CLASSIFYING LIFTS IN THE TRACE-KERNEL EXTENSION

JOINT WORK WITH J. GABE, C. SCHAFHAUSER, A. TIKUISIS, AND S. WHITE

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APPROXIMATE CLASSIFICATION OF MORPHISMS

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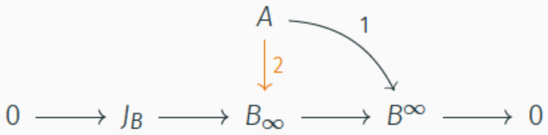

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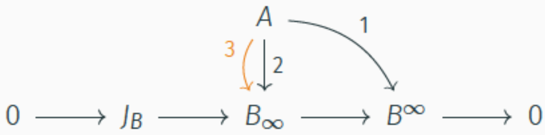

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3. couch classification in terms of invariant; K-theoretic computation involving $J_B := \{(b_n) : \|b_n\|_{2,u} \to 0\}$

This talk will focus on lifts

Existence and uniqueness results: in terms of $KK(A, B_\infty)$ and $KK(A, J_B)$. Used in argument:

- nuclearity (either of A or of the maps)
- $\mathcal{Z}$-stability/enough comparison for B
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[†] Disclaimer: J_B is actually "separably" stable:

$I \subseteq J_B$ separable $\Rightarrow$ $\exists$ separable and stable $I_0 \subseteq J_B$ s.t. $I \subseteq I_0$.

SAMPLE UNIQUENESS RESULT

Think of Voiculescu's Theorem:

$$\begin{array}{ccccccccc} & & & & A & \xrightarrow{\quad\theta\quad} & & & \\ & & & & \psi\downarrow\downarrow\varphi & \searrow & & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{B}(\mathcal{H}) & \longrightarrow & \mathcal{Q}(\mathcal{H}) & \longrightarrow & 0 \end{array}$$

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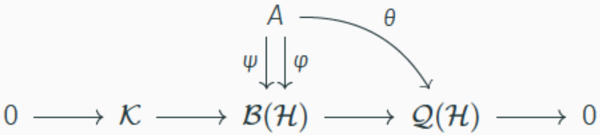

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In fact:

Theorem (Dadarlat-Eilers '01)

Suppose: $\varphi, \psi \colon A \to \mathcal{B}(\mathcal{H})$ *are ample lifts of* θ.

$$[\varphi, \psi] = 0 \in KK(A, \mathcal{K}) \quad \Longleftrightarrow \quad \varphi \approxeq \psi$$

Def.

- *(A,B)-Cuntz pair:* *-hom's $\varphi^+, \varphi^- : A \to M(B \otimes \mathcal{K})$ with $\mathrm{im}(\varphi^+ - \varphi^-) \subseteq B \otimes \mathcal{K}$

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Some KK-elements

- zero element: $[\varphi, \varphi]_{KK}$, any $\varphi\colon A \to M(B \otimes \mathcal{K}) \in KK(A, B)$
- if $\pi\colon A \to B$ and $p \in \mathcal{K}$ rank-one projection:

$$[\pi]_{KK} := \big[\pi(\,\cdot\,) \otimes p, 0\big]_{KK} \quad \in KK(A, B)$$

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- Consider an extension $0\to I\xrightarrow{j}E\to D\to 0$
  where $I\cong I\otimes\mathcal{K}$. Let $\lambda\colon E\to M(I)$ canonical map.

$$\text{given } \varphi,\psi\colon A\to E,\quad \operatorname{im}(\varphi-\psi)\subseteq I,$$

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Moreover: j gives $KK(A, j)\colon KK(A, I) \to KK(A, E)$ and

$$KK(A, j)\Big([\varphi, \psi]_{KK}\Big) = [\varphi]_{KK} - [\psi]_{KK}$$

Suppose $I \cong I \otimes \mathcal{K}$.

Theorem (Dadarlat-Eilers)

$\varphi, \psi \colon A \to M(I)$ Cuntz pair.

$$[\varphi, \psi]_{KK} = 0 \iff \varphi \oplus \sigma \approxeq \psi \oplus \sigma$$

for some $\sigma \colon A \to M(I)$.

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For our application: need to get rid of the summand σ.
Central notion:

Def.

$\varphi \colon A \to M(I)$ is *absorbing* if $\varphi \oplus \pi \approx_u \varphi$ for all $\pi \colon A \to M(I)$.

$\mathcal{Z}$ is an important ingredient. Here is a glimpse of how it enters the uniqueness argument:

Proposition (CGSTW)

Suppose $I \cong I \otimes \mathcal{K}$ and
$\varphi, \psi \colon A \to M(I)$ is a Cuntz pair of *absorbing* *-hom's.

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- Is every unital properly infinite C^*-algebra is K_1-injective?

GETTING ABSORPTION

- Voiculescu: any ample $\varphi\colon A \to \mathcal{B}(H)$ is absorbing
- Kasparov: A or $I \otimes \mathcal{K}$ nuclear $\Rightarrow$ any ample $\varphi\colon A \to \mathcal{B}(H) \subset M(I)$ is absorbing
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Suppose $T(B) \neq \varnothing$ is compact, B has strict comp. w.r.t. $T(B)$. Then:

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(φ full: $0 \neq a \in A_+ \Rightarrow \varphi(a)$ not contained in any proper ideal)

MAIN LIFTING THEOREM

$$\begin{array}{c} A \xrightarrow{\theta} B^\infty \\ 0 \longrightarrow J_B \xrightarrow{j} B_\infty \xrightarrow{q} B^\infty \longrightarrow 0 \end{array}$$

Theorem (CGSTW)

- *B is $\mathcal{Z}$-stable, $T(B) \neq \varnothing$ compact, B strict comp. w.r.t. $T(B)$*
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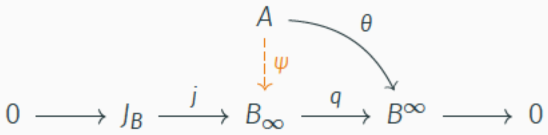

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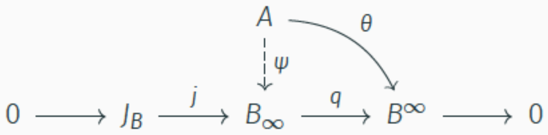

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Theorem

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2. *If φ is another unitizably full lift with $[\varphi, \psi] = 0 \in KK(A, J_B)$, and then $\psi \approxeq \varphi$.*

Note that $[\varphi, \psi] = 0$ implies

$$0 = KK(A, j)\Big([\varphi, \psi]\Big) = [\varphi]_{KK} - [\psi]_{KK} = [\varphi]_{KK} - \alpha.$$

Related to step 3:

If A satisfies UCT, can get uniqueness from $[\varphi]_{KK} = [\psi]_{KK}$ and $\overline{K}_1^{\mathrm{alg}}(\varphi) = \overline{K}_1^{\mathrm{alg}}(\psi)$ instead.

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In fact, need stronger uniqueness statement to prove existence.

If ψ is a lift of θ as in (1) and $\kappa \in KK(A, J_B)$, then
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- $\mathcal{Z}$-stability of $B \rightsquigarrow \varphi \approxeq \varphi'$

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  $\rightsquigarrow e_\theta$ splits, and θ lifts to ψ'
- use stronger uniqueness to get lift ψ with
  $[\psi] - [\psi'] = \alpha - [\psi']$, i.e. $[\psi] = \alpha$.

Have lifting theorem for extensions with "trace-kernel features":

Def.

[redacted]

[redacted]

This is enough to get absorption when needed. Still need some $\mathcal{Z}$-stability for classification.

[redacted]