Quasi-representations, almost flat bundles and K-theory

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joint work with Marius Dadarlat

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Some motivation

- Idea: obtain numerical invariants by pushing-forward K-theory elements via *approximately* multiplicative maps.

$$\pi\colon B \to M_n(\mathbb{C}) \quad \rightsquigarrow \quad \pi_\sharp\colon K_0(B) \to \mathbb{Z}$$

(e.g., $B = \ell^1 \Gamma or C^*(\Gamma)$)

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Why *approximately* multiplicative?

Proposition (Dadarlat)

Suppose Γ satisfies Baum-Connes and is torsion-free. If
$\pi\colon C^*(\Gamma) \to M_n(\mathbb{C})$
is a (unital) representation, then the induced map π_* on K_0 equals $n \cdot \iota_*$ where $\iota =$ trivial rep. of Γ.

Definition of a quasi-representation

$A =$ unital C^*-algebra (with a tracial state τ)

Definition

Let $\mathcal{F} \subset \Gamma$ be finite, $\varepsilon > 0$.
$\pi \colon \Gamma \to U(A)$ is an $(\mathcal{F}, \varepsilon)$-*representation* if $\forall\ s, t \in \mathcal{F}$:

- $\pi(1) = 1_A$
- $\|\pi(s^{-1}) - \pi(s)^*\| < \varepsilon$
- $\|\pi(st) - \pi(s)\pi(t)\| < \varepsilon$

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- *Quasi-representation:* $\mathcal{F}$, ε not necessarily specified.
- May extend a quasi-rep π of Γ to a unital, linear, *approximately multiplicative* contraction on $\ell^1(\Gamma)$
  (in the obvious way).

Pushing-forward using quasi-representations

Let $\pi\colon \Gamma \to U(A)$ be a quasi-representation. (Extend to $\ell^1(\Gamma)$.)

How to push-forward $x \in K_0(\ell^1(\Gamma))$

- choose idempotents e_0, e_1 in matrices over $\ell^1(\Gamma)$ s.t. $x = [e_0] - [e_1]$.
- π multiplicative enough $\Rightarrow \pi(e_i) \approx$ idempotent in matrices over A $\Rightarrow \pi(e_i)$ may be perturbed to idempotent f_i over A
- define $\pi_\sharp(x) = [f_0] - [f_1] \in K_0(A)$.

Existence of quasi-representations

Proposition (C-Dadarlat)

Suppose that

- $\Gamma =$ *discrete countable group with the Haagerup property*
- $B\Gamma =$ *finite simplicial complex*
- *all the elements of* $K^0(B\Gamma)$ *are almost flat.*

If $h\colon K_0(C^*(\Gamma)) \to \mathbb{Z}$ *is a group homomorphism and* $\mathcal{F} \subset K_0(C^*(\Gamma))$ *is finite, then there are* $m \in \mathbb{N}$ *and quasi-rep's* $\pi^+, \pi^-\colon \Gamma \to U(m)$ *s.t.*

$$h(y) = \pi^+_\sharp(y) - \pi^-_\sharp(y) \qquad \forall\, y \in \mathcal{F}.$$

Example: Almost commuting unitaries

Suppose $u, v \in U(n)$ and $\|uvu^{-1}v^{-1} - 1_n\|$ is small.

- Associate a quasi-representation $\pi\colon \mathbb{Z}^2 \to U(n)$ s.t.

$$s \mapsto u, \quad t \mapsto v, \quad st \mapsto uv$$

- $K_0(\ell^1(\mathbb{Z}^2)) \cong K_0(C(\mathbb{T}^2)) = \mathbb{Z}[1] \oplus \mathbb{Z}\beta.$ (β = Bott element)
- $\|[u, v] - 1_n\|$ small enough $\Rightarrow \pi$ multiplicative enough so that

$$\kappa(u, v) := \operatorname{Tr} \pi_\sharp(\beta) \in \mathbb{Z}$$

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Theorem (Exel-Loring '91)

If $\|[u, v] - 1_n\|$ is small enough, then

$$\kappa(u, v) = \frac{1}{2\pi i} \operatorname{Tr} \log([u, v]).$$

Setup for main result

Can we generalize Exel-Loring? In what context?

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M = g-holed torus (for some $g \in \mathbb{N}$)

$\Gamma := \pi_1(M)$. Known that:

$$\Gamma = \left\langle s_1, t_1, \ldots, s_g, t_g : \prod_{i=1}^{g} [s_i, t_i] \right\rangle$$

$$\left([s_i, t_i] := s_i t_i s_i^{-1} t_i^{-1} \right)$$

A = unital C^*-algebra with a tracial state τ

Fix a quasi-representation $\pi \colon \Gamma \to U(A)$.

The analogue of the invariant $\kappa(u, v)$

$$K_0(M) \xrightarrow{\mu} K_0\big(\ell^1(\Gamma)\big)$$

(Lafforgue's ℓ^1-version of) the assembly map μ is an isomorphism in this case.

