On groups with quasidiagonal C^*-algebras

José R. Carrión

Purdue University

joint work with Marius Dadarlat and Caleb Eckhardt

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Quasidiagonality

Quasidiagonality for operators

Quasidiagonality for operators was introduced by Halmos.

- An **operator** $T \in \mathcal{B}(\mathcal{H})$ is **quasidiagonal** if $\exists$ finite rank projections $P_1 \leq P_2 \leq \cdots$ with $P_n \to 1_{\mathcal{H}}$ and $\|P_n T - T P_n\| \to 0$.
- (Equivalently, $T = D + K$ where K is compact and D is block-diagonal.)
- If $\mathcal{H}$ is separable, a (separable) **set** $\Omega \subset \mathcal{B}(\mathcal{H})$ is **quasidiagonal** if $\exists$ a sequence (P_n) as above that works simultaneously for all $T \in \Omega$.

Quasidiagonality for C^*-algebras

A **C*-algebra** is **quasidiagonal** if it has a faithful representation as a quasidiagonal set of operators.

Quasidiagonality (cont.)

- Quasidiagonality is a *local finite-dimensional approximation property* (Voiculescu)
- Connections to BDF and KK-theory, classification theory for nuclear C^*-algebras, AF-embeddability of C^*-algebras, ...

Our focus: quasidiagonality and group C^*-algebras.

From now on: all groups (Γ, Λ, Δ, etc.) are discrete and countable.

Rosenberg's theorem

Recall:

- $\lambda_s \in \mathcal{B}(\ell^2\Gamma) =$ left translation by $s \in \Gamma$.
- $C^*_\lambda(\Gamma) = C^*$-algebra generated by $\lambda(\Gamma) \subset \mathcal{B}(\ell^2\Gamma)$.

When is $C^*_\lambda(\Gamma)$ quasidiagonal?
Same as asking for which Γ is $\lambda(\Gamma)$ a QD set.

Theorem (Rosenberg '87)

If $\exists$ a sequence of projections $(P_n) \subset \mathcal{K}(\ell^2\Gamma)$ such that

$$\|P_n\lambda_s - \lambda_s P_n\| \to 0$$

$\forall$ *$s \in \Gamma$, then Γ is amenable.*
In particular, **for $C^*_\lambda(\Gamma)$ to be QD, Γ must be amenable.**

Rosenberg's theorem (cont.)

For finite $F \subset \Gamma$, let

$$C_F = \inf_{\substack{P \in \mathcal{K}(\ell^2\Gamma) \\ 0 \neq P = P^* = P^2}} \max_{s \in F} \|P\lambda_s - \lambda_s P\|.$$

Note: $C^*_\lambda(\Gamma)$ quasidiagonal $\Rightarrow C_F = 0$ for all finite $F \subset \Gamma$.

Alternate formulation of Rosenberg's theorem

Γ not amenable $\Rightarrow C_F > 0$ for some $F \subset \Gamma$.

A “quantitative” version of Rosenberg’s theorem

Paradoxical decompositions

Recall: Γ is not amenable iff $\exists$ disjoint $X_1, \ldots, X_n, Y_1, \ldots, Y_m \subset \Gamma$ and $s_1 = 1, \ldots, s_n, t_1 = 1, \ldots, t_m \in \Gamma$ s.t.

$$\left(\bigsqcup_{i=1}^{n} X_i\right) \sqcup \left(\bigsqcup_{j=1}^{m} Y_j\right) = \Gamma = \bigsqcup_{i=1}^{n} s_i X_i = \bigsqcup_{j=1}^{m} t_j Y_j$$

Theorem (C-Dadarlat-Eckhardt)

Suppose Γ has a paradoxical decomposition as above. Then

$$C_F \geq \frac{1}{n+m-2},$$

where $F = \{s_1, \ldots, s_n, t_1, \ldots, t_n\}$.

For example, $\langle a, b \rangle = \mathbb{F}_2 \subseteq \Gamma \Rightarrow C_F \geq 1/2$ for $F = \{a, b\}$.

Quasidiagonality and groups

Γ amenable $\Rightarrow C^*_\lambda(\Gamma)$ quasidiagonal **?**

There are several results in this direction. For example:

Theorem (Bekka '90)

Suppose Γ is amenable. Then

$$\Gamma \hookrightarrow U\left(\prod M_n(\mathbb{C})\right) \quad \Leftrightarrow \quad C^*_\lambda(\Gamma) \hookrightarrow \prod M_n(\mathbb{C}).$$

(That is, $\quad \Gamma$ *is MAP* $\quad \Leftrightarrow \quad C^*_\lambda(\Gamma)$ *is RFD.)*

In particular, Γ amenable MAP $\Rightarrow C^*_\lambda(\Gamma)$ is QD.

