G-odometers and Classification of C^*-algebras

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Odometers

One picture: addition with carry over

Space: $X = \prod_{i=1}^{\infty}\{0,1\}$; action: $+1$ with carry over:

$$(1\ 1\ 0\ 0\ \cdots) \overset{+1}{\longmapsto} (0\ 0\ 1\ 0\ \cdots) \overset{+1}{\longmapsto} (1\ 0\ 1\ 0\ \cdots).$$

Connection with 2-adic integers

Associate such sequences with formal sums:

$$(1\ 1\ 0\ 0\ \cdots) \longleftrightarrow 1\cdot 2^0 + 1\cdot 2^1 + 0\cdot 2^2 + 0\cdot 2^3 + \cdots$$
$$(0\ 0\ 1\ 0\ \cdots) \longleftrightarrow 0\cdot 2^0 + 0\cdot 2^1 + 1\cdot 2^2 + 0\cdot 2^3 + \cdots$$

Action $\longleftrightarrow$ (formal) addition of 1.

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Action $\longleftrightarrow$ (formal) addition of 1.

Odometers (cont.)

Regard partial sums as elements of $\mathbb{Z}/2^i\mathbb{Z}$:

$$1 \cdot 2^0 = 1 \in \mathbb{Z}/2\mathbb{Z}, \quad 1 \cdot 2^0 + 1 \cdot 2^1 = 3 \in \mathbb{Z}/4\mathbb{Z},$$
$$1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 = 3 \in \mathbb{Z}/8\mathbb{Z}, \quad \text{etc.}$$

Associate $(1\ 1\ 0\ \cdots) \in X$
with $(1, 3, 3, \dots) \in \varprojlim \mathbb{Z}/2^i\mathbb{Z}$.

Alternate picture: inverse limits and (left) translation

Space:

$$\varprojlim \mathbb{Z}/2^i\mathbb{Z} = \left\{ (s_i) \in \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z} \;\middle|\; s_i = s_{i+1} \mod 2^i \right\}$$

Action: coordinate-wise addition of $(1, 1, \dots)$.

An example: the discrete Heisenberg group

The discrete Heisenberg group $\mathbb{H}_3$: 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbb{Z}$ (usual matrix multiplication). It is a central extension of $\mathbb{Z}$ by $\mathbb{Z}^2$.

$\mathbb{H}_3$ is amenable and *residually finite.* Let $L_i := \mathbb{H}_3(2^i\mathbb{Z})$; then

$$\mathbb{H}_3 \supset L_1 \supset L_2 \supset \cdots$$

is a nested sequence of finite index normal subgroups satisfying

$$\bigcap_{i \geq 1} L_i = \{1\}.$$

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A crossed product associated with $\mathbb{H}_3$

Form the *profinite completion* w.r.t. this sequence:

$$\widetilde{\mathbb{H}}_3 := \varprojlim \mathbb{H}_3/L_i \quad \Big(\subseteq \prod_{i\geq 1} \mathbb{H}_3/L_i\Big).$$

This is a compact, Hausdorff and totally disconnected group.

Our main example this talk:

$\mathbb{H}_3$ acts on $\widetilde{\mathbb{H}}_3$ by left multiplication. The crossed product

$$C(\widetilde{\mathbb{H}}_3) \rtimes \mathbb{H}_3$$

is an example of a *generalized Bunce-Deddens algebra.*

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Definition of a generalized Bunce-Deddens algebra

Definition (Orfanos, 2008)

$G =$ discrete, countable, amenable and residually finite group

$(L_i) =$ nested sequence of finite index normal subgroups with trivial intersection

Form the profinite completion $\tilde{G}$ just as above. Call $C(\tilde{G}) \rtimes G$ a **generalized Bunce-Deddens algebra** associated with G.

Theorem (Orfanos, 2008)

A generalized Bunce-Deddens algebra

- *is simple, separable and nuclear;*
- *has real rank zero, stable rank one, and a unique trace;*
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Are these algebras classified by their Elliott invariant?

Definition

Elliott invariant for a generalized B-D algebra A:

$$\mathrm{Ell}(A) = \big(K_0(A), K_0(A)^+, [1_A]_0, K_1(A)\big).$$

For generalized B-D algebras A, B:

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \overset{?}{\Longrightarrow} A \cong B$$

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Some background and motivation

Example: Bunce-Deddens algebras

- $G = \mathbb{Z}$: "classical" Bunce-Deddens algebras. Classification was carried out by J. Bunce and J. Deddens (1973).
- Classification using modern tools could be obtained by recalling that a Bunce-Deddens algebra is an inductive limit of C^*-algebras of the form $M_r(\mathbb{C}) \otimes C(\mathbb{T})$, and noting that $\mathbb{T}$ has covering dimension 1.

