Some non-trivial PL knots whose complements are homotopy circles

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Abstract

We show that there exist non-trivial piecewise-linear (PL) knots with isolated singularities $S^{n-2} \subset S^n$, $n \geq 5$, whose complements have the homotopy type of a circle. This is in contrast to the case of smooth, PL locally-flat, and topological locally-flat knots, for which it is known that if the complement has the homotopy type of a circle, then the knot is trivial.

It is well-known that if the complement of a smooth, piecewise linear (PL) locally-flat, or topological locally-flat knot $K \subset S^n$, $K \cong S^{n-2}$, $n \geq 5$, has the homotopy type of a circle, then K is equivalent to the standard unknot in the appropriate category (see Stallings [11] for the topological case and Levine [6] and [8, §23] for the smooth and PL cases). This is also true of classical knots $S^1 \hookrightarrow S^3$ (see [10, §4.B]), for which these categories are all equivalent, and in the topological category for locally-flat knots $S^2 \hookrightarrow S^4$ by Freedman [2, Theorem 6].

By contrast, Freedman and Quinn showed in [3, §11.7] that any classical knot with Alexander polynomial 1 bounds a topological locally-flat D^2 in D^4 whose complement is a homotopy circle, and by collapsing the boundary, one obtains a singular S^2 in S^4 with the same property. In the same dimensions, Boersema and Taylor [1] constructed a specific example of a PL knot with an isolated singularity whose complement is a homotopy circle. It follows by taking iterated suspensions that there are nontrivial PL embeddings $S^{n-2} \hookrightarrow S^n$ in all dimensions $n \geq 4$ whose complements are homotopy circles, though this process will lead to increasingly more complicated singularities. In this note, we construct PL knots for any $n \geq 5$ that are locally-flat except at one point and whose complements are homotopy circles.

To construct the knots with the desired properties, it will suffice to construct for each $n \geq 5$ a PL locally-flat disk knot $L \subset D^n$, such that $D^n - L \sim_{h.e.} S^1$ and such that the PL locally-flat boundary sphere knot $\partial L \subset \partial D^n$ is non-trivial. By a PL locally-flat disk knot $L \subset D^n$, we mean the image of a PL locally-flat embedding $D^{n-2} \hookrightarrow D^n$ such that $\partial L \subset \partial D^n$ is a locally-flat sphere knot and $int(L) \subset int(D^n)$. This will suffice since, if such a disk knot exists, we may then adjoin the cone on the boundary pair $(\partial D^n, \partial L)$ to obtain a PL sphere knot $K \subset S^n$ that is locally-flat except at the cone point:

$$\begin{array}{ccccc} K & = & L & \cup_{\partial L} & c(\partial L) \\ \cap & & \cap & & \cap \\ S^n & = & D^n & \cup_{\partial D^n} & c(\partial D^n) \end{array} .$$

It is clear that $S^n - K \sim_{h.e.} D^n - L$, so if the complement of L is a homotopy circle then so will be that of K. Furthermore, K will be non-trivial since the link pair of the cone point will be non-trivially knotted, which is impossible in the unknot, which is locally-flat.

So we construct such a disk knot. The procedure will be based upon that given by the author in [4] for constructing certain Alexander polynomials of disk knots, which in turn was a generalization of Levine's construction of sphere knots with given Alexander polynomials in [7]. All spaces and maps will be in the PL category without further explicit mention.

Suppose that $n \geq 5$, and let U be the trivial disk knot $U \subset D^n$, i.e. D^n may be identified with the unit ball in $\mathbb{R}^n$ such that U is the intersection of D^n with the coordinate plane $\mathbb{R}^{n-2} \subset \mathbb{R}^n$. We can assume that U bounds an embedded $n-1$ disk V in D^n, that ∂U bounds an $n-2$ disk F in ∂D^n, that $\partial V = U \cup F$, and that $int(V) \subset int(D^n)$. Embed an unknotted S^{n-3} into

$\partial D^n = S^{n-1}$ so that it is not linked with ∂U and does not intersect F (in fact, we may assume that the new S^{n-3} and F are in opposite hemispheres of ∂D^n). We use the standard framing of the new unknotted S^{n-3} to attach an $n-2$ handle to D^n, obtaining a space homeomorphic to $S^{n-2} \times D^2$ and containing V in a trivial neighborhood of some point on the boundary. Let $C_0 = S^{n-2} \times D^2 - U$. Since $\pi_1(D^n - U) \cong \mathbb{Z}$, $\pi_1(C_0) \cong \mathbb{Z}$ by the Seifert-van Kampen theorem. Let $\tilde{C}_0$ be the infinite cyclic cover of C_0 associated with the kernel of the homomorphism $\pi_1(C_0) = \mathbb{Z} \to \mathbb{Z}$ determined by linking number with U. Let $X_0 = \partial(S^{n-2} \times D^2) - \partial U$, and let $\tilde{X}_0$ be the infinite cyclic cover of X_0 in $\tilde{C}_0$.

