On invariants of almost-flat bundles associated with almost-homomorphisms of groups

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joint work with Marius Dadarlat

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The Exel-Loring formula

Theorem (Exel-Loring '91)

Suppose $u, v \in U(n)$ satisfy $\|uv - vu\| < \varepsilon$. Associate a K_0-element $\mathcal{K}(u, v)$ to (u, v) as follows:

- regard this pair as an approximately multiplicative map $\pi\colon C^*(\mathbb{Z}^2) \to M_n(\mathbb{C})$ ($s \mapsto u$, $t \mapsto v$ etc.);
- write $K_0(C^*(\mathbb{Z}^2)) \cong \mathbb{Z} \oplus \mathbb{Z}x$ and let q be a projection s.t. $[q] - 1 = x \in K_0(C^*(\mathbb{Z}^2))$;
- set

$$\mathcal{K}(u, v) = \pi_\sharp(q) - \pi_\sharp(1) \in K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$$

Then

$$\mathcal{K}(u, v) = \frac{1}{2\pi i} \operatorname{Tr} \operatorname{Log}(uvu^{-1}v^{-1}).$$

A result of Dadarlat

Some notation

- M : compact connected orientable manifold with $\pi_1(M) = G$
- $\ell \in KK(\mathbb{C}, C(M) \otimes C^*(G))$: element corresponding to Mischenko line bundle
- $\mu\colon KK(C(M), \mathbb{C}) \to K_0(\ell^1(G))$: Lafforgue version of assembly map
- A : unital C^*-algebra with trace τ
- $z \in KK(C(M), \mathbb{C})$

Theorem (Dadarlat)

There are $\mathcal{F} \subset\subset G$ and $\delta > 0$ s.t. if $\pi\colon G \to \mathrm{GL}(A)$ is an $(\mathcal{F}, \delta)$-morphism, then

$$\tau(\pi_\sharp(\mu(z))) := \langle \mathrm{ch}_\tau((1 \otimes \pi)_\sharp(\ell)), \mathrm{ch}(z) \rangle.$$

A special case

For the rest of the talk, $M = 2$ dimensional.

Dadarlat's result in this case

- $\mathrm{ch}\colon KK(C(M),\mathbb{C}) \to H_0(M,\mathbb{Z}) \oplus H_2(M\mathbb{Z})$ is a bijection, as is the assembly map $\mu\colon KK(C(M),\mathbb{C}) \to K_0(C^*(G))$.
- $K_0(M)$ is generated by a character $\iota\colon C(M) \to \mathbb{C}$ and by its fundamental class $[M] \in K_0(M)$ satisfying $\mathrm{ch}[M] = [M]$.
- Therefore may write $K_0(C^*(G)) \cong \mathbb{Z} \oplus \mathbb{Z}\mu[M]$.

The theorem says that

$$\pi_\sharp(\mu[M]) = \langle \mathrm{ch}_\tau(1 \otimes \pi)_\sharp(\ell), [M] \rangle.$$

Can we compute this?

Example result

Riemann surface M_g

- $M_g = g$-holed torus (compact connected orientable surface of genus $g \geq 1$)
- $\Gamma_g = \pi_1(M_g) \cong \langle s_1, t_1, \ldots, s_n, t_n : \prod_{i=1}^{g} [s_i, t_i] = 1 \rangle$

Theorem (C-Dadarlat)

$\exists\ \varepsilon$ s.t. $\forall\ \{u_1, v_1, \ldots, u_g, v_g\} \subset U(A)$ satisfying

$$\| \prod_{i=1}^{g} [u_i, v_i] - 1 \| < \varepsilon,$$

there is an almost-homomorphism $\pi \colon \Gamma_g \to U(A)$ [[definition of π?]] with

$$\tau(\pi_\sharp(\mu[M_g])) = \frac{1}{2\pi i} \tau(\log(\prod_{i=1}^{g} [u_i, v_i])).$$

The Mischenko Line Bundle

$$\begin{array}{c} EG \times_G \ell^1(G) \\ \downarrow \\ BG \end{array}$$

This is a bundle P over BG with fibers all equal to $\ell^1(G)$. Introduced by Mischenko in ??.

