AN ABSTRACT APPROACH TO THE CLASSIFICATION OF NUCLEAR C*-ALGEBRAS

JOINT WORK WITH J. GABE, C. SCHAFHAUSER,
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INTRODUCTION

Theorem ("Many hands")

The class of separable, simple, unital, nuclear and $\mathcal{Z}$-stable C^*-algebras that satisfy the UCT is classified by K-theoretic invariants.

- No traces: Kirchberg-Phillips (1990s).
- We focus only on the case $T(A) \neq \varnothing$.
- Classifying invariant:

$$\mathrm{Ell}(A) := \Big(K_0(A),\, K_0(A)_+,\, [1_A]_0,\, K_1(A),\, T(A),\, T(A) \times K_0(A) \to \mathbb{R}\Big)$$

Impossible to summarize decades of work in a slide.

Some recent components relevant here:

Classification of “model” algebras

- Gong-Lin-Niu ’15: classified C^*-algebras with a certain internal tracial approximation structure.
- The class exhausts range of $\mathrm{Ell}(-)$.

Realizing the approximations

- Elliott-Gong-Lin-Niu ’15: abstract conditions on a C^*-algebra $\Rightarrow$ concrete tracial approximations of GLN.
- Tikuisis-White-Winter ’17: the abstract conditions are the ones stated in the classification theorem.

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We will (mostly) ignore the difficulties that arise from non-separability or the lack of a unit.

Rough scheme: produce invariant (functor) $\mathrm{inv}(-)$ s.t.

- (existence)
  $\alpha\colon \mathrm{inv}(A) \to \mathrm{inv}(B) \Longrightarrow \exists\, \varphi\colon A \to B$ s.t. $\mathrm{inv}(\varphi) = \alpha$;
- (uniqueness)
  $\varphi, \psi\colon A \to B$ and $\mathrm{inv}(\varphi) = \mathrm{inv}(\varphi) \Longrightarrow \varphi \approx_u \psi$.

End result: $\mathrm{inv}(A) \cong \mathrm{inv}(B) \Longrightarrow A \cong B$.

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End result: $\mathrm{inv}(A) \cong \mathrm{inv}(B) \implies A \cong B$.

Would also want:

- $\mathrm{Ell}(A) \xrightarrow{\sim} \mathrm{Ell}(B)$ yields $\mathrm{inv}(A) \xrightarrow{\sim} \mathrm{inv}(B)$.

Existence and uniqueness for morphisms of AF algebras

A, B = (unital) AF algebras.

1. $\alpha\colon (K_0(A), K_0(A)_+, [1_A]_0) \to \big(K_0(B), K_0(B)_+, [1_B]_0\big)$
   $\Longrightarrow \exists$ $*$-hom. $\varphi\colon A \to B$ s.t. $\alpha = K_0(\varphi)$.
2. $\varphi, \psi\colon A \to B$ and $K_0(\varphi) = K_0(\psi) \quad \Longrightarrow \quad \varphi \approx_u \psi$.

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Classification of AF Algebras

If $\exists$ isomorphism

$$\alpha\colon (K_0(A), K_0(A)_+, [1_A]) \to (K_0(B), K_0(B), [1_B]_0),$$

then $\exists$ isomorphism $\varphi\colon A \to B$ s.t. $K_0(\varphi) = \alpha$.

SECOND EXAMPLE: A$\mathbb{T}$ ALGEBRAS

Thomsen-Nielsen (early 90s): different proof of Elliott's classification of simple unital A$\mathbb{T}$ algebras. Need refined inv.

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$\overline{K}_1^{\mathrm{alg}}(A)$ came up in Thomsen's work on the role of the relationship between and K-theory and traces in classification theory.

K_0 and traces

Briefly: $[p]_0 \in K_0(A) \rightsquigarrow$ the affine map $\tau \mapsto \tau(p)$ on $T(A)$.
Write $\rho_A \colon K_0(A) \to \operatorname{Aff} T(A)$ for this function.

Thomsen-Nielsen provided existence and uniqueness theorems for morphisms in the A$\mathbb{T}$ case using

$$\left(K_0(-), \overline{K}_1^{\mathrm{alg}}(-), \operatorname{Aff} T(-)\right)$$

as their invariant.

Examples show that $\varphi \approx_u \psi$ might fail if φ and ψ only agree on K_0, K_1, and traces.