As in the usual construction of infinite cyclic covers in knot theory (see, e.g., Rolfsen [10]), we can form $\tilde{C}_0$ by a cut and paste procedure: we cut C_0 along V to obtain Y_0 and then glue a countably infinite number of copies of Y_0 together along the copies of V. Since $C_0 - V \sim_{h.e.} S^{n-2}$, we have $\tilde{H}_{n-2}(\tilde{C}_0) = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ - where t represents a generator of the group of covering translations - and all other reduced homology groups are trivial. Similarly, since $\partial(S^{n-2} \times D^2) - F$ is a punctured $S^{n-2} \times S^1$, $\tilde{H}_*(\tilde{X}_0)$ is $\mathbb{Z}[\mathbb{Z}]$ in dimensions $n-2$ and 1, and trivial otherwise.

It is also apparent that $\pi_*(\tilde{C}_0)$ is trivial for $* < n-2$, while $\pi_1(\tilde{X}_0)$ is free on a countably infinite number of generators. Thus, since $n \geq 5$, $\pi_2(\tilde{C}_0, \tilde{X}_0)$ is also free on a countably infinite number of generators. Meanwhile, for X_0, itself, $\pi_1(X_0)$ is the free group on two generators: one generator corresponds to the generator of $\pi_1(\partial(S^{n-2} \times D^2)) = \pi_1(S^{n-2} \times S^1) = \mathbb{Z}$ and the other corresponds to the meridian of the unknotted ∂U (this can be demonstrated by an easy Seifert-van Kampen argument, by considering ∂U to lie in a ball neighborhood of some point). Let a represent the generator corresponding to the meridian of ∂U, and let b represent the other described generator. We note that the generator of $\pi_1(C_0) \cong \mathbb{Z}$ is also given by a, while b is contractible in this larger space.

Consider now the element γ of $\pi_1(X_0)$ given by $b^2aba^{-1}b^{-1}ab^{-1}a^{-1}$. Since $b = 1$ in $\pi_1(C_0)$ and a occurs with total exponent 0 in γ, the image of γ in $\pi_1(C_0)$ is trivial, so any representative of γ is the boundary of a 2-disk Γ in C_0. Since $n \geq 5$, we can assume that Γ is properly embedded (see [5, Corollary 8.2.1]). Furthermore, γ can be lifted to a closed curve in $\tilde{X}_0$; if we let c_i represent the generators of $\pi_1(\tilde{X}_0)$, then any lift of a is a path between adjoining lifts of X_0 in the cut and paste construction, and γ lifts to $\tilde{\gamma} = c_0^2c_1c_0^{-1}c_1^{-1} \in \pi_1(\tilde{X}_0)$. In the abelianization $H_1(\tilde{X}_0)$, the image of $\tilde{\gamma}$ is the same as the image of c_0, which is a $\mathbb{Z}[\mathbb{Z}]$-module generator of $H_1(\tilde{X}_0)$.

Let N denote an open regular neighborhood of Γ in C_0. We claim that $S^{n-2} \times D^2 - N$ is homeomorphic to D^n. In fact, observe that in $S^{n-2} \times S^1$, γ is homotopic to the standard generator $b = * \times S^1$ of $\pi_1(S^{n-2} \times S^1)$ (with an appropriate choice of orientations). Thus, in $(S^{n-2} \times D^2, S^{n-2} \times S^1)$, the pair (Γ, γ) is homotopic to the standard generator $* \times D^2$ of $\pi_2(S^{n-2} \times D^2, S^{n-2} \times S^1)$. These homotopies can be realized by ambient isotopies by [5, Theorem 10.2]. Then it is clear that $S^{n-2} \times D^2 - N \cong D^{n-2} \times D^2 \cong D^n$.

Fixing a homeomorphism $S^{n-2} \times D^2 - N \to D^n$, the image of U is a new disk knot, which we christen L. We claim that L is no longer trivial but that its complement is a homotopy circle.

Let C be the complement of an open regular neighborhood of L in D^n (the disk knot exterior). Thus C is homotopy equivalent to $D^n - L$. Similarly, let X be the exterior of ∂L in $\partial D^n = S^{n-1}$. We must study the homotopy and homology of C, X, and their coverings.

