On the classification problem for a class of crossed product C^*-algebras

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An example: the discrete Heisenberg group

The discrete Heisenberg group $\mathbb{H}_3$: 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbb{Z}$ (usual matrix multiplication). It is a central extension of $\mathbb{Z}$ by $\mathbb{Z}^2$.

$\mathbb{H}_3$ is amenable and *residually finite*. Let $L_i := \mathbb{H}_3(2^i\mathbb{Z})$; then

$$\mathbb{H}_3 \supset L_1 \supset L_2 \supset \cdots$$

is a nested sequence of finite index normal subgroups satisfying

$$\bigcap L_i = \{1\}.$$

A crossed product associated with $\mathbb{H}_3$

Form the *profinite completion* w.r.t. this sequence:

$$\widetilde{\mathbb{H}}_3 := \varprojlim \mathbb{H}_3/L_i \quad \Big(\subseteq \prod_{i\geq 1} \mathbb{H}_3/L_i\Big).$$

This is a compact, Hausdorff and totally disconnected group.

Our main example this talk:

$\mathbb{H}_3$ acts on $\widetilde{\mathbb{H}}_3$ by left multiplication. The crossed product

$$C(\widetilde{\mathbb{H}}_3) \rtimes \mathbb{H}_3$$

is an example of a generalized Bunce-Deddens algebra.

Definition of a generalized Bunce-Deddens algebra

Definition (Orfanos, 2008)

$G =$ discrete, countable, amenable and residually finite group

$(L_i) =$ nested sequence of finite index normal subgroups with trivial intersection

Form the profinite completion $\tilde{G}$ just as above. Call $C(\tilde{G}) \rtimes G$ a **generalized Bunce-Deddens algebra** associated with G.

Theorem (Orfanos, 2008)

A generalized Bunce-Deddens algebra

- *is simple, separable and nuclear;*
- *has real rank zero, stable rank one, and a unique trace;*
- *is quasidiagonal.*

Are these algebras classified by their Elliott invariant?

Definition

Elliott invariant for a generalized B-D algebra A:

$$\mathrm{Ell}(A) = \big(K_0(A), K_0(A)^+, [1_A]_0, K_1(A)\big).$$

For generalized B-D algebras A, B:

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \overset{?}{\Longrightarrow} A \cong B$$

A little background and motivation

Example: Bunce-Deddens algebras

- $G = \mathbb{Z}$: "classical" Bunce-Deddens algebras. Classification was carried out by J. Bunce and J. Deddens (1973).
- Classification using modern tools could be obtained by recalling that a Bunce-Deddens algebra is an inductive limit of C^*-algebras of the form $M_r(\mathbb{C}) \otimes C(\mathbb{T})$, and noting that $\mathbb{T}$ has covering dimension 1.

A generalized B-D algebra is an inductive limit in a similar way:

$$C(\tilde{G}) \rtimes G \cong \varinjlim C(G/L_i) \rtimes G \cong \varinjlim M_{r_i}(\mathbb{C}) \otimes C^*(L_i).$$

We will use a version of noncommutative covering dimension due to Kirchberg and Winter: the *decomposition rank.*

The main result

$\mathcal{G}$ = class of all (discrete) groups G s.t. there exist finitely generated abelian groups N, Q with G a central extension

$$1 \to N \to G \to Q \to 1.$$

(The groups in $\mathcal{G}$ are residually finite.)

Theorem

If $A = C(\tilde{G}_1) \rtimes G_1$ and $B = C(\tilde{G}_2) \rtimes G_2$ are generalized Bunce-Deddens algebras associated to groups G_i in $\mathcal{G}$, then

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \Longrightarrow A \cong B.$$

Classification via decomposition rank

General classification result we rely on (Winter, 2005):

Under the "right" conditions (including e.g. simplicity, real rank zero), finite decomposition rank is enough for classification.

Orfanos' results $\Rightarrow$ generalized B-D algebras satisfy the "right" conditions. Therefore, only need:

Proposition

$G \in \mathcal{G} \Rightarrow C(\tilde{G}) \rtimes G$ has finite decomposition rank.

What happens in our example?

Back to $\mathbb{H}_3$

$C(\widetilde{\mathbb{H}}_3) \rtimes \mathbb{H}_3 =$ inductive limit of $C^*(L_i) \otimes M_{r_i}(\mathbb{C})$.

