Quasi-homomorphisms and Surface Groups

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Some motivation

- Idea: obtain numerical invariants by pushing-forward K-theory elements via *approximately* multiplicative maps.

$$\pi\colon B \to M_n(\mathbb{C}) \quad \rightsquigarrow \quad \pi_\sharp\colon K_0(B) \to \mathbb{Z}$$

- Connes-Gromov-Moscovici: push-forward equivariant index of elliptic operators over M by quasi-representations of $\pi_1(M)$. Applications to Novikov conjecture.

Why *approximately* multiplicative?

Proposition (Dadarlat)

Suppose G satisfies Baum-Connes and is torsion-free. If
$\pi\colon C^*(G) \to M_n(\mathbb{C})$
is a (unital) representation, then the induced map π_* on K_0 equals $n \cdot \iota_*$ where $\iota =$ trivial rep. of G.

Definition of a quasi-homomorphism

$A =$ unital C^*-algebra (with a tracial state τ)

Definition

Let $\mathcal{F} \subset G$ be finite, $\varepsilon > 0$.
$\pi\colon G \to U(A)$ is an $(\mathcal{F}, \varepsilon)$*-homomorphism* if $\forall\ s, t \in \mathcal{F}$:

- $\pi(1) = 1_A$
- $\|\pi(st) - \pi(s)\pi(t)\| < \varepsilon$

- *Quasi-homomorphism:* $\mathcal{F}$, ε not necessarily specified.
- May extend a quasi-hom. π of G to a unital, linear, *approximately multiplicative* contraction on $\ell^1(G)$ (in the obvious way).

Pushing-forward via quasi-homomorphisms

Let $\pi\colon G \to U(A)$ be a quasi-homomorphism. (Extend to $\ell^1(G)$.)

How to push-forward $x \in K_0(\ell^1(G))$

- choose idempotents e_0, e_1 in matrices over $\ell^1(G)$ s.t. $x = [e_0] - [e_1]$.
- π multiplicative enough $\Rightarrow \pi(e_i) \approx$ idempotent in matrices over A
  $\Rightarrow \pi(e_i)$ may be perturbed to idempotent f_i over A
- define $\pi_\sharp(x) = [f_0] - [f_1] \in K_0(A)$.

To obtain numerical invariants, apply trace $\tau \in T(A)$.

Example: Almost commuting unitaries

Suppose $u, v \in U(n)$ and $\|uvu^{-1}v^{-1} - 1_n\|$ is small.

- Associate a quasi-homomorphism $\pi\colon \mathbb{Z}^2 \to U(n)$ s.t.

$$s \mapsto u, \quad t \mapsto v, \quad st \mapsto uv$$

- $K_0(\ell^1(\mathbb{Z}^2)) \cong K_0(C(\mathbb{T}^2)) = \mathbb{Z}[1] \oplus \mathbb{Z}\beta. \quad (\beta = \text{Bott element})$
- $\|[u, v] - 1_n\|$ small enough $\Rightarrow \pi$ multiplicative enough so that

$$\kappa(u, v) := \operatorname{Tr} \pi_\sharp(\beta) \in \mathbb{Z}$$

  is defined.

Theorem (Exel-Loring '91)

If $\|[u, v] - 1_n\|$ is small enough, then

$$\kappa(u, v) = \frac{1}{2\pi i} \operatorname{Tr} \log([u, v]).$$

Setup for main result

Can we generalize Exel-Loring? In what context?

M = closed, connected, orientable surface

$\Rightarrow M$ is an m-holed torus (for some $m \in \mathbb{N}$)

$G := \pi_1(M)$. Known that:

$$G = \left\langle s_1, t_1, \ldots, s_m, t_m : \prod_{i=1}^{m} [s_i, t_i] \right\rangle$$

$$\left(\, [s_i, t_i] := s_i t_i s_i^{-1} t_i^{-1} \, \right)$$

A = unital C^*-algebra with a tracial state τ

Fix a quasi-homomorphism $\pi \colon G \to U(A)$.

