Order zero approximations of nuclear C^*-algebras

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joint work with N. Brown and S. White

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Nuclear C^*-algebras

Theorem/definition (Choi-Effros, Kirchberg)

A C^*-algebra A is nuclear iff $\exists$ f.d. algebras F_i and c.c.p. maps

$$\begin{array}{ccc} A & \xrightarrow{\ \mathrm{id}_A\ } & A \\ & {\scriptstyle \psi_i} \searrow \quad \nearrow {\scriptstyle \phi_i} & \\ & F_i & \end{array}$$

s.t.

- $\|(\phi_i \circ \psi_i)(a) - a\| \to 0 \qquad \forall a \in A$

Explore refinements to CPAP arising from imposing conditions on the ψ_i's or the ϕ_i's.

Nuclearity and quasidiagonality

Quasidiagonality

A is QD if $\exists$ asymp. multiplicative and asymp. isometric c.c.p. maps $\theta_i \colon A \to F_i$.

For example:

$C_0(0,1] \otimes A$ (Voiculescu);
$C^*(\Gamma)$ for any discrete amenable group Γ (Tikuisis-White-Winter)

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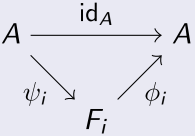
Theorem (Blackadar-Kirchberg)

A is nuclear and quasidiagonal iff $\exists$ f.d. algebras F_i and c.c.p. maps



s.t.

- $\|(\phi_i \circ \psi_i)(a) - a\| \to 0$
- the ψ_i are asymp. multiplicative

Nuclear dimension

Definition (Winter-Zacharias)

$\dim_{\text{nuc}}(A) \leq n$ if $\exists$ f.d. algebras F_i and c.p. maps

$$\begin{array}{ccc} A & \xrightarrow{\quad \mathrm{id}_A \quad} & A \\ & {\scriptstyle \psi_i} \searrow \quad \nearrow {\scriptstyle \phi_i} & \\ & F_i & \end{array}$$

s.t.

- $\|(\phi_i \circ \psi_i)(a) - a\| \to 0$
- each ϕ_i "decomposes" into $n+1$ c.c.p. order zero maps (order zero: $a \perp b \;\Rightarrow\; \theta(a) \perp \theta(b)$)

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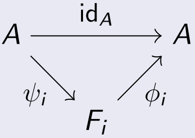

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Theorem (Many hands; see TWW)

For unital, simple, separable C^*-algebras satisfying the UCT: finite nuclear dimension $\Rightarrow$ classifiability.

A refinement of the CPAP

Theorem A (Brown-C-White)

A is nuclear iff $\exists$ f.d. algebras F_i and c.c.p. maps

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s.t.

1. $\|(\phi_i \circ \psi_i)(a) - a\| \to 0$
2. $\phi_i = \sum_{\text{finite}}$ c.c.p. order zero maps
3. the ψ_i are asymp. order zero:
   $a, b \geq 0, ab = 0 \Rightarrow \|\psi_i(a)\psi_i(b)\| \to 0$

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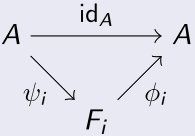

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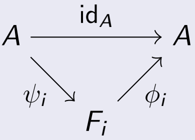

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- Was known for $\dim_{\mathrm{nuc}}(A) < \infty$ (Winter-Zacharias).
- For nuclear A, approximations satisfying 1 and 2 were known to exist (Hirshberg-Kirchberg-White).

Theorem B (Brown-C-White)

Suppose A is nuclear. Then A is QD and $T(A) = T_{\mathrm{qd}}(A)$ iff $\exists$ f.d. algebras F_i and c.c.p. maps

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ & {\scriptstyle \psi_i} \searrow \quad \nearrow {\scriptstyle \phi_i} & \\ & F_i & \end{array}$$

s.t.

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3′. the ψ_i are asymp. multiplicative

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Deep result (Tikuisis-White-Winter): $T(A) = T_{\mathrm{qd}}(A)$ if A is separable, unital, nuclear, QD and satisfies the UCT.

Strategy

Theorem B $\Rightarrow$ Theorem A...

... by factoring through the cone in "an order zero way": $A \to CA \to A$.

Reduce Thm. B to a σ-weak version

Want a "σ-weak" version of the factorization for $\iota\colon A \hookrightarrow A^{**}$ instead of id_A. That is, want ϕ_i and ψ_i satisfying $\phi_i \circ \psi_i \to \iota$ pt.-σ-weakly (and s.t. the other conditions hold unchanged).

$A^{**} = A^{**}_{\mathrm{inf}} \oplus A^{**}_{\mathrm{fin}}$

Decompose proof accordingly.

Strategy (cont.)

$$\begin{array}{ccc} A & \xrightarrow{\quad \pi \quad} & N \\ & {\scriptstyle \psi_i} \searrow \quad \nearrow {\scriptstyle \phi_i} & \\ & F_i & \end{array}$$

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Properly ∞ case

Combine Blackadar and Kirchberg's NF algebras and techniques from Haagerup's proof of injective $\Rightarrow$ hyperfinite in the prop. ∞ case.

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Finite case

Can reduce to $N = \pi_\tau(A)''$. Use that τ is qd. Use that two $*$-hom's from A to a finite vN algebra M are (σ-strong*) approx. u-equiv. if they agree on all normal traces on M.