Example to keep in mind

$$\begin{array}{c} \mathbb{R}^2 \times_{\mathbb{Z}^2} \ell^1(\mathbb{Z}^2) \\ \downarrow \\ \mathbb{T}^2 \end{array}$$

Quotient of $\mathbb{R}^2 \times \ell^1(\mathbb{Z}^2)$ by diagonal action of $\mathbb{Z}^2$

Will need more explicit description in terms of a convenient cover of BG and gluing maps (transition functions).

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Notation, triangulations

Let $M = BG$. Add w/e notation is missing after explaining Marius's index thm

- Λ : triangulation of M
- $\Lambda^0 = \{v_0, \dots, v_n\}$: vertices of Λ
- fix partial order on Λ^0 such that the vertices of each simplex are totally ordered.

Triangulation of $\mathbb{T}^2$

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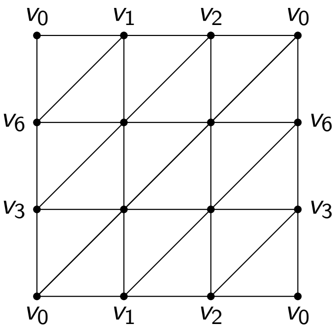
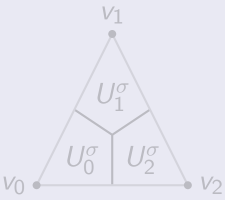


Triangulation of $\mathbb{T}^2$

Dual cells and cell blocks

Dual cell blocks

$\sigma = \langle v_0, v_1, v_2 \rangle$



define the *dual cell block*

$U_i^\sigma = \{\sum_{l=0}^{2} t_l v_l \in \sigma : t_i \geq t_l \ \forall \ l\}$

and the *dual cell* to v_i

$U_i = \cup\{U_i^\sigma : v_i \in \sigma\}$

The U_i's "cover" M. Will describe P as the bundle obtained from the disjoint union $\bigcup U_i \times \ell^1(G)$ by gluing along intersections $U_i \cap U_j$.

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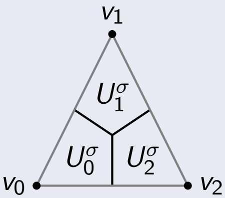

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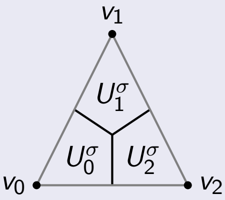

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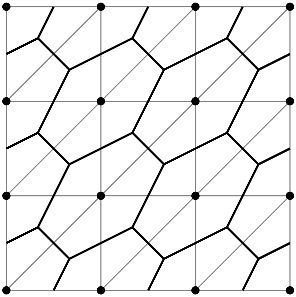

From edges to group elements

Our transition functions will be constant. To construct them:

- Fix a spanning tree T of Λ.
- An oriented edge $v_i v_j$ determines a loop γ_{ij} based at v_0.
- Let $s_{ij} = [\gamma_{ij}] \in \pi_1(M) = G$.

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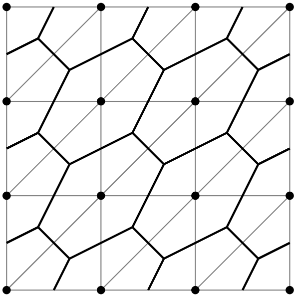

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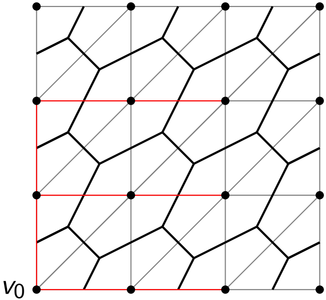

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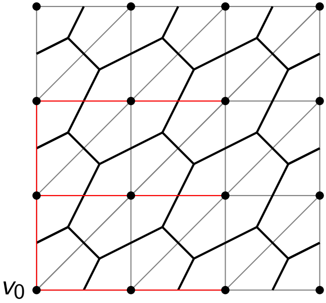

