Orthogonal Polynomials
TCU Seminar Lecture Notes

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Before beginning, it seems appropriate to mention Matthew Watkins' web page, A Directory of All Known Zeta Functions,
`http://empslocal.ex.ac.uk/people/staff/mrwatkin//zeta/directoryofzetafunctions.htm`

And a disclaimer: I've made lots of changes of variables throughout. Expect some mistakes. I would appreciate hearing about any you find.

I want to look at two different topics that have to do with orthogonal polynomials. The first has to do with approximation; the second, with Mellin transforms and zeta functions.

If we have an inner product on $\mathbb{R}[x]$, we can use Gram-Schmidt to convert $\{1, x, x^2, \ldots\}$ into an orthogonal basis of monic polynomials $\{p_n(x)\}$. Our inner products will have the form

$$\langle p, q\rangle = \int_a^b p(x)\, q(x)\, w(x)\, dx$$

for some *weight function* w. A family of orthogonal polynomials will have p_n of degree n, but not necessarily monic. For a given weight function, we may always multiply each polynomial by an arbitrary constant to get another family. Standard choices are monic, normalized to $\langle p_n, p_n\rangle = 1$, or to coincide with nice generating functions.

Examples

Hermite Polynomials

In Igor Prokhorenkov's talk last fall, he spoke of the Hamiltonian for the one-dimensional quantum mechanical harmonic oscillator:

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$$

with eigenvalues $n + 1/2$, $n = 0, 1, 2, \ldots$ (see [20, p. 181]), leading to the Hurwitz zeta function. The eigenfunctions are $\psi_n(x) = H_n(x)e^{-x^2/2}$, where $H_n(x)$ is the Hermite polynomial of degree n.

Weight Function: $w(x) = e^{-x^2}$ on the interval $(-\infty, \infty)$ (normal distribution)

Differential Equation: $y'' - 2xy' + 2ny = 0$

Rodrigues' type Formula: $H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n}\left(e^{-x^2}\right)$

Generating Function: $\sum_{n=0}^{\infty} \dfrac{H_n(x)}{n!} z^n = e^{2xz - z^2}$

Recursion: $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \qquad H_0(x) = 1, H_1(x) = 2x$

Remark. The Hermite polynomials form a complete basis for $L^2(\mathbb{R}, w(x)\,dx)$ [33].

Laguerre Polynomials

Looking at the wave equation for the hydrogen atom leads to an expression involving generalized Laguerre polynomials $L_n^\alpha(x)$ (see [20, p. 188], [28, Ch. 5]).

Weight Function: $w(x) = x^\alpha e^{-x}$ on $[0, \infty)$, $\alpha > -1$ (gamma distribution)

Differential Equation: $xy'' + (\alpha + 1 - x)y' + ny = 0$

Rodrigues' type Formula: $L_n^\alpha(x) = \dfrac{e^x}{n! x^\alpha} \dfrac{d^n}{dx^n} \left(x^{n+\alpha} e^{-x}\right)$

Generating Function: $\sum_{n=0}^{\infty} L_n^\alpha(x) z^n = \dfrac{e^{xz/(z-1)}}{(1-z)^{\alpha+1}}$

Recursion: $(n+1)L_{n+1}^\alpha(x) = (2n + 1 + \alpha - x)L_n^\alpha(x) - (n + \alpha)L_{n-1}^\alpha(x)$
$L_0^\alpha(x) = 1, L_1^\alpha(x) = 1 + \alpha - x$

Remark. The Hermite polynomials relate to $L_n^{\pm 1/2}$ by a change of variables. $\{L_n^\alpha\}$ forms a complete basis for $L^2([0, \infty), w(x)\, dx)$ [33].

Legendre Polynomials

Legendre's expansion of Newtonian potential or solutions to Laplace's equation in spherical coordinates lead to Legendre polynomials.

Weight Function: 1 on $[-1, 1]$ (uniform distribution)

Differential Equation: $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$

Rodrigues' Formula (1816): $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n} \left[(x^2 - 1)^n\right]$

Generating Function: $\sum_{n=0}^{\infty} P_n(x) z^n = (1 - 2xz + z^2)^{-1/2}$

$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} z^n = e^{xz} J_0(z\sqrt{1 - x^2}) \qquad \text{(Bessel function)}$$

Recursion: $(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x), \qquad P_0(x) = 1, P_1(x) = x$

Chebyshev Polynomials (of the First Kind)

There is a unique polynomial of degree at most n passing through any $n+1$ points in the plane with distinct x-coordinates. One formula is called Lagrange interpolation (though I just read that Waring preceded him by 16 years). Suppose we use Lagrange interpolation to approximate f on an interval, say $[-1,1]$. If we choose points $x_1, \ldots, x_{n+1}$, the error has the form

$$\frac{f^{(n+1)}(\xi_x)}{(n+1)!}\,(x-x_1)\cdots(x-x_{n+1}) \quad [3],[42].$$

What is the best choice of $x_1, \ldots, x_{n+1}$ if, say, we want to minimize use the sup norm on $[-1,1]$? we can't control $f^{(n+1)}$ so seek to minimize $\max\limits_{x\in[-1,1]} |(x-x_1)\cdots(x-x_{n+1})|$. As we'll see later, the best choice is the roots of the Chebyshev polynomial $T_{n+1}(x)$.