MORE ON $\overline{K}_1^{\mathrm{alg}}(A)$

Thomsen's extension

$$0 \to \frac{\operatorname{Aff} T(A)}{\overline{\operatorname{im} \rho_A}} \xrightarrow{\mathrm{Th}_A} \overline{K}_1^{\mathrm{alg}}(A) \to K_1(A) \to 0$$

Th_A is the inverse of an isomorphism

$$\ker\left(\overline{K}_1^{\mathrm{alg}}(A) \to K_1(A)\right) \to \frac{\operatorname{Aff} T(A)}{\overline{\operatorname{im} \rho_A}}$$

defined using the de la Harpe-Skandalis determinant.

Definition

$\underline{K}(A) = \bigoplus_{n=0}^{\infty} K_0(A; \mathbb{Z}/n\mathbb{Z}) \oplus K_1(A; \mathbb{Z}/n\mathbb{Z})$

Can think of $K_i(A; \mathbb{Z}/n\mathbb{Z})$ as $K_i(A \otimes D_n)$,
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Slogan

Can check "closeness" of $KK(\varphi)$ and $KK(\psi)$ by checking that $\underline{K}(\varphi)$ and $\underline{K}(\psi)$ agree on large finite subsets of $\underline{K}(A)$.

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A *compatible triple* $(\underline{\alpha}, \beta, \gamma) \colon \mathrm{inv}(A) \to \mathrm{inv}(E)$ consists of

$$\underline{\alpha} \colon \underline{K}(A) \to \underline{K}(E), \quad \beta \colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(E), \quad \gamma \colon \mathrm{Aff}\,T(A) \to \mathrm{Aff}\,T(E)$$

such that

$$\begin{array}{ccccccc}
K_0(A) & \xrightarrow{\rho_A} & \mathrm{Aff}\,T(A) & \xrightarrow{\mathrm{Th}_A} & \overline{K}_1^{\mathrm{alg}}(A) & \longrightarrow & K_1(A) \\
\downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha_1} \\
K_0(E) & \xrightarrow{\rho_E} & \mathrm{Aff}\,T(E) & \xrightarrow{\mathrm{Th}_E} & \overline{K}_1^{\mathrm{alg}}(E) & \longrightarrow & K_1(E)
\end{array}$$

commutes.

Define $B_\infty := \prod_{n=1}^\infty B / \sum_{n=1}^\infty B$.

Theorem (C-Gabe-Schafhauser-Tikuisis-White)

- *A : sep., exact, UCT*
- *B : sep., $\mathcal{Z}$-stable, strict comparison, $T(B) \neq \varnothing$ & compact, no unbounded traces*
- *$\varphi, \psi \colon A \to B_\infty$ full[†] nuclear $*$-hom's*

Then:

$$\mathrm{inv}(\varphi) = \mathrm{inv}(\psi) \quad \Longrightarrow \quad \varphi \approx_u \psi$$

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Theorem (CGSTW)

A and B as above.

$(\underline{\alpha}, \beta, \gamma)\colon \mathrm{inv}(A) \to \mathrm{inv}(B_\infty)$: *compatible triple that is "faithful and amenable on traces"*† *(and unital*‡ *in unital case)*

Then: $\exists$ *a full nuclear* $*$*-hom.* $\varphi\colon A \to B_\infty$ *s.t.*

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- Higson-Kasparov: Γ satisfies Baum-Connes.
- Lück: range of $K_0(\tau)$ is contained in $\mathbb{Q} \cong K_0(\mathcal{Q})$.
- Tu: $C_r^*(\Gamma)$ satisfies UCT.

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$\exists$ trace-preserving $C^*_r(\Gamma) \hookrightarrow \mathcal{Q}$ $\Leftrightarrow$ Γ is amenable $\Leftrightarrow$ $\exists$ trace-preserving $L\Gamma \hookrightarrow \mathcal{R}$

STRATEGY

The *trace-kernel ideal* is

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Analogy with TAF case: $B^\infty \sim$ "tracially large corner"
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Approximate classification of morphisms: major steps

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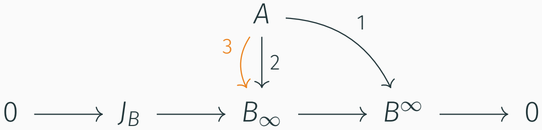
1. Classify morphisms into B^∞
2. Classify lifts of morphisms to B_∞



1. Classify morphisms into B^∞
2. Classify lifts of morphisms to B_∞
3. Adjust the K-theory, exploiting J_B

TECHNIQUES, STEP BY STEP

Think of B^∞ as a II_1 factor—a tracial ultrapower of $\pi_\tau(B)''$.

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(It's not.)

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Moreover:

$$f \colon T(B^\infty) \to T_{\mathrm{amen}}(A) \Longrightarrow \exists \text{ nuclear } \theta \colon A \to B^\infty \text{ s.t. } T(\theta) = f$$

CLASSIFICATION OF LIFTS

Adapt Schafhauser's approach to the TWW theorem:
think in terms of extensions, *KK*-theory.