Lemma 1. *$\pi_1(C) = \mathbb{Z}$.*

Proof. $C \sim_{h.e.} D^n - L \cong (C_0 - N) \cup_{D^2 \times S^{n-3}} N$. Since $\pi_1(C_0) \cong \mathbb{Z}$ and $N \cong D^n$, $\pi_1(C) \cong \mathbb{Z}$ by the Seifert-van Kampen Theorem. □

Lemma 2. *$\pi_1(X) \cong \langle a, b \mid b^2aba^{-1}b^{-1}ab^{-1}a^{-1} \rangle$.*

Proof. The effect of the handle subtraction $C_0 - N$ on the boundary X_0 is that of a surgery on the embedded curve γ. Since $\pi_1(X_0)$ is free on the generators a and b, the result of the surgery is the given group. (Proof: The result of the surgery is $(X_0 - S^1 \times D^{n-2}) \cup D^2 \times S^{n-3}$, where the S^1 represents γ. But since $n \geq 5$, $\pi_1(X_0 - S^1 \times D^{n-2}) \cong \pi_1(X_0)$. So by Seifert-van Kampen, π_1 of the result of the surgery is $\pi_1(X_0)/\pi_1(S^1 \times S^{n-3}) \cong \pi_1(X_0)/\mathbb{Z}$, where the $\mathbb{Z}$ is generated by $S^1 \times *$ in $S^1 \times S^{n-3}$, which is the boundary of the neighborhood of γ. But any such curve is homotopic to γ, which represents $b^2aba^{-1}b^{-1}ab^{-1}a^{-1}$.) □

Lemma 3. *The Alexander modules $\tilde{H}_*(\tilde{C})$, $\tilde{H}_*(\tilde{X})$, and $\tilde{H}_*(\tilde{C}, \tilde{X})$ are all trivial.*

Proof. Let $\tilde{\gamma}$ be the lift of γ considered above. We can also lift Γ to a 2-disk $\tilde{\Gamma}$ in $\tilde{C}_0$. In fact, we can find a countable number of lifts $\tilde{\gamma}_i$ and $\tilde{\Gamma}_i$, and, since Γ is embedded, the $\tilde{\Gamma}_i$ are all disjoint. If $\tilde{N}_i$ then represent the lifts of the regular neighborhood N, $\tilde{C}_0 - \amalg_i \tilde{N}_i$ will be the infinite cyclic cover of $C_0 - N \cong D^n - L$.

Now consider $\tilde{X}_0 \cup \amalg_i \tilde{N}_i$. Each intersection $\tilde{X}_0 \cap N_i$ is homotopy equivalent to a translate of $\tilde{\gamma}_i$, which we know represents the $\mathbb{Z}[\mathbb{Z}]$-module generator of $H_1(\tilde{X}_0)$. It thus follows from the Mayer-Vietoris sequence that $\tilde{H}_*(\tilde{X}_0 \cup \amalg_i \tilde{N}_i)$ is trivial except in dimension $n-2$, where it is $\mathbb{Z}[\mathbb{Z}]$. Meanwhile, we already know that $\tilde{H}_*(\tilde{C}_0)$ is trivial except in dimension $n-2$, where it is also $\mathbb{Z}[\mathbb{Z}]$. Consider the map $H_{n-2}(\tilde{X}_0 \cup \amalg_i \tilde{N}_i) \rightarrow H_*(\tilde{C}_0)$. In each module, a $\mathbb{Z}[\mathbb{Z}]$-module generator is represented by a choice of $S^{n-2} \times * \subset S^{n-2} \times S^1 \subset S^{n-2} \times D^2$ that is disjoint from V. Thus this homology map is an isomorphism, and it follows that $H_*(\tilde{C}_0, \tilde{X}_0 \cup \amalg_i \tilde{N}_i)$ is trivial. But by excision, $H_*(\tilde{C}_0, \tilde{X}_0 \cup \amalg_i \tilde{N}_i) \cong H_*(\tilde{C}, \tilde{X})$.

Similarly, it follows from easy homological calculations that $\tilde{H}_*(\tilde{X})$ is trivial. In fact, it can be seen that the construction of X from X_0 is by a surgery, and upon restriction of our construction to its effect on X_0, we obtain the construction of Levine for producing smooth sphere knots with given Alexander polynomials in [7]. In this case, the Alexander polynomial is trivial (since $\tilde{\gamma}$ generates $H_1(\tilde{X}_0)$), and it follows from Levine's calculations that $\tilde{H}_*(\tilde{X}) = 0$.