Weight Function: $(1-x^2)^{-1/2}$ on $[-1,1]$ (with connections to Fourier series after an obvious change of variables)

Differential Equation: $(1-x^2)y'' - xy' + n^2 y = 0$

Rodrigues' type Formula: $P_n(x) = \dfrac{(-1)^n \pi^{1/2}(1-x^2)^{1/2}}{2^n \Gamma(n+1/2)} \dfrac{d^n}{dx^n}\left[(1-x^2)^{n-1/2}\right]$

Generating Function: $\sum\limits_{n=0}^{\infty} P_n(x) z^n = \dfrac{1-xz}{1-2xz+z^2}$

Recursion: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad T_0(x) = 1, T_1(x) = x$

Remarks. An alternative definition: $T_n(x) = \cos(n \arccos x) = \cosh(n \text{ arccosh } x)$ and the n real roots are easily explicitly computed. Furthermore, $T_m \circ T_n = T_{mn}$.
Along with Chebyshev polynomials of the second kind $U_n(x)$ (weight $(1-x^2)^{1/2}$), we get solutions to the Pellian equation $T_n^2(x) - (x^2-1)U_{n-1}^2(x) = 1$.

Summary

There are explicit formulas in terms of binomial coefficients or gamma functions. They are examples of hypergeometric functions.

The Jacobi polynomials $P^{(\alpha,\beta)}(x)$ are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = (1-x)^\alpha (1+x)^\beta$ (linear change of variables from a beta distribution). They generalize the Legendre and Chebyshev polynomials (with ultraspherical or Gegenbauer polynomials, $\alpha = \beta$, in between) and, along with the Hermite and Laguerre polynomials, form the classical classical classical orthogonal polynomials. However, there are even more families of orthogonal polynomials than there are zeta functions.

Zeros and Recursion

Theorem. *Let $\{p_n(x)\}$ be a family of orthogonal polynomials (indexed by their degree). The zeros of $p_n(x)$ are real, simple, and lie in the support of the weight function $w(x)$.*

Proof. Let $q_n(x)$ have the odd-order roots of $p_n(x)$ as simple roots. Note that $p_n(x)\,q_n(x)$ has no sign changes in the support $[a,b]$ of $w(x)$. Therefore,

$$\int_a^b p_n(x)\,q_n(x)\;dx \neq 0.$$

However, if the theorem fails, the degree of $q_n(x)$ is less than n and the integral is 0. □

In fact, the argument shows $p_n(x) + cp_{n-1}(x)$ has real, simple roots.

Theorem. *Let $\{p_n(x)\}$ be a family of monic orthogonal polynomials. Then there exist real constants $\{c_n\}$ and positive constants $\{\lambda_n\}$ such that*

$$p_{n+1}(x) = (x - c_n)p_n(x) - \lambda_n p_{n-1}(x).$$

Proof. We have, using monotonicity,

$$xp_n(x) = p_{n+1}(x) + a_n p_n(x) + a_{n-1}p_{n-1}(x) + \cdots + a_0 p_0(x).$$

We have

$$a_k = \frac{\langle xp_n(x), p_k(x)\rangle}{\langle p_k(x), p_k(x)\rangle} = \frac{\langle p_n(x), xp_k(x)\rangle}{\langle p_k(x), p_k(x)\rangle} = \begin{cases} 0 & k < n-1 \\[2ex] \dfrac{\langle p_n(x), p_n(x)\rangle}{\langle p_{n-1}(x), p_{n-1}(x)\rangle} & k = n-1 \\[2ex] \dfrac{\langle p_n(x), xp_n(x)\rangle}{\langle p_n(x), p_n(x)\rangle} & k = n. \end{cases}$$

Solve for $p_{n+1}(x)$. □

Theorem. *For a family of orthogonal polynomials $\{p_n(x)\}$, the zeros of $p_n(x)$ are real, simple, and interlace those of $p_{n-1}(x)$.*

Proof. Plug the zeros of $p_n(x)$ into the recursion and use induction. □

The converse (see Favard 1935 [11]):

Theorem. *Let $p_{-1}(x) = 0, p_0(x) = 1,$ and*

$$p_{n+1}(x) = (x - c_n)p_n(x) - \lambda_n p_{n-1}(x),$$

with c_n is real and $\lambda_n > 0$. Then there exists a unique positive-definite linear functional $\mathcal{L} : \mathbb{R}[x] \to \mathbb{R}$ such that $\mathcal{L}(1) = 1$ and $\mathcal{L}(p_n(x)p_m(x)) = 0$ for $m \neq n$. In particular, $\{p_n(x)\}$ is a family of orthogonal polynomials.