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Main result

Theorem (C-Dadarlat)

Fix a genus $g \in \mathbb{N}$. There exist $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset \Gamma$ s.t. the following holds:

If A is a unital C^*-algebra, $\tau \in T(A)$ and $\pi : \Gamma \to U(A)$ is any $(\mathcal{F}, \varepsilon)$-representation, then

$$\tau(\pi_\sharp(\mu[M])) = \frac{1}{2\pi i}\tau\left(\log\left(\prod_{i=1}^{g}[\pi(s_i), \pi(t_i)]\right)\right).$$

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Remark

$\forall\ \varepsilon > 0$, finite $\mathcal{F} \subset \Gamma\ \ \exists\ \delta > 0$ s.t.
given A and $u_1, v_1, \ldots, u_g, v_g \in U(A)$ satisfying

$$\left\| \prod_{i=1}^{g} [u_i, v_i] - 1 \right\| < \delta,$$

then there is an $(\mathcal{F}, \varepsilon)$-representation $\pi \colon \Gamma \to U(A)$ with $\pi(s_i) = u_i$ and $\pi(t_i) = v_i$.

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Example: NC tori

$A_\theta := C^*(u, v \mid u, v \text{ are unitaries s.t. } vu = e^{2\pi i\theta} uv)$.

θ is small enough $\Rightarrow \exists$ quasi-representation $\pi\colon \mathbb{Z}^2 \to U(A_\theta)$ s.t. $\pi(s) = u$, $\pi(t) = v$ and

$$\tau\big(\pi_\sharp(\beta)\big) = \frac{1}{2\pi i} \tau\big(\log\,[u, v]\big) = -\theta.$$

The context for the formula

Recall the formula:

$$\tau(\pi_\sharp(\mu[M])) = \frac{1}{2\pi i}\tau\left(\log\left(\prod_{i=1}^{g}[\pi(s_i), \pi(t_i)]\right)\right)$$

The assembly map $\mu\colon K_0(M) \to K_0(\ell^1(\Gamma))$ is implemented by the *Mishchenko line bundle* ℓ:

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$$\mu[M] \longmapsto \tau\big(\pi_\sharp(\mu[M])\big)$$

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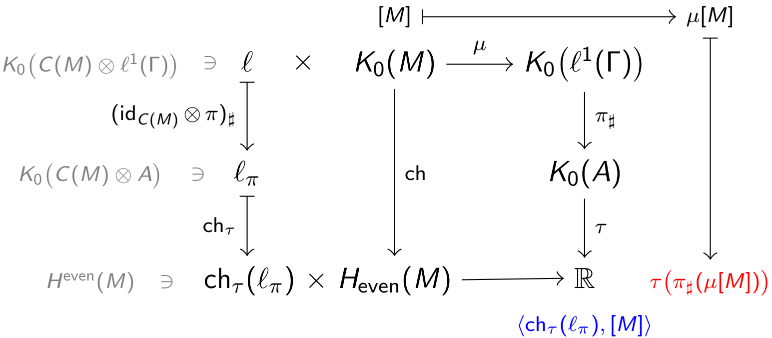

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K_0\big(C(M)\otimes A\big) & \ni & \ell_\pi & & \mathrm{ch} & & K_0(A) & \Big\downarrow \\
 & & \mathrm{ch}_\tau \Big\downarrow & & \Big\downarrow & & \Big\downarrow \tau & \Big\downarrow \\
H^{\mathrm{even}}(M) & \ni & \mathrm{ch}_\tau(\ell_\pi) & \times & H_{\mathrm{even}}(M) & \longrightarrow & \mathbb{R} & \tau\big(\pi_\sharp(\mu[M])\big) \\
 & & & & & & \langle \mathrm{ch}_\tau(\ell_\pi), [M]\rangle &
\end{array}$$

Special case of a Theorem of Dadarlat:

$$\tau\big(\pi_\sharp(\mu[M])\big) = \langle \mathrm{ch}_\tau(\ell_\pi), [M]\rangle$$

Hilbert A-module bundles

- $\ell \in K_0(C(M) \otimes \ell^1(\Gamma)) \Rightarrow$ push-forward $\ell_\pi \in K_0(C(M) \otimes A)$
- $K_0(C(M) \otimes A) =$ Grothendieck group of isomorphisms classes of f.g.p. *Hilbert A-module bundles*

Hilbert A-module bundle $E \to M$

- $E \to M$: fibers $\cong$ Hilbert A-modules
- $E =$ f.g.p. $\Rightarrow E$ has (unique) *smooth structure; curvature* Ω

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- For the proof of the theorem, we construct bundle $E_\pi \to M$ s.t. $[E_\pi] = \ell_\pi$.
- Construction is explicit enough that we may apply Chern-Weil theory, using $\mathrm{ch}_\tau(\ell_\pi) = \tau(i\Omega/2\pi) \in \Omega^2(M, \mathbb{C})$, to get

$$\langle \mathrm{ch}_\tau(\ell_\pi), [M] \rangle = \int_M \tau\left(\frac{i\Omega}{2\pi}\right).$$