MF groups

Definition

Γ is **MF** if

$$\Gamma \hookrightarrow U\left(\frac{\prod M_{n_k}(\mathbb{C})}{\sum M_{n_k}(\mathbb{C})} \right)$$

for some increasing sequence (n_k).

($\prod$ means ℓ^∞-direct sum, $\sum$ means c_0-direct sum.)

Theorem (C-Dadarlat-Eckhardt)

Suppose Γ is amenable. Then

$$\Gamma \hookrightarrow U\left(\frac{\prod M_{n_k}(\mathbb{C})}{\sum M_{n_k}(\mathbb{C})} \right) \quad \Leftrightarrow \quad C^*_\lambda(\Gamma) \hookrightarrow \frac{\prod M_{n_k}(\mathbb{C})}{\sum M_{n_k}(\mathbb{C})}.$$

(That is, $\quad \Gamma$ *is MF* $\quad \Leftrightarrow \quad C^*_\lambda(\Gamma)$ *is QD.)*

Example: LEF groups ($\subset$ MF groups)

Definition (Gordon-Vershik)

Γ is *locally embeddable into the class of finite groups* (**LEF**) if $\forall$ finite $F \subset \Gamma$ $\exists$ a finite group Λ and a function $\phi\colon \Gamma \to \Lambda$ s.t. $\phi|_F$ is injective and multiplicative.

Every (locally) residually finite group is LEF.

Proposition

Γ is LEF $\Rightarrow$ Γ is MF.

Proposition: MF $\supsetneq$ LEF

Abels ('79) constructed a finitely presented solvable group Γ that is not residually finite. This group is not LEF. However, Γ is MF.

Example: topological full groups ($\subset$ LEF $\subset$ MF)

Definition

Let ϕ = minimal homeomorphism of the Cantor set X.
The *topological full group* $[[\phi]] := \{$all homeomorphisms that are locally equal to some power of $\phi\}$.

- *Giordano-Putnam-Skau:* two CMS (X, ϕ) and (X, ψ) are flip conjugate $\Leftrightarrow$ $[[\phi]] \cong [[\psi]]$.
- *Matui:* $[[\phi]]'$ is simple; $[[\phi]]$ is f.g. $\Leftrightarrow$ (X, ϕ) is a minimal subshift.
- *Grigorchuk-Medynets:* $[[\phi]]$ is LEF.
- *Juschenko-Monod:* $[[\phi]]$ is amenable.
- Used to provide first examples of finitely generated, simple, infinite amenable groups.

Using the above:

$C^*_\lambda\big([[\phi]]\big)$ is QD for any Cantor minimal system (X, ϕ).

Strong quasidiagonality

Definition (Hadwin '87)

A C^*-algebra A is **strongly quasidiagonal** if A/I is QD $\forall\ I \lhd A$.

Example

Every irrational rotation algebra (in fact every NC torus) is strongly QD.

Theorem (Kirchberg-Winter '04)

A has finite decomposition rank $\Rightarrow$ A is strongly QD.

(Decomposition rank: a NC analog of covering dimension, important in the classification theory of nuclear C^*-algebras.)

Groups and strong quasidiagonality

Theorem (C)

If $1 \to \Delta \to \Gamma \to \Lambda \to 1$ is an exact sequence of groups s.t.

- *$\Delta \leq Z(\Gamma)$ is f.g., and*
- *Λ is f.g. abelian*

Then $C^*_\lambda(\Gamma)$ has finite decomposition rank, so it is strongly QD.

For example:

C^*(integer Heisenberg group) is strongly QD.

On the other hand:

$C^*(\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z})$ is *not* strongly QD (it is QD, however)

This follows from a more general result.

Groups and strong quasidiagonality (cont.)

Theorem (C-Dadarlat-Eckhardt)

Let A = unital C^*-algebra, Γ = amenable group. If

- $\exists\, x \in A$, *rep's* π_0, π_1 *of* A *with* $\pi_0(x) = 0$, $\pi_1(x) = 1$, *and*
- $\mathbb{Z} \hookrightarrow \Gamma$,

then $A^{\otimes \Gamma} \rtimes_\beta \Gamma$ *is not strongly QD.*

(β = Bernoulli action of Γ on $A^{\otimes \Gamma}$)

This applies to $C^*_\lambda(\Delta \wr \Gamma)$ if e.g.

- $\exists$ non-trivial finite dim'l rep. of Δ, or
- Δ has a finite conjugacy class

(assuming $\mathbb{Z} \hookrightarrow \Gamma$.)