Quasi-homomorphisms and Surface Groups

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(joint work with Marius Dadarlat)

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Some motivation

- Idea: obtain numerical invariants by pushing-forward K-theory elements via *approximately* multiplicative maps.

$$\pi\colon B \to M_n(\mathbb{C}) \quad \rightsquigarrow \quad \pi_\sharp\colon K_0(B) \to \mathbb{Z}$$

- Connes-Gromov-Moscovici: push-forward equivariant index of elliptic operators over M by quasi-representations of $\pi_1(M)$. Applications to Novikov conjecture.

Why *approximately* multiplicative?

Proposition (Dadarlat)

Suppose G satisfies Baum-Connes. If $\pi\colon C^*(G) \to M_n(\mathbb{C})$ is a (unital) representation, then the induced map π_* on K_0 equals $n \cdot \iota_*$ where $\iota =$ trivial rep. of G.

$G =$ always countable and discrete

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Definition of a quasi-homomorphism

$A =$ unital C^*-algebra (with a tracial state τ)

Definition

Let $\mathcal{F} \subset G$ be finite, $\varepsilon > 0$.
$\pi \colon G \to U(A)$ is an $(\mathcal{F}, \varepsilon)$-*homomorphism* if

- $\pi(1) = 1_A$
- $\pi(s^{-1}) = \pi(s)^*$
- $\|\pi(st) - \pi(s)\pi(t)\| < \varepsilon \quad \forall\, s, t \in \mathcal{F}$.

- *Quasi-homomorphism:* $\mathcal{F}$, ε not necessarily specified.
- May extend a quasi-hom. π of G to a unital, linear, *approximately multiplicative* contraction on $\ell^1(G)$ (in the obvious way).

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Pushing-forward via quasi-homomorphisms

Let $\pi\colon G \to U(A)$ be a quasi-homomorphism. (Extend to $\ell^1(G)$.)

How to push-forward $x \in K_0(\ell^1(G))$

- choose idempotents e_0, e_1 is matrices over $\ell^1(G)$ s.t. $x = [e_0] - [e_1]$.
- π multiplicative enough $\Rightarrow \pi(e_i) \approx$ idempotent in matrices over A $\Rightarrow \pi(e_i)$ may be perturbed to idempotent f_i over A
- define $\pi_\sharp(x) = [f_0] - [f_1] \in K_0(A)$.

To obtain numerical invariants, apply trace $\tau \in T(A)$.

Example: Almost commuting unitaries

Suppose $u, v \in U(n)$ and $\|uv - vu\|$ is small.

- Associate a quasi-homomorphism $\pi \colon \mathbb{Z}^2 \to U(n)$ such that $s \mapsto u$, $t \mapsto v$ and $st \mapsto uv$.
- $K_0(\ell^1(\mathbb{Z}^2)) \cong K_0(C(\mathbb{T}^2)) = \mathbb{Z}[1] \oplus \mathbb{Z}\beta$ where β is the Bott element.
- If u and v commute enough, then π is multiplicative enough to consider

$$\kappa(u, v) := \operatorname{tr} \pi_\sharp(\beta) \in \mathbb{Z}$$

(tr = normalized trace on $M_n(\mathbb{C})$).

Theorem (Exel-Loring '91)

If $\|uv - vu\|$ is small enough, then

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Setup for main result

Can we generalize Exel-Loring? What is the right context?

M = closed, connected, orientable surface
$\Rightarrow M$ is a m-holed torus (for some $m \in \mathbb{N}$)

$G := \pi_1(M)$. Known that

$$G = \left\langle s_1, t_1, \ldots, s_m, t_m : \prod_{i=1}^{m} [s_i, t_i] \right\rangle$$

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The analogue of the invariant $\kappa(u, v)$

$$K_0(M) \xrightarrow{\mu} K_0\big(\ell^1(G)\big)$$

The (Lafforgue version of the) assembly map μ is an isomorphism in this case.

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Statement of the main result

Theorem (C-Dadarlat)

Let $m \in \mathbb{N}$. There exist $\varepsilon_0 > 0$ and a finite subset $\mathcal{F}_0 \subset G$ s.t. $\forall\ 0 < \varepsilon < \varepsilon_0$ and finite $\mathcal{F}_0 \subset \mathcal{F} \subset G$, the following holds:

If A is a unital C^*-algebra, $\tau \in T(A)$ and $\pi\colon G \to U(A)$ is any $(\mathcal{F}, \varepsilon)$-homomorphism, then

$$\tau\left(\pi_\sharp(\mu[M])\right) = \frac{1}{2\pi i}\tau\left(\log\left(\prod_{i=1}^{m}\left[\pi(s_i), \pi(t_i)\right]\right)\right).$$

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Remark

$\forall\ \varepsilon > 0$, finite $\mathcal{F} \subset G\ \ \exists\ \delta > 0$ s.t.
given A and $u_1, v_1, \ldots, u_m, v_m \in U(A)$ satisfying

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then there is an $(\mathcal{F}, \varepsilon)$-homomorphism $\pi \colon G \to U(A)$ with $\pi(s_i) = u_i$ and $\pi(t_i) = v_i$.