Monotone Polynomial Interpolation

Our February 2008 Problem of the Month (to which we got four student solutions!): Show that there is no (everywhere) increasing cubic polynomial passing through the points $(0, 0)$, $(1, 1)$, and $(2, 16)$.

In the 1951, Wolibner [41] showed one may interpolate $\{(x_i, y_i)\}$ $(y_{i-1} \neq y_i)$ with a polynomial on that is monotone on each subinterval $[x_{i-1}, x_i]$. Passow and Raymon [27] obtained bounds on the degree.

Theorem. *Suppose $x_0 < x_1 < \cdots < x_n$ and $y_0 < y_1 < \cdots < y_n$.*

(a) (Rubinstein 1970) [31] There is an everywhere increasing polynomial through $\{(x_i, y_i)\}$.

(b) [9] Even for $n = 3$, the degree of this polynomial can be arbitrarily large.

In fact, Ford and Roulier showed in 1974 [12] that if the slopes $(y_i - y_{i-1})/(x_i - x_{i-1})$ are increasing, i.e. the polynomial could be concave up, then one may interpolate by polynomials that are everywhere increasing and concave up on $[x_0, x_n]$ or that are increasing on $[x_0, x_n]$ and everywhere concave up.

Proof. (a) (sketch) Step 1. By the Weierstrass Approximation Theorem, we may approximate a smoothed square pulse of height 1 supported on $[x_{i-1}, x_i]$ by a polynomial q_i that approximates it to within a suitable ε on $[x_0, x_n]$. Rubinstein actually approximates a delta function on each subinterval by using beta distributions in a way reminiscent of using the Bernstein polynomials to prove the Weierstrass Approximation Theorem.

Step 2. Form the polynomial

$$p(x) = y_0 + \int_{x_0}^{x} \left[a_1 q_1^2(u) + \cdots + a_n q_n^2(u)\right] du.$$

Step 3. Show that by choosing $a_i \approx \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}$ carefully, we pass through the $n + 1$ points.

(b) Consider the points $(-1, -1)$, $(1, 1)$, (a, b) with $a > 1$, $b > 1$ and let $p(x)$ be our increasing interpolating polynomial and suppose it has degree $2n + 1$.

Step 1. Show that $p'(x)$ is a sum of squares of polynomials. In fact, two polynomials suffice, so $p'(x) = c_1^2 q_1^2(x) + c_2^2 q_2^2(x)$ where the $q_i(x)$ have norm 1 and $c_1^2 + c_2^2 = \int_{-1}^{1} p'(u)\, du = 2$ (graduate student exercise). Fixing q_i, it is clear that the largest $p(a)$ occurs when $c_1 c_2 = 0$, so assume $c_2 = 0$.

Step 2. Express $q = q_1$ as a linear combination of orthonormal polynomials $u_i(x)$ on $[-1, 1]$ of degree up to n, e.g. Legendre polynomials scaled to have norm 1. Say

$$p'(x) = \left[\sum_{k=0}^{n} c_i u_i(x)\right]^2,$$

where $\sum_{k=0}^{n} c_i^2 = 2$.

Step 3. Because $(c_0, \ldots, c_n)$ runs over a compact set, the possible values of b, a homogeneous quadratic polynomial in $(c_0, \ldots, c_n)$, are bounded. □

We can get an effective lower bound on the degree.

Consider a monotone increasing cubic polynomial $p(x)$ passing through $(-1,-1)$ and $(1,1)$. Take $a > 1$. Then ([9])

$$p(a) \le \frac{a^3 + a + (a^2-1)\sqrt{3+a^2}}{2},$$

with equality iff

$$p(x) = \frac{(a+\sqrt{3+a^2})x^3 + 3x^2 + (\sqrt{3+a^2}-a)x - 3}{2\sqrt{3+a^2}}.$$

For $p(x)$ of degree $2n+1 \ge 3$, we have

$$p(a) \le 1 + \frac{\binom{2n}{n}^2}{2^{2n}}(a+1)^{2n+1} < \frac{2^{2n+1}}{4n}(a+1)^{2n+1}.$$

In fact, certain monotone polynomials of degree $2n+1$ derived in a simple way from the Legendre polynomials satisfy $p(\pm 1) = \pm 1$ and, as $a \to \infty$,

$$p(a) \sim \frac{\binom{2n}{n}^2}{2^{2n}} a^{2n+1}.$$

If $a^n \ge b > a$, we can average x and x^n, hence the minimal degree for interpolation is at most $\lceil \log_a b + 1 \rceil$. The above shows we can't do better than degree

$$\frac{\log_a b + \log_a 4}{1 + \log_a 4} > \frac{1}{1+\log_a 4} \log_a b, \tag{1}$$

so we are within a constant multiplicative factor (that goes to 1 as $a \to \infty$) on the minimal degree.