Simplicial complexes and evaluation of the integral

- To construct E_π and deal with $\int_M \tau(i\Omega/2\pi)$ we work with a triangulation of M.
- Edges in simplicial complex $\rightsquigarrow$ elements of $\Gamma = \pi_1(M)$

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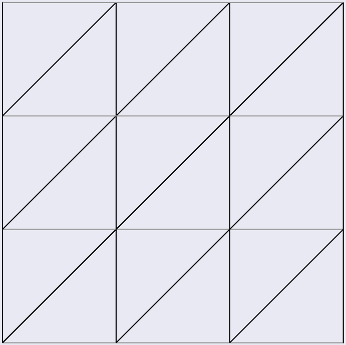

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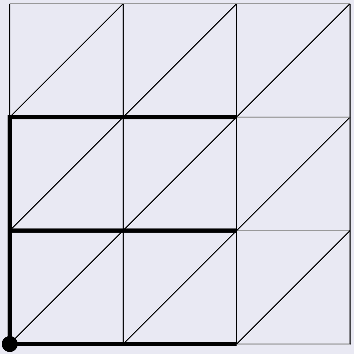

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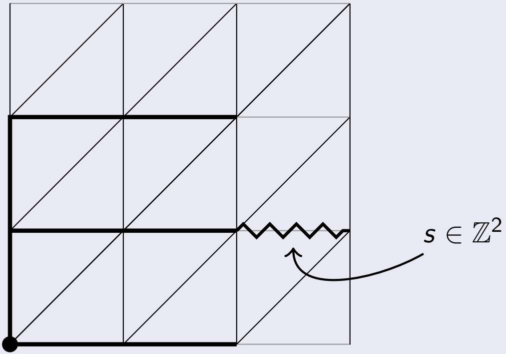

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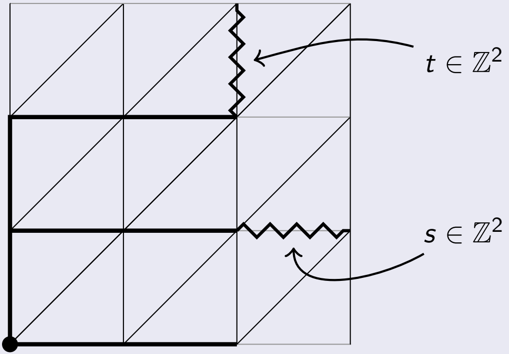
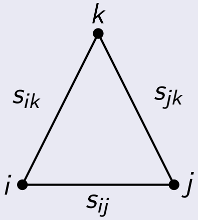


Evaluation of the integral (cont.)

On a 2-simplex $\sigma = \langle i, j, k \rangle \dots$



- s_{ij} := element of Γ corresp. to edge $i\,j$
- have "cocycle condition": $s_{ij}s_{jk} = s_{ik}$
- π quasi-representation
  $\Rightarrow \pi(s_{ij})\pi(s_{jk}) \approx \pi(s_{ik})$
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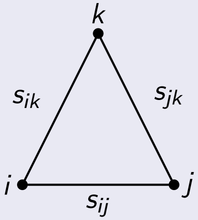

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Let ξ_σ = segment $\pi(s_{ij})\pi(s_{jk}) \rightsquigarrow \pi(s_{ik})$ in $\mathrm{GL}_\infty(A)$. Then

$$\int_M \tau(\Omega) = \qquad \tilde{\Delta}_\tau(\xi_\sigma).$$

where $\tilde{\Delta}_\tau$ = de la Harpe-Skandalis determinant.

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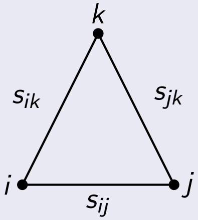

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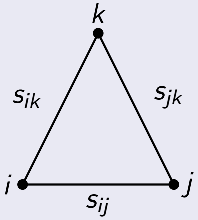

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To summarize

- Interested in how a quasi-rep π acts on $K_0(\ell^1(\Gamma))$, where Γ = surface group
- $K_0(\ell^1(\Gamma)) \cong \mathbb{Z} \oplus \mathbb{Z}\mu[M]$

To summarize

- Interested in how a quasi-rep π acts on $K_0(\ell^1(\Gamma))$, where $\Gamma =$ surface group
- $K_0(\ell^1(\Gamma)) \cong \mathbb{Z} \oplus \mathbb{Z}\mu[M]$

- Push-forward $\mu[M]$ via π and apply trace τ : $\tau\big(\pi_\sharp(\mu[M])\big)$
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- Use Chern-Weil theory for Hilbert A-module bundles to deal with $\mathrm{ch}_\tau(\ell_\pi)$.