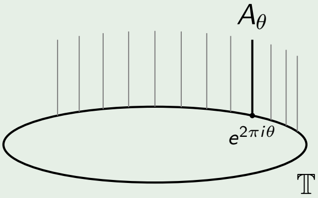
Since

$$\mathrm{dr}\left(\varinjlim C^*(L_i) \otimes M_{r_i}(\mathbb{C})\right) \leq \underline{\lim}\, \mathrm{dr}\, C^*(L_i),$$

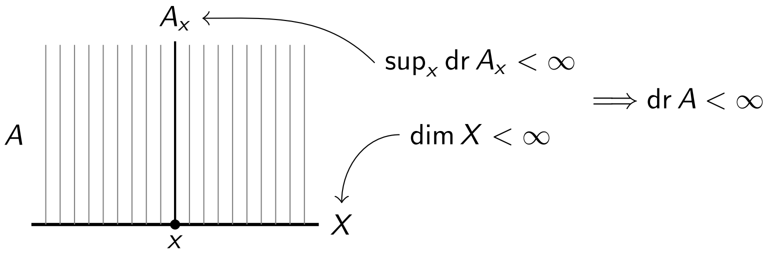
can focus on $C^*(L_i)$. *Motivation:* $L_i = \mathbb{H}_3(2^i\mathbb{Z})$ “not that different” from $\mathbb{H}_3$.

What do we know about $C^*(\mathbb{H}_3)$?

$C^*(\mathbb{H}_3) \cong$ sections of continuous field over $\mathbb{T}$ with fibers A_θ (Anderson-Paschke ’89)



The decomposition rank of a $C(X)$-algebra



Proposition

Let X be a compact metric space and A a (separable) $C(X)$-algebra. If $\dim X \leq l$ and $\sup_x \operatorname{dr} A_x \leq k$, then

$$\operatorname{dr} A \leq (l+1)(k+1) - 1.$$

What is decomposition rank?

A contractive completely positive (c.c.p.) map $\varphi\colon A \to B$ has **order zero** if $\varphi(a)\varphi(b) = 0$ whenever $a, b \geq 0$ with $ab = 0$.

Definition (Kirchberg-Winter, 2003)

A has **decomposition rank** at most n if there exist

- finite dimensional C^*-algebras F_k,
- c.c.p. maps $A \xrightarrow{\varphi_k} F_k \xrightarrow{\psi_k} A$ with $\psi_k \circ \varphi_k \to \mathrm{id}_A$ pointwise, and
- a partition $F_k = F_k^{(0)} \oplus \cdots \oplus F_k^{(n)}$ such that $\psi_k|_{F_k^{(i)}}$ has order zero for every $0 \leq i \leq n$.

C^*-algebras with finite decomposition rank

Some examples

- $\mathrm{dr}\, C(X) = \dim X$ (Winter)
- $\mathrm{dr}\, A = 0 \Leftrightarrow A$ is an AF algebra (Winter)
- $\mathrm{dr}\, A_\theta \leq 2n+1$, where A_θ = noncommutative n-torus (uses Phillips' result that every *simple* NC torus is an A$\mathbb{T}$ algebra)

Proposition

Let G be a central extension

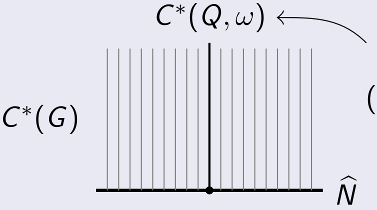
$$1 \to N \to G \to Q \to 1$$

where N and Q are finitely generated abelian groups of ranks n and m, respectively (i.e. $G \in \mathcal{G}$). Then

How this ties in with our example

$C^*(G)$ as a continuous field

G = central extension of N by Q. Then (by Packer-Raeburn, '92):



(e.g., $C^*(\mathbb{Z}^m,\omega) = A_\theta$)

Have that dr $C^*(Q,\omega) \leq (m+1)(2m+2) - 1$ for Q finitely generated abelian of rank $\leq m$.

To summarize in a few steps

Classification $\rightsquigarrow$ dr $C^*(G) < \infty$

- Want to classify $C(\tilde{G}) \rtimes G$ (for $G \in \mathcal{G}$).
- Enough to prove $C(\tilde{G}) \rtimes G$ has finite decomposition rank.
- Reduces to proving $C^*(G)$ has finite decomposition rank for every $G \in \mathcal{G}$

dr $C^*(G) < \infty \rightsquigarrow$ dr of continuous fields

- $A =$ continuous field over a finite dimensional space with fibers of uniformly bounded dr $\Rightarrow$ dr $A < \infty$.
- $C^*(G)$ is the C^*-algebra of a continuous field over a finite dimensional space with fibers of the form $C^*(Q, \omega)$.
- $Q =$ f.g. abelian of rank $\leq m \Rightarrow$ dr $C^*(Q, \omega) < 2m^2 + 4m + 1$.