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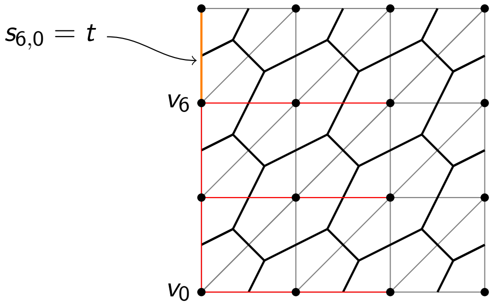

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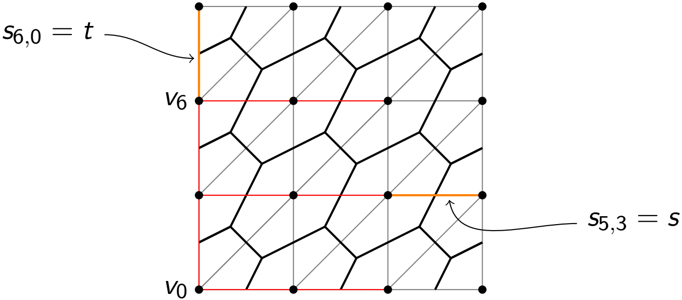

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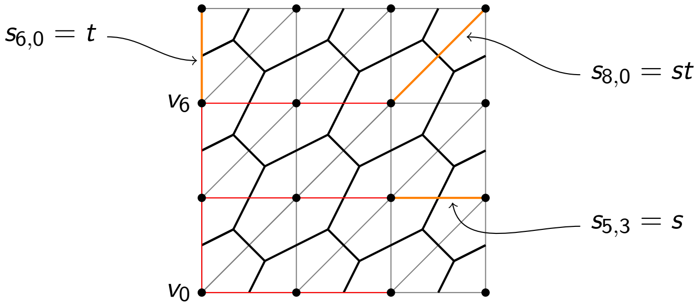

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Convenient description of P

The bundle

$P \cong$ the bundle obtained from the disjoint union $\bigcup U_i \times \ell^1(G)$ by identifying (x, a) with $(x, s_{ij}a)$ whenever $x \in U_i \cap U_j$.

The projection

Let $\{\chi_i\}$ = partition of unity $\prec \{U_i\}$. Identify P with the projection

$$\sum e_{ij} \otimes \chi_i^{1/2}\chi_j^{1/2} \otimes s_{ij} \in M_n(\mathbb{C}) \otimes C(M) \otimes \ell^1(G)$$

(e_{ij} = standard matrix units).

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The push-forward of P

Let $\pi\colon G \to U(A)$ be an $(\{s_{ij}\}, \varepsilon)$-multiplicative map.

Example: $G = \mathbb{Z}^2$

Write $\mathbb{Z}^2 = \langle s, t : sts^{-1}t^{-1} = 1 \rangle$. Then $\{s_{ij}\} = \{1, s, t, st, s^{-1}, t^{-1}, (st)^{-1}\}$. If $u, v \in U(A)$ with $[u, v]$ almost $= 1_A$, then $\pi(s^l t^k) := u^l v^k$, $(l, k \geq 0)$ and $\pi(\cdot^{-1}) = \pi(\cdot)^*$ defines an almost-homomorphism.

The push forward

$$(1 \otimes \pi)(P) = \sum e_{ij} \otimes \chi_i^{1/2} \chi_j^{1/2} \otimes \pi(s_{ij})$$

is nearly a projection in $M_n(\mathbb{C}) \otimes C(M) \otimes A$. Perturb it to a projection $\chi((1 \otimes \pi)(P))$ using functional calculus. Define

$$\pi_\sharp(P) = \text{ the } K_0 \text{ class of } \chi((1 \otimes \pi)(P)).$$

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The bundle corresponding to $\pi_\sharp(P)$

Definition

A *Hilbert A-module bundle* E over M is a topological space E with a projection $E \to X$ s.t. the fiber over each point has the structure of a Hilbert A-module V, and with local trivializations $\phi \colon E|_U \xrightarrow{\sim} U \times V$ that are fiberwise Hilbert A-module isomorphisms.

- $K_0(C(M, A)) \cong$ the K-theory group of (finitely generated projective) Hilbert A-module bundles over M.
- What is the Hilbert A-module bundle corresponding to $\pi_\sharp(P)$?