On the other hand, for a close to 1 the lower degree bound (1) is nearly vacuous. The upper bound $\lceil \log_a b + 1 \rceil$, if large, is also nowhere close to the true story. One can show the sharp $p'(1) \le (n+1)^2$, so for a close to 1, the largest b behaves more like $a^{(n+1)^2}$ and our minimal degree is more like $2\sqrt{\log_a b}$. This last calculation led me to something called the Henyey-Greenstein phase function from physics.

The Boutroux-Cartan Lemma

For degree 2, the level curves $|p(z)| = r$ are ovals of Cassini. These also arise in estimating eigenvalues of an $n \times n$ matrix in Brauer's strengthening of Gerschgorin's circle theorem. I have a Mathematica notebook that shows what some look like for degree 3.

Theorem. *(Boutroux-Cartan Lemma) For a monic polynomial $p(z)$, $|p(z)| > r^n$ outside of n balls whose radii sum to at most $2er$.*

The obvious result is that if $|p(z)| \leq r^n$, then z is within r of some root of p, and these balls have radii summing to nr. The remarkable part of the theorem is removing the dependence on n.

Proof. (sketch)

Let $z_1, \ldots, z_n$ be the roots of p and let $\rho > 0$. Note that the disks $|z - z_i| \leq \rho$ contains at least one root.

Step 1. If a disk $|z - a| \leq j\rho$ contains more than j roots of p, then, increasing j by 1 at a time, some disk $|z - a| \leq k\rho$ contains exactly k roots. Choose a_1 whose k_1 to be the maximal such k. Iterate among the remaining roots. Note that if some disk $|z - a| \leq k\rho$ contains k roots, then at least one of these roots is contained in some $|z - a_i| \leq k_i\rho$ with $k_i \geq k$.

Step 2. The disks $|z - a_i| \leq k_i\rho$ are disjoint because otherwise they, along with their $k_i + k_j$ roots, would be contained within a disk of radius $(k_i + k_j)\rho$.

Step 3. Double the radius of each disk. For ζ outside their union, $|z - \zeta| \leq k\rho$ contains at most $k - 1$ roots. Thus, the closest zero to ζ is at least ρ away, the second closest at least 2ρ away, and so forth. Thus, $|p(\zeta)| \geq n!\rho^n$. Use the fact that $(1 + 1/n)^n$ increases to e and induction to conclude that $n! > (n/e)^n$, so $|p(\zeta)| > (n\rho/e)^n$. Setting $r = n\rho/e$, the sum of the radii of the balls is $2n\rho = 2er$.

□

We could have obtained a better estimate for $n!$ using Stirling's formula. With just a wee bit more care, we can take advantage of this and we can shrink the sum of the radii to $er = 2.71828\ldots r$.

Theorem. *View a monic polynomial of degree n over $\mathbb{R}$ as a function from $\mathbb{R} \to \mathbb{C}$. Then $p^{-1}(\{|z| < r^n\})$ is contained in n intervals of total width at most $2^{-1/n} \cdot 2r$. This is sharp.*

Proof. (sketch)

Step 1. Replace each root by its real part. This can only decrease $|p(x)|$.

Step 2. If $p^{-1}(\{|z| < r\})$ is disconnected, translate the set of roots in one of the outside components towards the other roots. This can only enlarge $\mu\left(p^{-1}(\{|z| < r\})\right)$ until the components become tangent.

Step 3. If the maximum of $|p(x)|$ between consecutive roots is not r, we can translate these two roots equal distances away from the critical point and enlarge $\mu\left(p^{-1}(\{|z| < r\})\right)$.

Step 4. At this point, we have $p(x) = \pm r$ at every critical point and have reached a characterization of scaled Chebyshev polynomials. □

Corollary. *For a monic polynomial $p(x)$ of degree n*

$$\max_{[a,b]} |p(x)| \geq 2 \left(\frac{b-a}{4} \right)^n .$$

Remark. Note that the the extremal polynomials of degree n for the sup norm “are” Chebyshev polynomials of the first kind. For L^2 minimization, the extremal polynomials “are” Legendre polynomials and L^1 norm “are” Chebyshev polynomials of the second kind. My source for these results has been [30], a 700-page encyclopedia on the analytic theory of polynomials.