The analogue of the invariant $\kappa(u, v)$

$$
\begin{array}{ccccc}
[M] & & \mathbb{Z} \oplus \mathbb{Z}\mu[M] & & \\
\pitchfork & & \wr\!\!\mid\mid & & \\
K_0(M) & \xrightarrow{\ \mu\ } & K_0\big(\ell^1(G)\big) & \ni & \mu[M] \\
 & & \downarrow \pi_\sharp & & \Big\downarrow \\
 & & K_0(A) & & \Big\downarrow \\
 & & \downarrow \tau & & \Big\downarrow \\
 & & \mathbb{R} & \ni & \tau\big(\pi_\sharp(\mu[M])\big)
\end{array}
$$

The (Lafforgue version of the) assembly map μ is an isomorphism in this case.

Main result

Theorem (C-Dadarlat)

Fix a genus $m \in \mathbb{N}$. There exist $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset G$ s.t. the following holds:

If A is a unital C^*-algebra, $\tau \in T(A)$ and $\pi \colon G \to U(A)$ is any $(\mathcal{F}, \varepsilon)$-homomorphism, then

$$\tau\left(\pi_\sharp(\mu[M])\right) = \frac{1}{2\pi i}\tau\left(\log\left(\prod_{i=1}^{m}\left[\pi(s_i), \pi(t_i)\right]\right)\right).$$

Remark

$\forall\ \varepsilon > 0$, finite $\mathcal{F} \subset G\ \ \exists\ \delta > 0$ s.t.
given A and $u_1, v_1, \dots, u_m, v_m \in U(A)$ satisfying

$$\left\| \prod_{i=1}^{m} [u_i, v_i] - 1 \right\| < \delta,$$

then there is an $(\mathcal{F}, \varepsilon)$-homomorphism $\pi\colon G \to U(A)$ with $\pi(s_i) = u_i$ and $\pi(t_i) = v_i$.

Example: NC tori

$A_\theta := C^*(u, v \mid u, v$ are unitaries s.t. $vu = e^{2\pi i\theta} uv)$.

θ is small enough $\Rightarrow \exists$ quasi-homomorphism $\pi\colon \mathbb{Z}^2 \to U(A_\theta)$ s.t. $\pi(s) = u$, $\pi(t) = v$ and

$$\tau\big(\pi_\sharp(\beta)\big) = \frac{1}{2\pi i}\tau\big(\log\,[u, v]\big) = -\theta.$$

The context for the formula

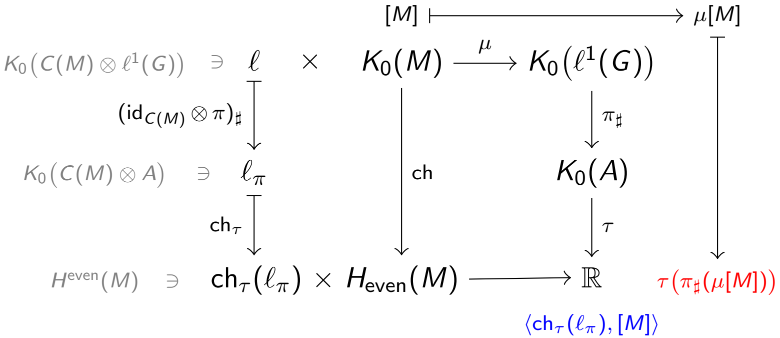
Recall the formula:

$$\tau\left(\pi_\sharp(\mu[M])\right) = \frac{1}{2\pi i}\tau\left(\log\left(\prod_{i=1}^{m}[\pi(s_i),\pi(t_i)]\right)\right)$$

The assembly map $\mu\colon K_0(M) \to K_0(\ell^1(G))$ is implemented by the *Mishchenko line bundle* ℓ:

$$\begin{array}{ccccc} K_0(C(M)\otimes \ell^1(G)) & & KK(C(M),\mathbb{C}) & & KK(\mathbb{C},\ell^1(G)) \\ \Cup & & \| & & \| \\ \ell & \times & K_0(M) & \xrightarrow{\ \mu\ } & K_0(\ell^1(G)) \end{array}$$

The context for the formula (cont.)