Idea: once we have bundle corresponding to $\pi_\sharp(P)$, can try to apply Chern-Weil theory to compute the appropriate Chern character.

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Almost flat bundles

Smooth structure

- A f.g.p. Hilbert A-module bundle E over M admits a unique smooth structure.
- There is a notion of a *connection* (covariant derivative) on E.
- A connection ∇ has an associated *curvature* Ω.
- For us, the fibers of E are all $\cong A$. So Ω is an A-valued 2-form:

$$\Omega \in \Omega^2(M, A) = \Gamma(\Lambda^2 T^* M \otimes A).$$

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A Hilbert A-module bundle is *α-flat* if there is a unitary connection ∇ s.t. $\|\Omega\| < \alpha$.

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Almost-flat bundles and almost-representations

Bundles $\rightsquigarrow$ representations

- Connes-Gromov-Moscovici '90: cyclic cohomology approach to the Novikov conjecture. Studied homotopy invariants of almost flat bundles.
- One idea used: associate almost representation π of $\pi_1(M)$ to an almost flat bundle E over M. (Here $A = M_k(\mathbb{C})$.)

Briefly: fix unitary connection ∇, base point $v_0 \in M$. For each element of $\pi_1(M)$, choose loop γ based at v_0 representing it. Then π "=" parallel transport along γ.

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Almost homomorphisms $\rightsquigarrow$ almost-flat bundles

We start with our almost homomorphism π and construct an almost flat bundle E_π. The two constructions end up being inverse to each other.

Related to computation of "topological charge" of lattice gauge fields and some work in theory of topological insulators (e.g. A. Phillips-Stone '90).

How to construct E_π?

- Have a cover $\{U_i\}$. Will define transition functions $c_{ij}\colon U_i \cap U_j \to \mathrm{GL}(A)$ with $c_{ij}(x)$ very close to s_{ij}.
- Because the c_{ij}'s will be almost constant, the bundle will be almost flat.

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How *not* to construct E_π

Try to define transition functions c_{ij} naively: $c_{ij}(x) \stackrel{?}{=} \pi(s_{ij})$.
Must check: $c_{ij}c_{jk} = c_{ik}$ on $U_i \cap U_j \cap U_k$

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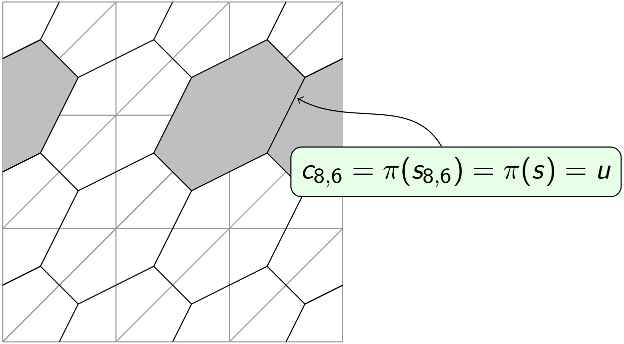

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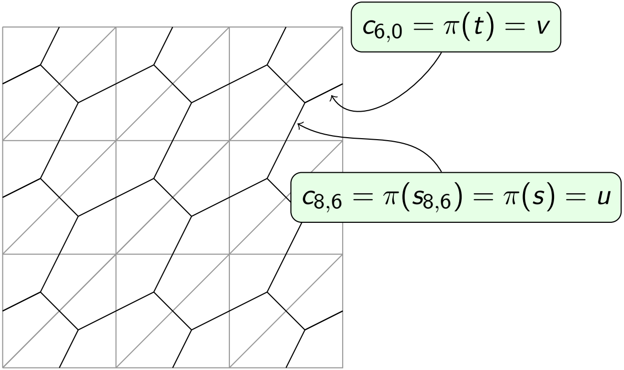