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Theorem (Existence for lifts)

$\theta\colon A \to B^\infty$ *full nuclear* $*$*-hom* *and* $\kappa \in KK_{\mathrm{nuc}}(A, B_\infty)$
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- $[e_\theta] = 0 \implies e_\theta \oplus$(trivial extension) $\approx$ a split extension.
- Weyl-von Neumann type absorption theorems $\implies e_\theta \oplus$(trivial extension) $\approx e_\theta$.

What if we have two lifts φ and ψ of θ?

$$\begin{array}{ccccccccc} & & & & A & & & & \\ & & & & \psi\downarrow\downarrow\varphi & \overset{\theta}{\searrow} & & & \\ 0 & \longrightarrow & J_B & \longrightarrow & B_\infty & \longrightarrow & B^\infty & \longrightarrow & 0 \end{array}$$

Want to guarantee (a strong form of) uniqueness with a condition that can be verified by comparing invariants.

Think of Voiculescu's Theorem:

$$
\begin{array}{ccccccccc}
 & & & & A & \xrightarrow{\quad\theta\quad} & & & \\
 & & & & \psi \downarrow\downarrow \varphi & & \searrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{B}(\mathcal{H}) & \longrightarrow & \mathcal{Q}(\mathcal{H}) & \longrightarrow & 0
\end{array}
$$

If φ, ψ are "admissible" (faithful, nondegenerate, and $\varphi(A) \cap \mathcal{K} = \{0\} = \psi(A) \cap \mathcal{K}$), then $\varphi \approx_u \psi$.

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More can be said:

Theorem (Dadarlat-Eilers '01)

Suppose: A is sep.; $\varphi, \psi : A \to \mathcal{B}(\mathcal{H})$ are admissible lifts of θ.

Then:

$[\varphi, \psi] = 0 \in KK(A, \mathcal{K}) \implies \varphi \approx_u \psi$ *via unitaries in* $\mathcal{K} + \mathbb{C}1_{\mathcal{H}}$.

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Theorem (Uniqueness for lifts)

A: sep., exact;

B: sep., $\mathcal{Z}$-stable, strict comparison, $T(B) \neq \varnothing$ & compact;

φ, ψ: full nuclear lifts of θ.

$$[\varphi, \psi] = 0 \in KL_{\mathrm{nuc}}(A, J_B) \quad \Longrightarrow \quad \varphi \approx_u \psi \text{ via unitaries in } \widetilde{J_B}\,.$$

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We'll answer this in terms of $\underline{K}(-)$ and $\overline{K}_1^{\mathrm{alg}}(-)$.

∃ morphism

$$j_*\colon KL_{\mathrm{nuc}}(A, J_B) \to \mathrm{Hom}_\Lambda\left(\underline{K}(A), \underline{K}(B_\infty)\right)$$
$$[\varphi, \psi] \mapsto \underline{K}(\varphi) - \underline{K}(\psi)$$

induced by $j\colon J_B \to B_\infty$.

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Subtle obstruction: even if $\varphi(u) \sim_h \psi(u)$ is true for $u \in U(A)$, the path ξ connecting them might have nonzero “winding number”.

This leads to a rotation map $R_{\varphi,\psi}$ which (roughly) assigns the function

$$\tau \mapsto \frac{1}{2\pi i} \int_0^1 \tau\Big(\frac{d\xi(t)}{dt}\xi(t)^{-1}\Big)\, dt$$

on $T(B_\infty)$ to $[u]_1 \in K_1(A)$.

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Punchline: assuming $\underline{K}(\varphi) = \underline{K}(\psi)$,

$$\overline{K}_1^{\mathrm{alg}}(\varphi) - \overline{K}_1^{\mathrm{alg}}(\psi) = 0 \quad \Longrightarrow \quad R\big([\varphi,\psi]\big) = 0 \quad \Longrightarrow \quad [\varphi,\psi] = 0.$$

This let us use the classification theorem for lifts.

A NON-UNITAL APPLICATION

Let

$$\mathrm{Ell}^+(A) = \Big(K_0(A),\, K_0(A)_+,\, \Sigma_A,\, K_1(A),\, T^+(A),\, r_A^+\Big)$$

Theorem

Suppose A and B are non-unital, simple, separable, nuclear, $\mathcal{Z}$-stable C^*-algebras satisfying the UCT, with $T^+(A) \neq \varnothing \neq T^+(B)$.

Any isomorphism $\mathrm{Ell}^+(A) \xrightarrow{\sim} \mathrm{Ell}^+(B)$ lifts to an isomorphism $A \xrightarrow{\sim} B$.

Thank you!