Special case of a Theorem of Dadarlat:

$$\tau(\pi_\sharp(\mu[M])) = \langle \mathrm{ch}_\tau(\ell_\pi), [M]\rangle$$

Hilbert A-module bundles

- $\ell \in K_0(C(M) \otimes \ell^1(G)) \Rightarrow$ push-forward $\ell_\pi \in K_0(C(M) \otimes A)$
- $K_0(C(M) \otimes A) =$ Grothendieck group of isomorphisms classes of f.g.p. *Hilbert A-module bundles*

Hilbert A-module bundle $\mathcal{E} \to M$

- $\mathcal{E} \to M$: fibers $\cong$ Hilbert A-modules
- $\mathcal{E} =$ f.g.p. $\Rightarrow \mathcal{E}$ has (unique) *smooth structure; curvature* Ω

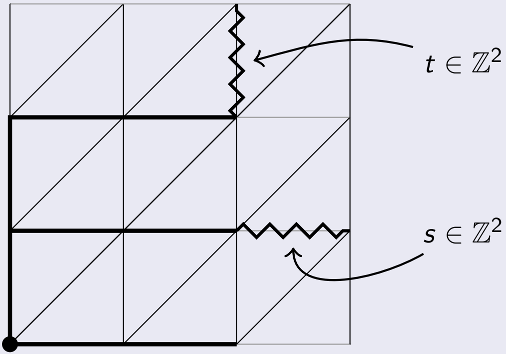
- For the proof of the theorem, we construct bundle $\mathcal{E}_\pi \to M$ s.t. $[\mathcal{E}_\pi] = \ell_\pi$.
- Construction is explicit enough that we may apply Chern-Weil theory, using $\mathrm{ch}_\tau(\ell_\pi) = \tau(i\Omega/2\pi) \in \Omega^2(M, \mathbb{C})$, to get

$$\langle \mathrm{ch}_\tau(\ell_\pi), [M] \rangle = \int_M \tau\left(\frac{i\Omega}{2\pi}\right) \overset{\cdots}{=} \frac{1}{2\pi i}\tau\left(\log\left(\prod_{i=1}^{m}[\pi(s_i), \pi(t_i)]\right)\right).$$

Simplicial complexes and evaluation of the integral

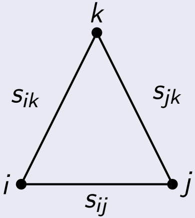
- To construct $\mathcal{E}_\pi$ and deal with $\int_M \tau(i\Omega/2\pi)$ we work with a triangulation of M.
- Edges in simplicial complex $\rightsquigarrow$ elements of $\pi_1(M)$

$m = 1$ case: $\mathbb{T}^2$



Evaluation of the integral (cont.)

On a 2-simplex $\sigma = \langle i, j, k \rangle \ldots$



- $s_{ij} :=$ element of G corresp. to edge $i\,j$
- have "cocycle condition": $s_{ij}s_{jk} = s_{ik}$
- π quasi-homomorphism
  $\Rightarrow \pi(s_{ij})\pi(s_{jk}) \approx \pi(s_{ik})$
- in the $\mathbb{T}^2$ case:
  $\pi(t)\pi(s) = vu \approx uv = \pi(ts)$

Let $\xi_\sigma =$ segment $\pi(s_{ij})\pi(s_{jk}) \rightsquigarrow \pi(s_{ik})$ in $\mathrm{GL}_\infty(A)$. Then

$$\int_M \tau(\Omega) = \sum_\sigma (-1)^{\mathbf{o}(\sigma)} \tilde{\Delta}_\tau(\xi_\sigma).$$

where $\tilde{\Delta}_\tau =$ de la Harpe-Skandalis determinant.

To summarize

- Interested in how a quasi-hom π acts on $K_0(\ell^1(G))$, where $G =$ surface group
- $K_0(\ell^1(G)) \cong \mathbb{Z} \oplus \mathbb{Z}\mu[M]$

- Push-forward $\mu[M]$ via π and apply trace τ : $\tau\big(\pi_\sharp(\mu[M])\big)$
- Push-forward ℓ, apply ch_τ etc. : $\langle \mathrm{ch}_\tau(\ell_\pi), [M] \rangle$.

Dadarlat: red invariant = blue invariant.

- Use Chern-Weil theory for Hilbert A-module bundles to deal with $\mathrm{ch}_\tau(\ell_\pi)$.