A generalized B-D algebra is an inductive limit in a similar way:

$$C(\tilde{G}) \rtimes G \cong \varinjlim C(G/L_i) \rtimes G \cong \varinjlim M_{r_i}(\mathbb{C}) \otimes C^*(L_i).$$

We will use a version of noncommutative covering dimension due to Kirchberg and Winter: the *decomposition rank.*

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We will use a version of noncommutative covering dimension due to Kirchberg and Winter: the *decomposition rank.*

The main result

$\mathcal{G}$ = class of all (discrete) groups G s.t. there exist finitely generated abelian groups N, Q with G a central extension

$$1 \to N \to G \to Q \to 1.$$

(The groups in $\mathcal{G}$ are residually finite.)

Theorem

If $A = C(\tilde{G}_1) \rtimes G_1$ and $B = C(\tilde{G}_2) \rtimes G_2$ are generalized Bunce-Deddens algebras associated to groups G_i in $\mathcal{G}$, then

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Classification via decomposition rank

General classification result we rely on (Winter, 2005):

Under the "right" conditions (including e.g. simplicity, real rank zero), finite decomposition rank is enough for classification.

Orfanos' results $\Rightarrow$ generalized B-D algebras satisfy the "right" conditions. Therefore, only need:

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What happens in our example?

Back to $\mathbb{H}_3$

$C(\widetilde{\mathbb{H}}_3) \rtimes \mathbb{H}_3 =$ inductive limit of $C^*(L_i) \otimes M_{r_i}(\mathbb{C})$.

Since

$$\operatorname{dr}\left(\varinjlim C^*(L_i) \otimes M_{r_i}(\mathbb{C})\right) \leq \underline{\lim}\, \operatorname{dr} C^*(L_i),$$

can focus on $C^*(L_i)$. *Motivation:* $L_i = \mathbb{H}_3(2^i\mathbb{Z})$ "not that different" from $\mathbb{H}_3$.

What do we know about $C^*(\mathbb{H}_3)$?

$C^*(\mathbb{H}_3) \cong$ sections of continuous field over $\mathbb{T}$ with fibers A_θ (Anderson-Paschke '89)

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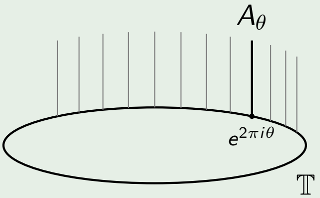
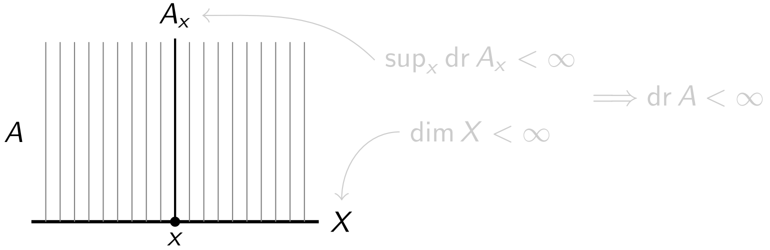


The decomposition rank of a $C(X)$-algebra



Proposition

Let X be a compact metric space and A a (separable) $C(X)$-algebra. If $\dim X \leq l$ and $\sup_x \operatorname{dr} A_x \leq k$, then

$$\operatorname{dr} A \leq (l+1)(k+1) - 1.$$

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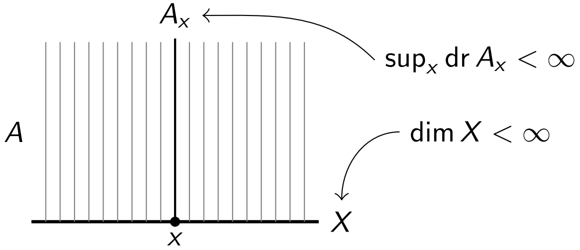

$\Longrightarrow \operatorname{dr} A < \infty$

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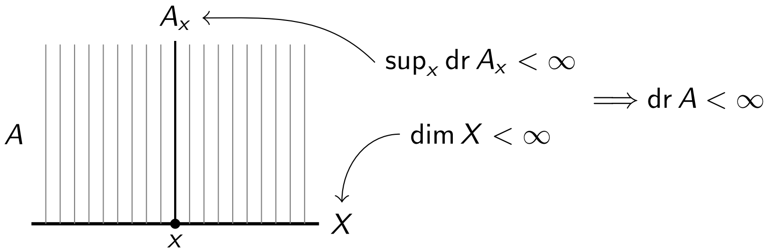

