Local embeddability of groups and quasidiagonality

José Carrión

Penn State University

based on joint work with Marius Dadarlat and Caleb Eckhardt

GPOTS
Kansas State University
May 28, 2014

Quasidiagonality

Quasidiagonality for operators

Quasidiagonality for operators was introduced by Halmos.

- An **operator** $T \in \mathcal{B}(\mathcal{H})$ is **quasidiagonal** if $\exists$ finite rank projections $P_1 \leq P_2 \leq \cdots$ with $P_n \to 1_{\mathcal{H}}$ and $\|P_n T - TP_n\| \to 0$.
- If $\mathcal{H}$ is separable, a (separable) **set** $\Omega \subset \mathcal{B}(\mathcal{H})$ is **quasidiagonal** if $\exists$ a sequence (P_n) as above that works simultaneously for all $T \in \Omega$.

Quasidiagonality for C^*-algebras

A **C*-algebra** is **quasidiagonal** if it has a faithful representation as a quasidiagonal set of operators.

Quasidiagonality (cont.)

- Quasidiagonality is a *local finite-dimensional approximation property* (Voiculescu)
- Connections to BDF and KK-theory, classification theory for nuclear C^*-algebras, AF-embeddability of C^*-algebras, ...

Our focus: quasidiagonality and group C^*-algebras.

From now on: all groups (Γ, Λ, Δ, etc.) are discrete and countable.

Rosenberg's theorem

Recall:

- $\lambda_s \in \mathcal{B}(\ell^2\Gamma)$: left translation by $s \in \Gamma$.
- $C^*_\lambda(\Gamma)$: C^*-algebra generated by $\lambda(\Gamma) \subset \mathcal{B}(\ell^2\Gamma)$.

Theorem (Rosenberg '87)

$C^*_\lambda(\Gamma)$ is QD $\quad\Rightarrow\quad$ *Γ is amenable.*

Conjecture (Rosenberg)

Γ is amenable $\quad\Rightarrow\quad$ $C^*_\lambda(\Gamma)$ is QD.

A result of Bekka

Theorem (Bekka '99)

Suppose Γ *is amenable. Then*

$$\Gamma \hookrightarrow U\left(\prod \mathrm{M}_n(\mathbb{C})\right) \quad \Leftrightarrow \quad C^*_\lambda(\Gamma) \hookrightarrow \prod \mathrm{M}_n(\mathbb{C}).$$

In particular, Γ amenable and *residually finite* $\Rightarrow$ $C^*_\lambda(\Gamma)$ is QD.

MF groups

Definition

Γ is **MF** if

$$\Gamma \hookrightarrow U\left(\frac{\prod \mathrm{M}_{n_k}(\mathbb{C})}{\sum \mathrm{M}_{n_k}(\mathbb{C})} \right)$$

for some increasing sequence (n_k).

($\prod$ means ℓ^∞-direct sum, $\sum$ means c_0-direct sum.)

Theorem (C-Dadarlat-Eckhardt '13)

Suppose Γ *is amenable. Then*

$$\Gamma \text{ is MF} \quad \Leftrightarrow \quad C^*_\lambda(\Gamma) \text{ is QD.}$$

That is,

$$\Gamma \hookrightarrow U\left(\prod \mathrm{M}_{n_k} \,/\, \sum \mathrm{M}_{n_k} \right) \quad \Leftrightarrow \quad C^*_\lambda(\Gamma) \hookrightarrow \prod \mathrm{M}_{n_k} \,/\, \sum \mathrm{M}_{n_k}$$

Example: topological full groups

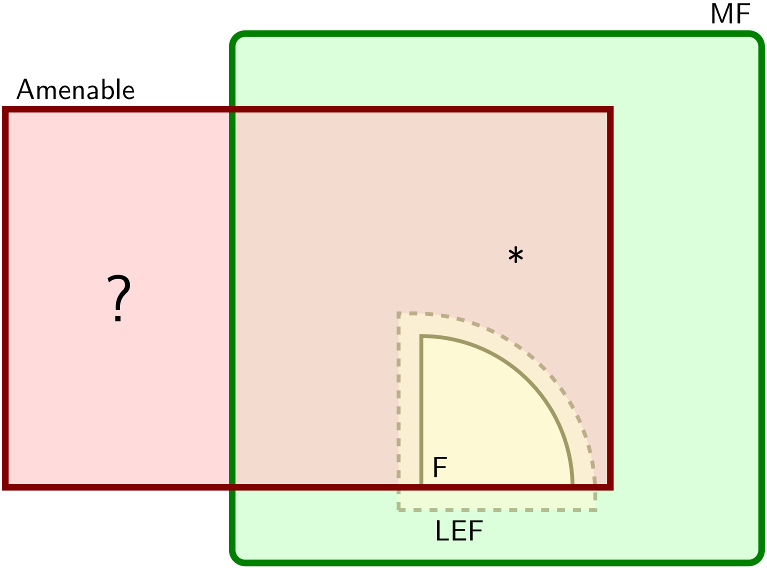
Definition

Let $\phi =$ minimal homeomorphism of the Cantor set X.
The *topological full group* $[[\phi]] :=$ {all homeomorphisms that are locally equal to some power of ϕ}.