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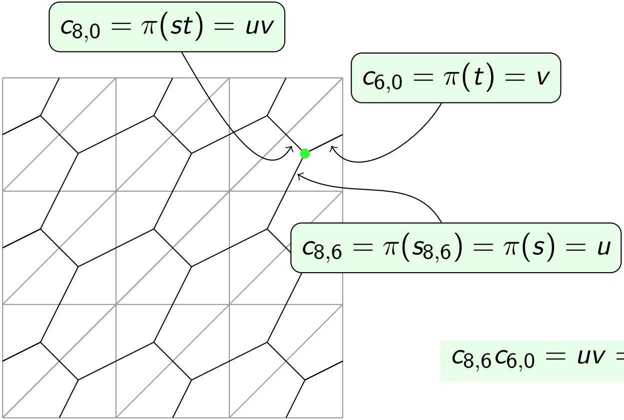

$$c_{8,6}c_{6,0} = uv = c_{8,0}$$

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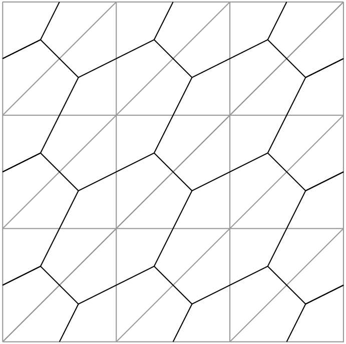

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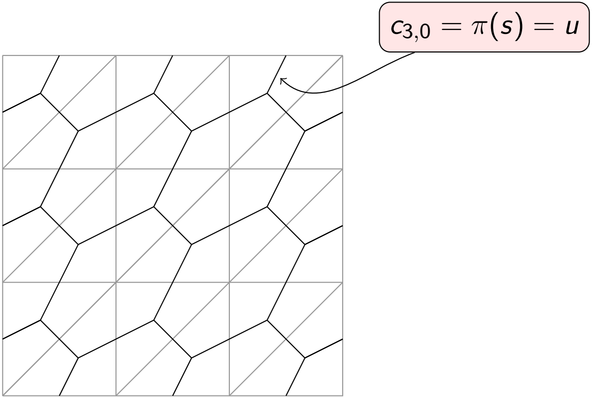

How *not* to construct E_π

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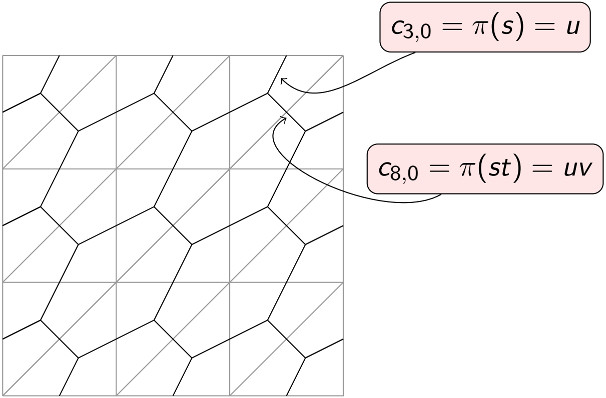

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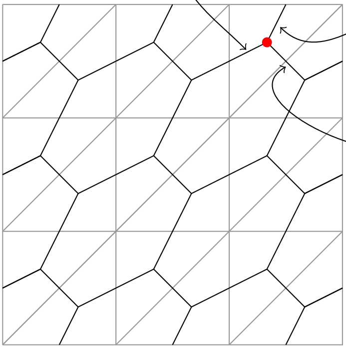
$c_{8,3} = \pi(st) = v$

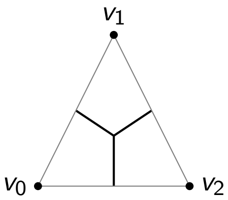
$c_{3,0} = \pi(s) = u$



$c_{8,0} = \pi(st) = uv$

$$c_{8,3}c_{3,0} = vu \neq uv = c_{8,0}$$

Define the transition functions



- Let $\sigma = \langle v_0, v_1, v_2 \rangle$ (written in increasing order).
- Notice that $U_0^\sigma \cap U_2^\sigma$: a segment

Definition: $c_{ij}^\sigma : U_i^\sigma \cap U_j^\sigma \to \mathrm{GL}(A)$

$$c_{0,1}^\sigma \equiv \pi(s_{0,1})$$

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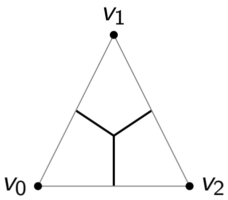