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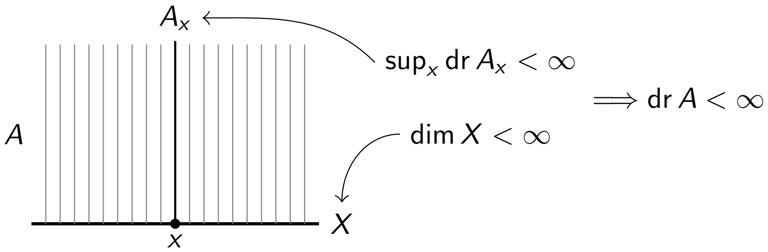

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What is decomposition rank?

A contractive completely positive (c.c.p.) map $\varphi\colon A \to B$ has **order zero** if $\varphi(a)\varphi(b) = 0$ whenever $a, b \geq 0$ with $ab = 0$.

Definition (Kirchberg-Winter, 2003)

A has **decomposition rank** at most n if there exist

- finite dimensional C^*-algebras F_k,
- c.c.p. maps $A \xrightarrow{\varphi_k} F_k \xrightarrow{\psi_k} A$ with $\psi_k \circ \varphi_k \to \mathrm{id}_A$ pointwise, and
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C^*-algebras with finite decomposition rank

Some examples

- dr $C(X) = \dim X$ (Winter)
- dr $A = 0 \Leftrightarrow A$ is an AF algebra (Winter)
- dr $A_\theta \leq 2n+1$, where $A_\theta =$ noncommutative n-torus (uses Phillips' result that every *simple* NC torus is an A$\mathbb{T}$ algebra)

Proposition

Let G be a central extension

$$1 \to N \to G \to Q \to 1$$

where N and Q are finitely generated abelian groups of ranks n and m, respectively (i.e. $G \in \mathcal{G}$). Then

$$\text{dr}\, C^*(G) \leq (n+1)(2m^2+4m+1) - 1.$$

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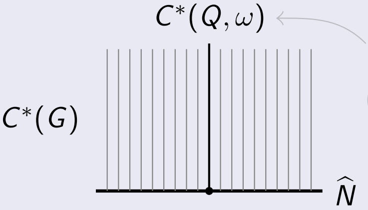
How this ties in with our example

$C^*(G)$ as a continuous field

G = central extension of N by Q. Then (by Packer-Raeburn, '92):



twisted group algebra

(e.g., $C^*(\mathbb{Z}^m,\omega) = A_\theta$)

Have that dr $C^*(Q,\omega) \leq (m+1)(2m+2)-1$ for Q finitely generated abelian of rank $\leq m$.

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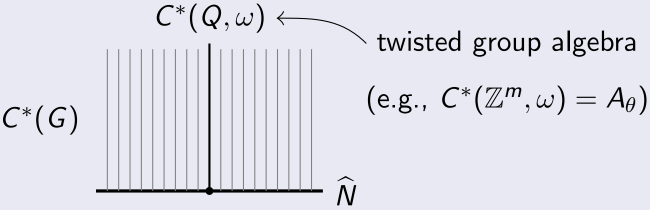

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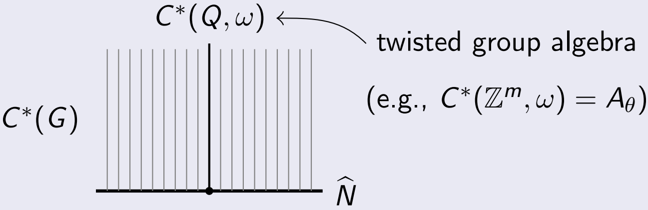

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- Want to classify $C(\tilde{G}) \rtimes G$ (for $G \in \mathcal{G}$).
- Enough to prove $C(\tilde{G}) \rtimes G$ has finite decomposition rank.
- Reduces to proving $C^*(G)$ has finite decomposition rank for every $G \in \mathcal{G}$

dr $C^*(G) < \infty \rightsquigarrow$ dr of continuous fields

- $A =$ continuous field over a finite dimensional space with fibers of uniformly bounded dr $\Rightarrow$ dr $A < \infty$.
- $C^*(G)$ is the C^*-algebra of a continuous field over a finite dimensional space with fibers of the form $C^*(Q, \omega)$.
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