Conjecture [19, p. 25] If $p(z)$ is monic , then, $p^{-1}(\{|z| < r^n\})$ is contained in $\deg p$ balls of total radius at most $2r$.

The Chebyshev examples show we can't do better. The best result to date,

$$(\pi e^{1/2}/2)r = 2.5898\ldots\ r,$$

is due to Ch'en in 1964 [5]. It seems to be a quick use of results of Pommerenke [29] on transfinite diameter.

A Riemann Hypothesis for Mellin Transforms

Integral Transforms

The Fourier transform is variously defined as

$$\hat{f}(y) = \mathcal{F}[f](y) = \frac{1}{c}\int_{-\infty}^{\infty} f(x)\, e^{-kixy}\, dx,$$

where $k = \pm 1, \pm 2, \pm\pi, \pm 2\pi$. The c is usually 1 or chosen to make the transform unitary. The inverse then has the form

$$f(x) = \frac{1}{\hat{c}}\int_{-\infty}^{\infty} \hat{f}(y)\, e^{kixy}\, dy.$$

I'll (for now anyway) use $k = 2\pi$ which is unitary with $c = \hat{c} = 1$. It is largely a matter of convenience since

$$\int_{-\infty}^{\infty} f(x)\, e^{-kixy}\, dx = \hat{f}\left((2\pi)^{-1}ky\right).$$

Observe

$$\mathcal{F}[f(ax)] = a^{-1}\hat{f}(y/a),$$

so, again, fiddling with constants hardly matters.

Define the Mellin transform of f by

$$\mathcal{M}(f, s) = \int_{0}^{\infty} x^s f(x)\, \frac{dx}{x}.$$

We view this as a transform on the positive reals under multiplication, with Haar measure dx/x. The exponential map is a homomorphism from the additive group of reals to the multiplicative group of positive reals. The substitution $x = e^u$, $s = -2\pi iy$ transforms the Mellin transform into a Fourier transform. (As an aside, the intermediate substitution $x = e^u$ turns the Mellin transform into the two-sided Laplace transform.)

The region of convergence depends on the behavior of f at 0 and ∞. The usual case is a vertical strip (a right-half plane for f of rapid decay). The inverse transform is then

$$\mathcal{M}^{-1}[\phi(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\phi(s) ds$$

for c in this vertical strip. For $f \in L^2((0, \infty, dx) \cap L^2((0, \infty), dx/x)$, this strip includes $\Re(s) = 1/2$, and in this case $\mathcal{M}$ is unitary in the sense that

$$\int_{0}^{\infty} f(x)\, g(x)\, \frac{dx}{x} = \int_{-\infty}^{\infty} \mathcal{M}(f, 1/2 + 2\pi it)\, \overline{\mathcal{M}(g, 1/2 + 2\pi it)}\, dt.$$

Note also that

$$\mathcal{M}(f(ax^r), s) = r^{-1}a^{-s}\mathcal{M}(f(x), s/r),$$

so this kind of fiddling with f has inconsequential effect. Similarly, changing s linearly is a matter only of convenience.

A broader viewpoint: The two quasi-characters ($\to \mathbb{C}^\times$) that extend $x \mapsto x^s$ on the positive reals are $x \mapsto |x|^s$ and $x \mapsto \operatorname{sgn}(x)|x|^s$. We have

$$\int_{-\infty}^{\infty} |x|^s f(x)\, \frac{dx}{x} = \begin{cases} 2\mathcal{M}(f,s) & f \text{ even} \\ 0 & f \text{ odd} \end{cases}$$

$$\int_{-\infty}^{\infty} \operatorname{sgn}(x)|x|^s f(x)\, \frac{dx}{x} = \begin{cases} 2\mathcal{M}(f,s) & f \text{ odd} \\ 0 & f \text{ even} \end{cases}$$

Some Zeta Functions

The Poisson Summation Formula asserts

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

whenever both sums are absolutely convergent. Suppose we choose any sufficiently rapidly decaying, even function $f(x)$ that is its own Fourier transform and apply Poisson summation to $f(xt)$. We obtain

$$\sum_{n=-\infty}^{\infty} f(nt) = t^{-1} \sum_{n=-\infty}^{\infty} f(n/t).$$

For $f(x) = e^{-\pi x^2}$, up to a somewhat unusual normalization on my part, this is just the transformation of the theta function that Ken Richardson used in his talk to recount the standard proof of the analytic continuation and functional equation of the Riemann zeta function. He views this in terms of the heat kernel; a number theorist might view the proof as exploiting a transformation formula for a modular/automorphic form. Alternatively, I'm going to repeat this calculation from a Fourier analysis point of view.

