also need: $KT_u(A) \cong KT_u(B) \Rightarrow \underline{K}T_u(A) \cong \underline{K}T_u(B)$; id_A suitable

Assuming we have this:

$KT_u(A) \cong KT_u(B) \Rightarrow \exists$ $\underline{K}T_u(A) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \underline{K}T_u(B)$ isom's

$\overset{\text{existence}}{\Longrightarrow} \exists$ $*$-hom's $A \underset{\psi}{\overset{\varphi}{\rightleftarrows}} B$ $\underline{K}T_u(\varphi \circ \psi) = \underline{K}T_u(id_B)$, $\psi\circ\varphi$ … id_A

$\overset{\text{uniqueness}}{\Longrightarrow} \varphi \circ \psi \approx_u id_B$, $\psi \circ \varphi \approx_u id_A$

$\overset{\text{intertwining}}{\Longrightarrow} A \cong B$.

The total invariant.

$$\underline{K}T_u(A) = (\underline{K}(A), T(A), \overline{K}_1^{alg}(A), [1_A]_0, \text{various connecting maps})$$

$\underline{K}(A)$: $\bigoplus_n K_*(A, \mathbb{Z}/n\mathbb{Z})$

$\overline{K}_1^{alg}(A) \cong U_\infty(A) / \overline{DU_\infty(A)}$

e.g. $K_0(A) \to \mathrm{Aff}\, T(A) \to \overline{K}_1^{alg}(A) \to K_1(A)$ . . .

Theorem. (C - Gabe - Schafhauser - Tikuisis - White) A: unital sep. nuclear UCT ; B: unital sep. simple nuclear $\mathcal{Z}$-stable ($T(A) \neq \emptyset$)

Given "unital faithful compatible" $\underline{K}T_u(A) \xrightarrow{\Phi} \underline{K}T_u(B)$ there is a unital faithful $*$-hom $\varphi : A \to B$ s.t. $\underline{K}T_u(\varphi) = \Phi$, unique up to $\approx_u$.

Corollary. (A: both domain & target) $\mathrm{Aut}(A) / \overline{\mathrm{Inn}}(A) \cong \mathrm{Aut}(\underline{K}T(A))$ ($\alpha \approx_u id_A$)

Ingredients / Broad outline.

- enough to classify maps $A \longrightarrow B_\infty$ $(= \ell^\infty(B)/c_0(B))$
- we'll consider the trace-kernel extension:

trace kernel ideal: "tracially-null" sequences in B_∞

$$0 \longrightarrow J_B \longrightarrow B_\infty \longrightarrow B^\infty \longrightarrow 0$$

- B_∞: inherits some nice prop's from B (regular, stable rank 1, …)
  B^∞: rich vNa-like structure (think II_1 factor ultrapower)
  J_B: helps run the KK-machine.

- broad outline:

$$\begin{array}{ccccccccc} 0 & \to & J_B & \to & E & \rightleftarrows & A & \to & 0 \\ & & \downarrow & & \downarrow & \swarrow & \downarrow \theta & & \\ 0 & \to & J_B & \to & B_\infty & \to & B^\infty & \to & 0 \end{array}$$

start. • $\underline{K}T_u(A) \to \underline{K}T_u(B)$. Get $T(B^\infty) \to T(A)$.
Glue together classif. of maps $A \to$ finite vNa's (Connes)
↳ Castillejos - Evington - Tikuisis - White. Get map $A \to B_\infty$.

! { • Have $\alpha \in KK(A, B_\infty)$, $\theta : A \to B^\infty$.
Use Ext / KK-based theory to show one can lift θ.

• package neatly in terms of $\underline{K}T_u$.