- Giordano-Putnam-Skau
- Matui
- Grigorchuk-Medynets: $[[\phi]]$ is **LEF**.
- Juschenko-Monod: $[[\phi]]$ is **amenable**.
- Give first examples of finitely generated, simple, infinite amenable groups.

Using the above:

$C^*_\lambda\big([[\phi]] \big)$ is QD for any Cantor minimal system (X, ϕ).



∗: Abels (C-D-E)

Elementary amenable groups

The class of elementary amenable groups EG

EG = smallest class of groups containing all finite and abelian groups that is closed under taking subgroups, quotients, extensions, and direct limits.

Theorem (Ozawa-Rørdam-Sato '14)

Γ *is elementary amenable* $\quad \Rightarrow \quad C^*_\lambda(\Gamma)$ *is QD.*

Their result is stronger and covers a class larger than EG.

Local embeddability after Vershik-Gordon

Definition

Let $\mathcal{C}$ be a class of groups. Say that Γ is *locally embeddable into* $\mathcal{C}$ (LE$\mathcal{C}$) if $\forall\ K \subset\subset \Gamma\ \exists\ \Lambda \in \mathcal{C}$ and a function $\phi\colon \Gamma \to \Lambda$ s.t.

- $\phi(s)\phi(t) = \phi(st) \quad \forall\ s, t \in K$; and
- $\phi|_K$ is injective.

Residually $\mathcal{C}$ vs. LE$\mathcal{C}$

Suppose $\mathcal{C}$ is closed under taking subgroups and finite direct products. Then a finitely presented group is residually $\mathcal{C}$ iff it is LE$\mathcal{C}$.

- LEF: Vershik-Gordon '97
- LEA: Gromov '99 ("initially subamenable")

Local embeddability and MF groups

Recall:

Γ is **MF** if

$$\Gamma \hookrightarrow U\left(\frac{\prod \mathrm{M}_{n_k}(\mathbb{C})}{\sum \mathrm{M}_{n_k}(\mathbb{C})} \right).$$

Theorem (C)

$$\Gamma \textit{ is LEMF} \quad \Rightarrow \quad \Gamma \textit{ is MF.}$$

MF vs. hyperlinear vs. sofic

Theorem (Rădulescu '00)

Γ *is* **hyperlinear** $\Leftrightarrow \forall\ K \subset\subset \Gamma,\ \varepsilon > 0\ \exists\ n \in \mathbb{Z}_{>0}$ *and* $\phi\colon K \to U(n)$ *s.t.*

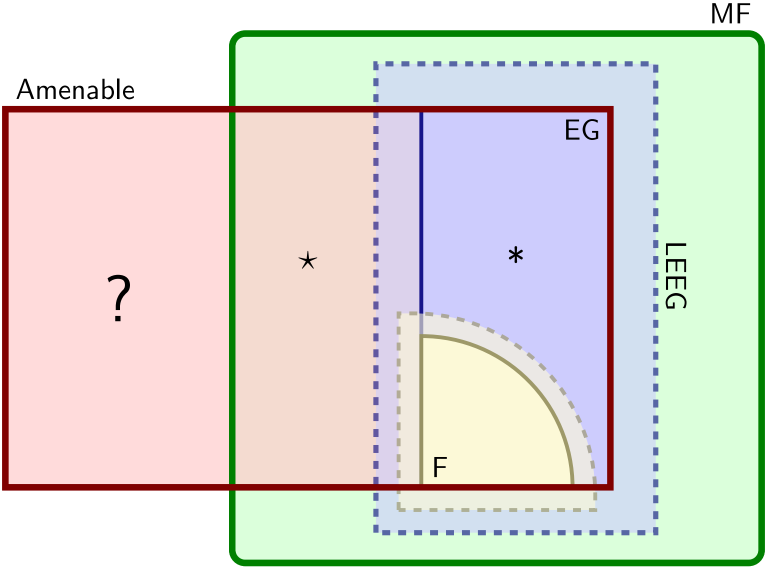
1. $\|\phi(st) - \phi(s)\phi(t)\|_{HS} < \varepsilon \quad \forall\ s, t \in K;$
2. $\|\phi(e) - 1_n\|_{HS} < \varepsilon \quad$ *if* $e \in K;$
3. $\|\phi(s) - \phi(t)\|_{HS} \geq 1/4 \quad \forall\ s, t \in K$ *with* $s \neq t.$

Theorem (Elek-Szabó '05)

Γ *is* **sofic** $\Leftrightarrow$ *. . . as above, but with* $\mathrm{Perm}(n)$ *instead of* $U(n)$.

Theorem (C)

Γ *is* **MF** $\Leftrightarrow$ *. . . as above but with* $\|\cdot\|$ *instead of* $\|\cdot\|_{HS}$.



$*$: Abels (C-D-E); $\star$: Grigorchuk, de Cornulier-Guyot-Pitsch (C)