Example: Rotation algebras

Recall: $A_\theta := C^*(u, v \mid u, v \text{ are unitaries s.t. } uv = e^{2\pi i\theta} vu)$.

θ is small enough $\Rightarrow \exists$ quasi-homomorphism $\pi \colon \mathbb{Z}^2 \to U(A_\theta)$ s.t. $\pi(s) = u$, $\pi(t) = v$ and

$$\tau\big(\pi_\sharp(\beta)\big) = \frac{1}{2\pi i}\tau\big(\log{[u, v]}\big) = \theta.$$

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The right context for the formula

Recall the formula:

$$\tau\left(\pi_\sharp(\mu[M])\right) = \frac{1}{2\pi i}\tau\left(\log\left(\prod_{i=1}^{m}\left[\pi(s_i), \pi(t_i)\right]\right)\right)$$

The assembly map $\mu\colon K_0(M) \to K_0(\ell^1(G))$ is implemented by the *Mishchenko line bundle* ℓ:

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H^{\mathrm{even}}(M) & \ni & \mathrm{ch}_\tau(\ell_\pi) & \times & H_{\mathrm{even}}(M) & \longrightarrow & \mathbb{R} & & \tau\big(\pi_\sharp(\mu[M])\big) \\
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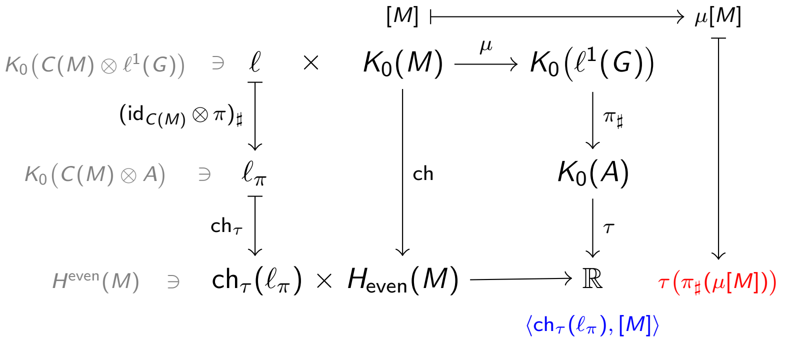

Special case of a Theorem of Dadarlat:

$$\tau(\pi_\sharp(\mu[M])) = \langle \mathrm{ch}_\tau(\ell_\pi), [M] \rangle$$

Hilbert A-module bundles

- $\ell \in K_0(C(M) \otimes \ell^1(G)) \Rightarrow$ push-forward $\ell_\pi \in K_0(C(M) \otimes A)$
- $K_0(C(M) \otimes A) =$ Grothendieck group of isomorphisms classes of f.g.p. *Hilbert A-module bundles*

Hilbert A-module bundle $E \to M$

- $E \to M$: fibers $\cong$ Hilbert A-modules
- $E =$ f.g.p. $\Rightarrow E$ has (unique) *smooth structure*; *curvature* Ω

- For the proof of the theorem, we construct bundle $E_\pi \to M$ s.t. $[E_\pi] = \ell_\pi$.
- Construction is explicit enough that we may use Chern-Weil theory, using $\mathrm{ch}_\tau(\ell_\pi) = \mathrm{ch}_\tau(E_\pi) = \tau(\Omega) \in \Omega^2(M, \mathbb{C})$, to get

$$\langle \mathrm{ch}_\tau(\ell_\pi), [M] \rangle = \int_M \tau(\Omega) \overset{\cdots}{=} \frac{1}{2\pi i} \tau\left(\log\left(\prod_{i=1}^m [\pi(s_i), \pi(t_i)]\right)\right).$$

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Simplicial complexes and evaluation of the integral

- To construct E_π and deal with $\int_M \tau(\Omega)$ we work with a triangulation of M.
- Edges in simplicial complex $\rightsquigarrow$ group elements

$m = 1$ case: $\mathbb{T}^2$

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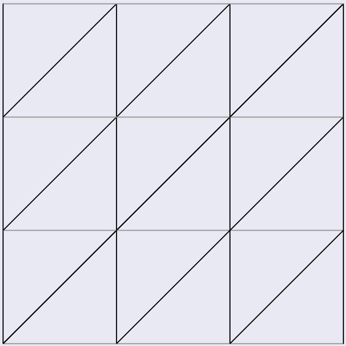