Let $g(x) = \sum_{n=-\infty}^{\infty} f(nx)$. Thus, $g(x) = x^{-1} g(1/x)$. Reversing summation and integration followed by simple substitution yields

$$\int_0^{\infty} x^s (g(x) - f(0))\, \frac{dx}{x} = 2\zeta(s) \int_0^{\infty} x^s f(x)\, \frac{dx}{x} = 2\zeta(s)\, \mathcal{M}(f,s). \tag{2}$$

We split the integral at 1 and use the transformation formula we obtained for Poisson summation on $[0,1]$ to see

$$2\zeta(s)\, \mathcal{M}(f,s) = \int_1^{\infty} (x^s + x^{1-s})(g(x) - f(0))\, \frac{dx}{x} - \frac{f(0)}{s} - \frac{f(0)}{1-s},$$

giving us both the analytic continuation (except for simple poles at 0 and 1 unless $f(0) = 0$) and functional equation. Why is $f(x) = e^{-\pi x^2}$, with

$$2\mathcal{M}(f,s) = \pi^{-s/2}\Gamma(s/2),$$

a good choice? We get analytic continuation and functional equation for all allowable f's, but this choice doesn't introduce any zeros. We'll look at some other choices shortly.

Tate's thesis [35] looks at the interplay between local Fourier analysis on the completions of a number field and the global analysis. Tao's blog has a nice synopsis. Although I wrote these notes following Tate's thesis because it is so clearly written, this approach over $\mathbb{R}$ goes back several years further [18], [37] . We find

$$\frac{\mathcal{M}(\hat{f}, 1-s)}{\pi^{-(1-s)/2}\,\Gamma((1-s)/2)} = \frac{\mathcal{M}(f,s)}{\pi^{-s/2}\,\Gamma(s/2)} \qquad \text{for } f \text{ even,} \tag{3}$$

$$\frac{\mathcal{M}(\hat{f}, 1-s)}{\pi^{s/2-1}\,\Gamma(1-s/2)} = -i\;\frac{\mathcal{M}(f,s)}{\pi^{-(s+1)/2}\,\Gamma((s+1)/2)} \qquad \text{for } f \text{ odd.} \tag{4}$$

For nice f, the left-side converges for $\Re(s) < 1$ and the right for $\Re s > 0$, giving the analytic continuation and functional equation (but not for $\zeta(s)$).

More generally for f, g both even or both odd,

$$\frac{\mathcal{M}(\hat{f}, 1-s)}{\mathcal{M}(\hat{g}, 1-s)} = \frac{\mathcal{M}(f,s)}{\mathcal{M}(g,s)}.$$

If $g(x) = e^{-\pi x^2}$ or $g(x) = xe^{-\pi x^2}$, respectively, we get the above.

As an example consider $f(x) = \hat{f}(x) = \dfrac{1}{e^{\pi x} + e^{-\pi x}}$. Then for $\Re s > 0$,

$$\mathcal{M}(f,s) = \pi^{-s}\Gamma(s)(1 - 3^{-s} + 5^{-s} - 7^{-s} + \cdots) = \pi^{-s}\Gamma(s) L\left(s, \left(\frac{-1}{n}\right)\right).$$

From (3) we get the analytic continuation and that

$$(2\pi)^{-(s+1)/2}\Gamma((s+1)/2) L\left(s, \left(\frac{-1}{n}\right)\right)$$

is invariant under $s \leftrightarrow 1-s$.

Paul Garrett has written some notes that show how to get this same result by "twisting" the theta function.

Combining them, one can prove that "half" of all primes are $\equiv 1 \pmod 4$ and "half" $\equiv 3 \pmod 4$ (a special case of Dirichlet's proof on primes in arithmetic progression).

If we apply (2) to this choice of f, we get meromorphic continuation and functional equation for the product of the Riemann zeta function and this L-function, i.e. the zeta function for the Gaussian integers. The is roughly the zeta function for the torus mentioned in Igor Prokhorenkov's talk.

Hecke originally got the analytic continuation and functional equation for the zeta function of number fields by at a Mellin transform of multidimensional theta functions with a transformation law obtained through Poisson summation.

Question Some (other) things I don't understand:

- What do we get when we "reverse engineer" other L-functions?
- Can we get these or other global objects working over the p-adics? If not, why not?