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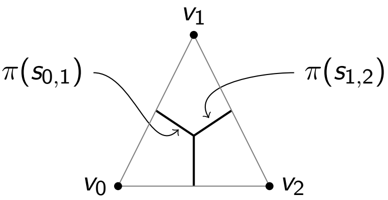

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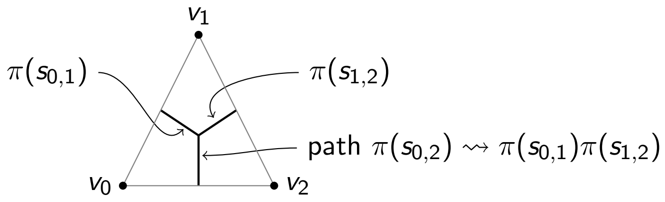

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The almost flat bundle E_π

For fixed i, j, the c_{ij}^σ's define a map $c_{ij}\colon U_i \cap U_j \to \mathrm{GL}(A)$. The c_{ij}'s form a family of transition functions.

Definition: E_π

$E_\pi =$ the bundle obtained from $\bigcup U_i \times A$ by identifying (x, a) with $(x, c_{ij}(x)a)$ whenever $x \in U_i \cap U_j$.

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$[E_\pi]_0 \in K_0(C(M, A))$ is the same as $\pi_\sharp(P)$.

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The de la Harpe-Skandalis determinant

Recall one of the invariants in Exel-Loring: $\operatorname{Tr}\operatorname{Log}([u, v])$.
Replacement: the de la Harpe-Skandalis determinant ('84).

Definition: $\tilde{\Delta}_\tau$

Fix $\tau \in T(A)$. For a C^1 path $\xi\colon [t_1, t_2] \to \mathrm{GL}_\infty(A)$ define

$$\tilde{\Delta}_\tau(\xi) = \frac{1}{2\pi i}\tau\left(\int_{t_1}^{t_2} \xi'(t)\xi(t)^{-1}dt\right)$$

- $\|\xi(t) - 1\| < 1 \Rightarrow 2\pi i\tilde{\Delta}_\tau(\xi) = \tau(\mathrm{Log}(t_2)) - \tau(\mathrm{Log}(t_1))$.
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Theorem (C-Dadarlat)

Recall setup:

- G : discrete countable group with $M = BG$ compact orientable surface
- A : unital C^*-algebra with trace τ
- s_{ij} : group element corresponding to edge $v_i v_j$
- $\pi : G \to U(A)$: an $(\{s_{ij}\}, \varepsilon)$-homomorphism
- P : Mischenko line bundle; E_π : bundle corresponding to $\pi_\sharp(P) \in K_0(C(M, A))$

For a simplex $\sigma = \langle x_i, x_j, x_k \rangle$ (written in increasing order), let ξ_σ be the linear path $\pi(s_{ij})\pi(s_{jk}) \rightsquigarrow \pi(s_{ik})$ in $\mathrm{GL}(A)$. Then

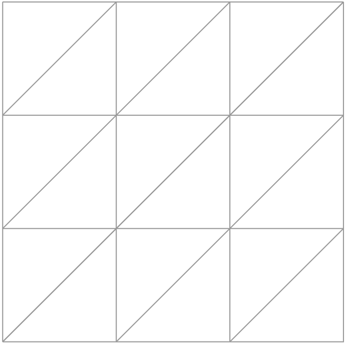
$$\frac{1}{2\pi i}\langle \tau(\pi_\sharp(P)), [M] \rangle = \frac{1}{2\pi i}\int_M \mathrm{ch}_\tau(E_\pi) = \sum_\sigma \tilde{\Delta}_\tau(\xi_\sigma).$$

Main application

state result with M_g

$G = \mathbb{Z}^2$

Recall: $\mathbb{Z}^2$ generated by s, t; let $u, v \in U(A)$ almost commute.
$\pi\colon \mathbb{Z}^2 \to U(A)$ is an almost homomorphism s.t. $\pi(s) = u$, $\pi(t) = v, \pi(st) = uv$.



