Dr. Friedman's Calculus III Notes, or
I'm looking at an integral - so what the heck do I do now?

Integrals on curves (one dimensional integrals)

1. I'm integrating a function: $\int_C f(x, y, z)\, dr$.

   (a) There's only one option: parameterize and evaluate:

   $$\int_C f(x, y, z)\, dr = \int_a^b f(x(t), y(t), z(t)) \left|\frac{dr}{dt}\right| dt$$

2. I'm integrating a vector field: $\int_C F \cdot dr$ or $\int_C M dx + N dy + P dz$.

   (a) C is *not* a closed curve

   - i. F is *not* conservative
     - A. Your only option is to parameterize and integrate: $\int_C F \cdot dr = \int_a^b F \cdot \frac{dr}{dt}\, dt$
   - ii. F *is* conservative: $F = \nabla f$
     - A. Option 1: You can parameterize and integrate: $\int_C F \cdot dr = \int_a^b F \cdot \frac{dr}{dt}\, dt$
     - B. Option 2: You can find the potential and use the fundamental theorem of line integrals: $\int_C F \cdot dr = f(r(b)) - f(r(a))$
     - C. Option 3: You can parametrize and integrate but using a different path with the same endpoints.

   (b) C *is* a closed curve

   - i. F is *not* conservative
     - A. Option 1: Parameterize and integrate: $\int_C F \cdot dr = \int_a^b F \cdot \frac{dr}{dt}\, dt$
     - B. Option 2: If C is a simple closed curve in the plane and F is a vector field in the plane, you could use Green's theorem: $\int_C M dx + N dy = \int_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$ if C is the boundary of R and C is oriented counterclockwise
     - C. Option 3: If C is a closed curve in space and it is the boundary of a surface S, then you can use Stokes's theorem: $\int_C F \cdot dr = \iint_S (\nabla \times F) \cdot N\, dS$, where N satisfies the right hand rule with respect to the orientation of C.
   - ii. F *is* conservative
     - A. The integral is 0.

Integrals on surfaces (two dimensional integrals)

1. I'm integrating a function: $\iint_S f(x, y, z)\, dS$.

(a) There's only one option: parameterize and evaluate:

$$\iint_S f(x,y,z)\,dS = \iint_R f(x(u,v),y(u,v),z(u,v)) \left|\frac{dr}{du} \times \frac{dr}{dv}\right| dA,$$

where R is the parametrizing region in the u-v plane.

2. I'm integrating a flux integral of a vector field $\iint F \cdot N\,dS$

(a) S is not orientable

i. The integral cannot be well-defined

(b) S is orientable but not the boundary of anything

i. Parametrize and evaluate:

$$\iint_S F(x,y,z) \cdot N\,dS = \iint_R F \cdot \left(\frac{dr}{du} \times \frac{dr}{dv}\right) dA,$$

ii. In the rare case that F is the curl of another vector field, say $F = \nabla \times W$, then you could use Stokes's theorem, but this doesn't come up very often.

(c) S is the boundary of a solid Q.

i. Option 1: Parametrize and evaluate:

$$\iint_S F \cdot N\,dS = \iint_R F \cdot \left(\frac{dr}{du} \times \frac{dr}{dv}\right) dA,$$

ii. Option 2: Use the divergence theorem

$$\iint_S F \cdot N\,dS = \iiint_Q \nabla \cdot F\,dV,$$

where N is the normal pointing out of the solid. If you want to compute flux into the solid, change the sign of the answer.

Integrals on solids (three dimensional integrals)

1. We *only* integrate functions on solids. Use chapter 14 methods to evaluate $\iiint_Q f(x,y,z)\,dV$

2. If you happen to know that the function f is a divergence $f = \nabla \cdot F$, then you *could* use the divergence theorem $\iiint_Q \nabla \cdot F\,dV = \iint_S F \cdot N\,dS$, where S is the boundary of Q and N is the outward pointing normal, but you'd almost never do this unless it really simplifies nicely for some reason.