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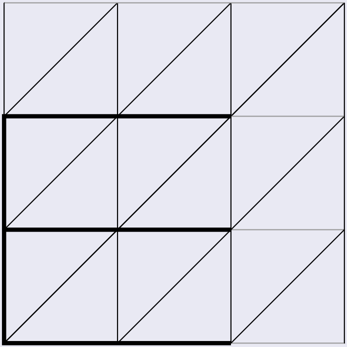

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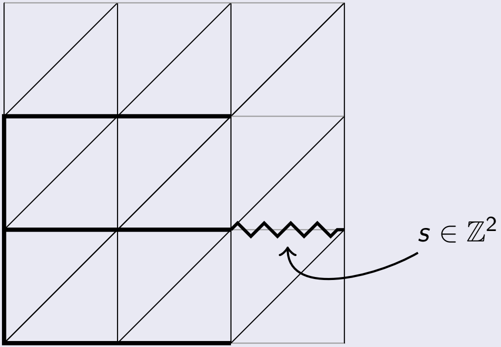

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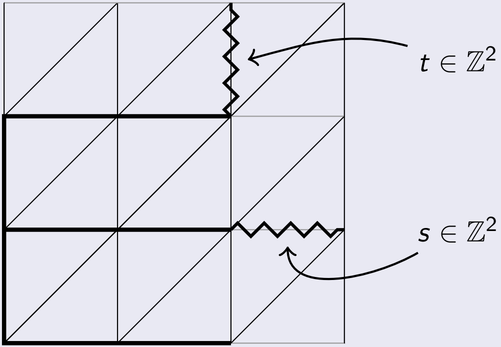
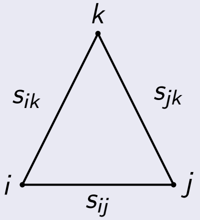


Evaluation of the integral (cont.)

On a 2-simplex $\sigma = \langle i, j, k \rangle \ldots$



- $s_{ij} :=$ element of G corresp. to edge $i\,j$
- have "cocycle condition": $s_{ij} s_{jk} = s_{ik}$
- π quasi-homomorphism
  $\Rightarrow \pi(s_{ij})\pi(s_{jk}) \approx \pi(s_{ik})$
- think of $\mathbb{T}^2$ case:
  $\pi(t)\pi(s) = vu \approx uv = \pi(ts)$

Let $\xi_\sigma =$ segment $\pi(s_{ij})\pi(s_{jk}) \rightsquigarrow \pi(s_{ik})$ in $\mathrm{GL}_\infty(A)$. Then

$$\int_M \tau(\Omega) = \sum_\sigma (-1)^{\mathfrak{o}(\sigma)} \tilde{\Delta}_\tau(\xi_\sigma).$$

where $\tilde{\Delta}_\tau =$ de la Harpe-Skandalis determinant.

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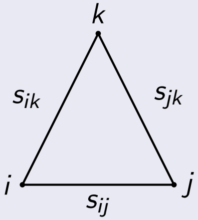

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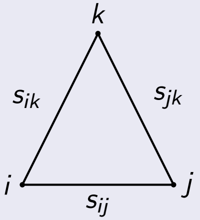

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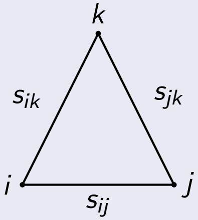
$$\int_M \tau(\Omega) = \sum_\sigma (-1)^{\mathbf{o}(\sigma)} \tilde{\Delta}_\tau(\xi_\sigma).$$

where $\tilde{\Delta}_\tau$ = de la Harpe-Skandalis determinant.

Evaluation of the integral (cont.)

On a 2-simplex $\sigma = \langle i, j, k \rangle \ldots$



- s_{ij} := element of G corresp. to edge $i\,j$
- have "cocycle condition": $s_{ij}s_{jk} = s_{ik}$
- π quasi-homomorphism
  $\Rightarrow \pi(s_{ij})\pi(s_{jk}) \approx \pi(s_{ik})$
- think of $\mathbb{T}^2$ case:
  $\pi(t)\pi(s) = vu \approx uv = \pi(ts)$

Let ξ_σ = segment $\pi(s_{ij})\pi(s_{jk}) \rightsquigarrow \pi(s_{ik})$ in $\mathrm{GL}_\infty(A)$. Then

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To summarize

- Interested in how a quasi-hom π acts on $K_0(\ell^1(G))$, where $G =$ surface group
- $K_0(\ell^1(G)) \cong \mathbb{Z} \oplus \mathbb{Z}\mu[M]$

- Push-forward $\mu[M]$ via π and apply trace τ : $\tau\big(\pi_\sharp(\mu[M])\big)$
- Push-forward ℓ, apply ch_τ etc. : $\langle \text{ch}_\tau(\ell_\pi), [M] \rangle$.

Dadarlat: red invariant $=$ blue invariant.

- Use Chern-Weil theory for Hilbert A-module bundles to deal with $\text{ch}_\tau(\ell_\pi)$.

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