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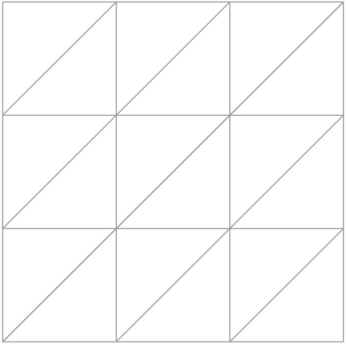

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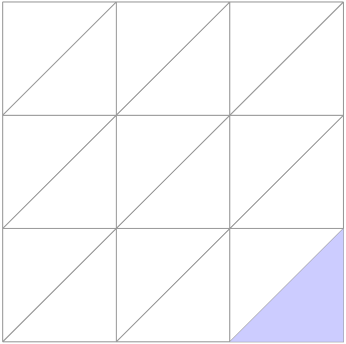

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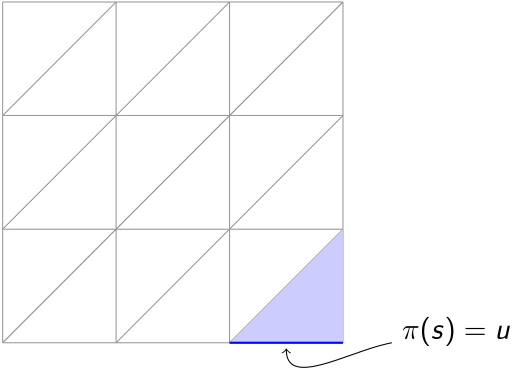

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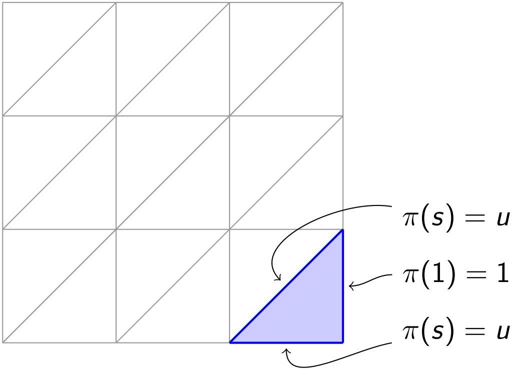

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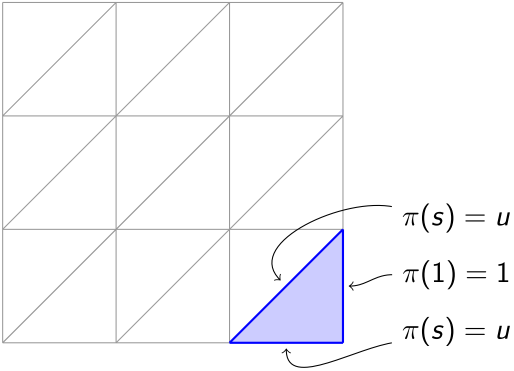

$\tilde{\Delta}_\tau(u \cdot 1 \rightsquigarrow u) = 0$

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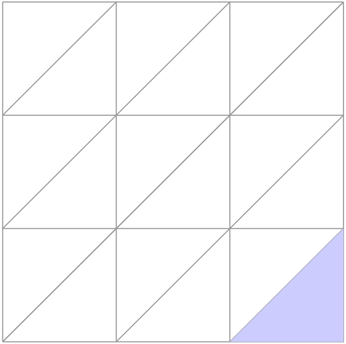

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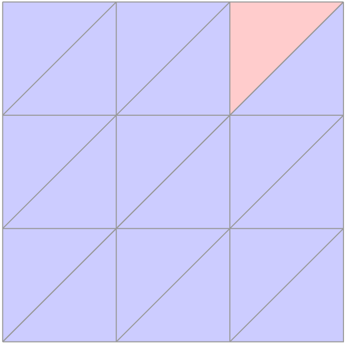

$$\tilde{\Delta}_\tau(vu \rightsquigarrow uv) = \tau(\mathrm{Log}([v, u]))$$

$$\tilde{\Delta}_\tau(\quad) = 0$$

$G = \Gamma_2$

$\Gamma_2 = \langle \alpha, \beta, \gamma, \delta \mid [\alpha, \beta][\gamma, \delta] = 1 \rangle$.
$B\Gamma_2 = M_2 =$ double torus.

figure with octagon (similar to last one on prev. slide)