Determinants, rotation maps, and classification of morphisms

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joint work (in progress) with
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First example: AF algebras

Existence and uniqueness for morphisms of AF algebras

A, B = AF algebras.

1. (Existence.) If $\alpha\colon K_0(A) \to K_0(B)$ is a group hom. s.t. $\alpha(\Sigma A) = \alpha(\Sigma B)$, then $\exists$ $*$-hom. $\phi\colon A \to B$ s.t. $\alpha = K_0(\phi)$.
2. (Uniqueness.) If $\phi, \psi\colon A \to B$ are $*$-hom's s.t. $K_0(\phi) = K_0(\psi)$, then $\phi \approx_u \psi$.

Classification of AF Algebras

If $\exists$ isomorphism $\alpha\colon K_0(A) \to K_0(B)$ s.t. $\alpha(\Sigma A) = \Sigma B$, then *exists* isomorphism $\phi\colon A \to B$ s.t. $K_0(\phi) = \alpha$.

A similar scheme for A$\mathbb{T}$ algebras

Thomsen-Nielsen (early 90s): different proof of Elliott's classification of simple unital A$\mathbb{T}$ algebras. Invariant used to classify morphisms is refined.

Definition

$$\overline{K}_1^{\mathrm{alg}}(A) := U^\infty(A)/CU^\infty(A)$$

where $CU^\infty(A)$ is the closure of the commutator subgroup of $U^\infty(A)$.

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$\overline{K}_1^{\mathrm{alg}}(A)$ came up in Thomsen's work on the role of the relationship between and K-theory and traces in classification theory.

K_0 and traces

Briefly: $[p]_0 \in K_0(A) \rightsquigarrow$ affine map $\tau \mapsto \tau(p)$ on $T(A)$.
Write $\rho_A \colon K_0(A) \to \mathrm{Aff}\, T(A)$ for this function.

A similar scheme for A$\mathbb{T}$ algebras (cont.)

Theorem (Thomsen-Nielsen)

1. (Existence) We can realize any* triple

$$\alpha\colon K_0(A) \to K_0(B), \quad \beta\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B), \quad \gamma\colon T(B) \to T(A)$$

   that is *compatible* using a $*$-hom $\phi\colon A \to B$.
2. (Uniqueness) If $\phi, \psi\colon A \to B$ induce the same maps on $K_0(A)$, $\overline{K}_1^{\mathrm{alg}}(A)$, and $T(B)$, then $\phi \approx_u \psi$.

Using this, an isomorphism $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ is shown to lift to an isomorphism $A \to B$ via an intertwining argument.

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Examples show that to conclude $\phi \approx_u \psi$ it is not enough that ϕ and ψ agree on K_0, K_1, and traces.

More on $\overline{K}_1^{\mathrm{alg}}(A)$

Thomsen's extension

$$0 \to \frac{\mathsf{Aff}\, T(A)}{\overline{\mathsf{im}\, \rho_A}} \xrightarrow{\mathsf{Th}_A} \overline{K}_1^{\mathrm{alg}}(A) \to K_1(A) \to 0$$

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To define Th_A we need the de la Harpe-Skandalis determinant: we'll use it to define a map

$$\ker\left(\overline{K}_1^{\mathrm{alg}}(A) \to K_1(A)\right) \to \frac{\mathrm{Aff}\, T(A)}{\overline{\mathrm{im}\, \rho_A}}$$

dlH-S determinants and Th_A

dlH-S determinant

Given piecewise smooth path ξ in $U^\infty(A)$ with $\xi(0) = 1$, define $\Delta(\xi) \in \mathrm{Aff}\ T(A)$ by

$$\Delta(\xi)(\tau) = \frac{1}{2\pi i} \int_0^1 \tau\Big(\xi'(t)\xi(t)^*\Big) dt, \qquad \tau \in T(A).$$

Facts: $\Delta(\,\cdot\,)$ is invariant under f.e.p. homotopy, and $\Delta(\xi_1\xi_2) = \Delta(\xi_1) + \Delta(\xi_2)$.

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Use Δ to define a homomorphism

$$U_0^\infty(A) \to \frac{\mathrm{Aff}\ T(A)}{\overline{\Delta\big(\pi_1 U^\infty(A)\big)}} = \frac{\mathrm{Aff}\ T(A)}{\overline{\rho_A(K_0(A))}}$$

Thomsen: the kernel consists of the closure of commutators.

dlH-S determinants and Th_A (cont.)

End result: an isomorphism

$$\ker\left(\overline{K}_1^{\mathrm{alg}}(A) \to K_1(A)\right) \to \frac{\mathrm{Aff}\, T(A)}{\overline{\mathrm{im}\,\rho_A}}$$

The inverse is Th_A.

Get the extension

$$0 \to \frac{\mathrm{Aff}\, T(A)}{\overline{\mathrm{im}\,\rho_A}} \xrightarrow{\mathrm{Th}_A} \overline{K}_1^{\mathrm{alg}}(A) \to K_1(A) \to 0$$

A lot of this applies to larger classes of C^*-algebras with nice internal approximations

An abundance of projections obscures these issues (Blackadar-Kumjian-Rørdam, Thomsen).

Theorem (Lin)

A: simple unital with tracial rank ≤ 1. Then

$$A \text{ is tracially AF} \quad \Leftrightarrow \quad \overline{\operatorname{im} \rho_A} = \operatorname{Aff} T(A) \quad \Leftrightarrow \quad CU(A) = U_0(A).$$

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Lin used similar scheme (with ever more refined invariants, including $\overline{K}_1^{\mathrm{alg}}(\,\cdot\,)$) for the classification of morphisms between simple sep. nuclear unital UCT C^*-algebras of TR≤ 1.

Gong-Lin-Niu: generalized tracial rank ≤ 1.

Towards existence in an abstract setting

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$\mathcal{Q}$-stable approximate existence (EGLN+TWW; CGSTW)

A: sep., simple, nuclear, UCT
B: simple, unital, nuclear, finite, $\mathcal{Q}$-stable
Suppose*

$$\alpha \in KK(A,B), \quad \beta\colon \overline{K}_1^{\mathrm{alg}}(A) \to \overline{K}_1^{\mathrm{alg}}(B), \text{ and } \quad \gamma\colon T(B) \to T(A)$$

are compatible: i.e. we have commutative diagram

$$\begin{array}{ccccccc}
K_0(A) & \xrightarrow{\rho_A} & \mathrm{Aff}\, T(A) & \xrightarrow{\mathrm{Th}_A} & \overline{K}_1^{\mathrm{alg}}(A) & \longrightarrow & K_1(A) \\
\downarrow \alpha_0 & & \downarrow \gamma^* & & \downarrow \beta & & \downarrow \alpha_1 \\
K_0(B) & \xrightarrow{\rho_B} & \mathrm{Aff}\, T(B) & \xrightarrow{\mathrm{Th}_B} & \overline{K}_1^{\mathrm{alg}}(B) & \longrightarrow & K_1(B)
\end{array}$$

Then there exists a $*$-homomorphism $\phi\colon A \to B_\omega$ satisfying

$$KK(\phi) = [\iota_B]\alpha, \quad \overline{K}_1^{\mathrm{alg}}(\phi) = \overline{K}_1^{\mathrm{alg}}(\iota_\beta)\beta, \quad \textit{and} \quad T(\phi) = \gamma T(\iota_B).$$