Mellin Transforms with Nice Zeros

Observe the relationships between the Fourier transform, multiplication by x, and differentiation:

$$\mathcal{F}[f^{(n)}(x)] = (2\pi i)^n y^n \hat{f}(y),$$

$$\mathcal{F}[x^n f(x)] = (-2\pi i)^{-n} \frac{d^n}{dy^n} \hat{f}(y).$$

Thus, the operator $D_n^{\pm} = \frac{1}{2}\left(\mp \frac{1}{(2\pi)^n} \frac{d^n}{dx^n} + x^n\right)$ satisfies the relation $\widehat{D_n f} = \mp i^n D_n \hat{f}$. D_2^+ is the Hamiltonian for the one-dimensional quantum mechanical oscillator that Igor mentioned (up to scaling x), with base energy eigenfunction $\psi_0(x) = e^{-\pi x^2}$. The raising (=creation=ladder) operator $D_+ = D_1^+$ [25] yields

$$\psi_n(x) = [D_+^{(n)} \psi_0](x) = (8\pi)^{-n/2} H_n(\sqrt{2\pi}x) e^{-\pi x^2},$$

where H_n is the Hermite polynomial of degree n, as eigenfunctions of both the Hamiltonian and the Fourier transform. They are even or odd as n is.

Theorem. *(Bump and Ng, Vaaler 1986 [4]) The Mellin transform of $\psi_n(x)$ is the product of exponential and gamma factors and polynomial satisfying a $s \leftrightarrow 1-s$ functional equation and the Riemann hypothesis.*

Bump and Ng set up the general proof and proved it for n even. In their paper, these polynomials also arose in a generating function they used in some estimates of values of the Riemann zeta function. From the raising operator, integration by parts yields the recursion

$$M_{n+1}(s) = \frac{1}{2} M_n(s+1) + \frac{s-1}{4\pi} M_n(s-1). \qquad (5)$$

They converted (5) to a recursion involving every other term. I'll say a bit more about their proof later. Jeff Vaaler observed that a slight adjustment in gamma factors gives the same conclusion for n odd, which they mentioned as an addendum.

Why the dichotomy between n even and odd? We saw this reflected in the cases (3), (4) above. (In fact, Poisson summation is trivial if f is odd.) Another reflection of the dichotomy in this particular case is that

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-1/2}(x^2)$$
$$H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{1/2}(x^2) [15, p.\ 1037],$$

where $L_n^{\pm 1/2}$ is the Laguerre polynomial.

This relationship suggests looking at Laguerre polynomials for a possible generalization.

Theorem. *(Gilbert 1991 [14]) The function*

$$p_n(s,\alpha) = \frac{\int_0^\infty x^{s-1} L_n^\alpha(x) x^{\alpha/2} e^{-x/2}\, dx}{2^{s+\alpha/2+n}\Gamma(s+\alpha/2)}$$

has analytic continuation to a monic polynomial that satisfies $p_n(1-s,\alpha) = (-1)^n p_n(s,\alpha)$. All zeros either lie on $\Re(s) = 1/2$ or are real. All zeros lie on $\Re(s) = 1/2$ for $\alpha \geq -1$ and zeros are simple for $\alpha > -1$.

Proof. (sketch) From the explicit formula for the Laguerre polynomials, obtain a functional equation similar to that of Bump and Ng and use induction to prove a three-term recursion. Alternatively,

$$(-2)^n \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}\, p_n(s,\alpha)$$

is the hypergeometric function $F(-n, s+\alpha/2+1/2; \alpha+1; 2)$ and one can use Gauss' recursion relations [15, p. 1039, 9.137.2 on p. 1044]. For $\alpha > -1$, setting $s = 1/2 + it$ yields the standard recursion for orthogonal polynomials. For $\alpha \leq -1$, the sign of the $(n-1)$st degree term is wrong (quasi-definite orthogonal polynomials), but we can adapt an approach of Erik van Doorn [40] to get the result. □

For α integral, these are related to Krawtchouk polynomials. More generally, they are members of the Meixner-Pollaczek family of orthogonal polynomials [7] (where the above theorem is reproved).

The first year of our REU, our two students looked at polynomials $\{p_n(x)\}$ orthogonal on $[-1,1]$ with respect to the weight $|x|^{2r}$. The Legendre polynomials are the case $r = 0$.

Theorem. *(Oberle, Scott, Gilbert, Hatcher, Addis 1993 [26]) The zeros and poles of the Mellin transform*

$$\int_{-1}^{1} \operatorname{sgn}^n(x)|x|^{s-1} p_n(x)|x|^r\, dx$$

are real and simple.

Proof. Use induction to compute the transforms explicitly as a ratio of gamma functions (or in terms of the Pochhammer symbol). □

I have counterexamples for some ultraspherical (Gegenbauer) polynomials though the $(1-x^2)^{\pm 1/4}$ has scared me away from the Chebyshev polynomials.

All Those Identities and Food for Thought

How far can we push this and where should we search? I don't know (and have no idea what's going on), but there are some reasons to be pessimistic.

Theorem. *(Bochner 1929 [2]) Suppose* $a_0(x), a_1(x), a_2(x)$ *are polynomials for which*

$$a_2(x)y'' + a_1(x)y' + a_0(x)y + \lambda_n y = 0,$$

has a solution that is a polynomial of degree n. Up to a linear change of variables and constant factors, the only orthogonal polynomial families are the Hermite, generalized Laguerre, and Jacobi polynomials.

The derivatives of the classical classical orthogonal polynomials are again orthogonal polynomials [6, p. 149]:

$$\begin{aligned} \frac{d}{dx} He_n(x) &= n He_{n-1}(x), \\ \frac{d}{dx} L_n^{(\alpha)}(x) &= -L_{n-1}^{(\alpha+1)}(x), \\ \frac{d}{dx} P_n^{(\alpha,\beta)}(x) &= \frac{n+\alpha+\beta+1}{2}\; P_{n-1}^{(\alpha+1,\beta+1)}(x). \end{aligned}$$

Theorem. *(Sonine [32], Hahn [16], [17], Krall [24] 1887 & late 1930's) If* $\{P_n(x)\}$ *and the* r*th derivatives* $\{P_n^{(r)}(x)\}$ *are both orthogonal families, once again the only possibilities are the Hermite, generalized Laguerre, and Jacobi polynomials up to a linear change of variables and constant factors.*

Theorem. *(Tricomi [39], Ebert [10], Cryer [8] 1948-1970) The only families of orthogonal polynomials satisfying Rodrigues' type formulas are essentially the Hermite, Laguerre, Jacobi polynomials.*

All of these results are summarized in [6, pp. 150-152], a great book.

Some Probable Dead Ends

Question Are other good weight functions and families of orthogonal polynomials out there?

- Are there any other potentially interesting operators from physics?
- The classical classical orthogonal polynomials have weights from the most important probability distributions. Any chance of other interesting probability distributions?
- Do nice properties of the zeros of the Mellin transform characterize (a subfamily of) the classical classical orthogonal polynomials?

Question Are there other interesting eigenfunctions of the Fourier transform?

- What happens if we use the square of the eigenfunction as a weight for a family of orthogonal polynomials?
- What happens if we apply Poisson summation?
- What happens if we apply the raising/lowering operators?
- Does looking at fractional derivatives give anything new?
- Are there other operators that nearly commute with the Fourier transform that one could use in place of the raising operator?

Question What other interesting transforms are there?

This is a good spot to mention Koornwinder's papers on transforms of orthogonal polynomials [22], [23].

Back to Bump and Ng's proof for the Mellin transforms of Hermite functions. Their polynomials p_n satisfied

$$p_{n+1}(s) = \frac{1}{8\pi}\left[(2s-1)p_n(s) + sp_n(s+2) + (s-1)p_n(s-2)\right].$$

Let $g(s) = f(s+1) + f(s-1)$ and $h(s) = sg(s+1) + (s-1)g(s-1)$. They prove that if f satisfies the Riemann hypothesis, then so does g, and that, if g does, so does h. If $f(s) = p_n(s)$, then $p_{n+1}(s) = \frac{1}{8\pi}h(s)$.

They tweaked a proof in Titchmarsch [36, 10.23.1]. Let me state a still stronger version, which can be proved in the same way from the Hadamard product.

Theorem. *(Zero Squeezing) Let ϕ and h be entire of genus 0 or 1, real for real s and satisfy*

(a) $\phi(1-s) = \pm\phi(s)$,

(b) $|h(s)| \geq |h(1-s)|$ for $\Re(s) \geq 1/2$.

Let $\sigma^* = \sup\{\Re(z) : \phi(z) = 0\}$ and $\delta > 0$. Define $\Phi(s) = h(s)\phi(s+\delta) \pm h(1-s)\phi(s-\delta)$. Then

(i) Φ is entire, genus 0 or 1, real on the real axis, and satisfies $\Phi(1-s) = \pm\Phi(s)$.

(ii) If $\Phi(z) = 0$ and $h(z)$, $h(1-z)$ are not both zero, then

$$|\Re(z) - 1/2| \leq \sqrt{\max\{0, (\sigma^* - 1/2)^2 - \delta^2\}}.$$

Note that $\dfrac{\phi(s+\delta) - \phi(s-\delta)}{2\delta}$ is an approximation to the derivative.

Remark. Toscano [38] used the operator $f(x) \mapsto f(x + i/2) - f(x - i/2)$ to find a generalized Rodrigues type formula for the Meixner-Pollaczek polynomials.

Question If we apply something like this to, say, $\xi(s)$ or some other L-function, can we prove anything about the zeros of the resulting function?

Question Any problems where we could apply Zero Squeezing?

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