Then $\tilde{H}_*(\tilde{C})$ is also trivial, by the long exact sequence of the pair $(\tilde{C}, \tilde{X})$. □

Proposition 4. $\pi_*(D^n - L) \cong \pi_*(S^1)$.

Proof. By Lemma 1, $\pi_1(C) = \mathbb{Z}$. Thus the infinite cyclic cover $\tilde{C}$ is simply connected, and since we also have $\tilde{H}_*(\tilde{C}) = 0$ by Lemma 3, it follows that $\pi_j(\tilde{C}) = 0$ for all $j > 1$ by Hurewicz's Theorem. Thus for $j > 1$, $\pi_j(C) = 0$, and $\pi_*(D^n - L) \cong \pi_*(C) \cong \pi_*(S^1)$. □

Theorem 5. *$D^n - L$ is a homotopy circle.*

Proof. By the preceding proposition, $D^n - L$ has the same homotopy groups as a circle. But $D^n - L$ is homotopy equivalent to C, which is homeomorphic to a finite simplicial complex. Since the inclusion $i : S^1 \rightarrow C$ of a meridian of L induces the isomorphism $\pi_1(S^1) \rightarrow \pi_1(C)$, we can conclude that i is a homotopy equivalence. Thus $C \sim_{h.e.} D^n - L$ is a homotopy circle. □

It only remains to show that L is non-trivial, which will follow once we show that the group $\pi_1(X)$ of the boundary knot ∂L is not $\mathbb{Z}$.

Lemma 6. *The group $G = \langle a, b \mid b^2 aba^{-1}b^{-1}ab^{-1}a^{-1} \rangle$ is not isomorphic to $\mathbb{Z}$.*

Proof. This lemma can be proven in a variety of ways. The following elegant demonstration was shown to me by Andrew Casson.

We adjoin an extra generator c, which we immediately set equal to aba^{-1}. Then

$$\begin{aligned}\langle a, b \mid b^2aba^{-1}b^{-1}ab^{-1}a^{-1}\rangle &\cong \langle a, b, c \mid b^2aba^{-1}b^{-1}ab^{-1}a^{-1}, cab^{-1}a^{-1}\rangle \\ &\cong \langle a, b, c \mid b^2cb^{-1}c^{-1}, cab^{-1}a^{-1}\rangle \\ &\cong \frac{\langle b, c \mid b^2cb^{-1}c^{-1}\rangle * \langle a\rangle}{\langle cab^{-1}a^{-1}\rangle}.\end{aligned}$$

Written this way, G has the form of an HNN extension of the Baumslag-Solitar group $H = \langle b, c \mid b^2cb^{-1}c^{-1}\rangle$, which is isomorphic to the semi-direct product $\mathbb{Z}[\frac{1}{2}] \rtimes \mathbb{Z}$. Thus H is a non-abelian subgroup of G, which hence cannot be $\mathbb{Z}$.

Alternatively, to apply an unnecessarily large hammer, once G is written as $\langle a, b, c \mid b^2aba^{-1}b^{-1}ab^{-1}a^{-1}, cab^{-1}a^{-1}\rangle$, it follows from [9] that G is not even residually finite.

A third proof would utilize Whitehead's theorem on one-relator groups [12]. □

Remark 7. There is nothing exceptionally special about the group G we have used in this construction, except that it turned out to be a fairly tractable example of a group with suitable properties. Any group possessing a two generator, one relator presentation with the properties employed above clearly would be sufficient.

References

[1] Jeff Boersema and Erica J. Taylor, *Knots in four dimensions and the fundamental group*, http://www.rose-hulman.edu/mathjournal/2003/vol4-n2/paper2/v4n2-2pd.pdf.

[2] Michael Freedman, *The disk theorem for four-dimensional manifolds*, Proceedings of the International Congress of Mathematicians: August 16-24, 1983, Warszawa, Volume 1 (Warszawa), Polish Scientific Publishers, 1984, pp. 647–664.

[3] Michael H. Freedman and Frank Quinn, *Topology of 4-manifolds*, Princeton University Press, Princeton, NJ, 1990.

[4] Greg Friedman, *Alexander polynomials of non-locally-flat knots*, Indiana Univ. Math. J. **52** (2003), 1479–1578.

[5] J.F.P. Hudson, *Piecewise linear topology*, W.A. Benjamin, Inc., New York, 1969.

[6] Jerome Levine, *Unkotting spheres in codimension two*, Topology **4** (1965), 9–16.

[7] ______, *Polynomial invariants of knots of codimension two*, Ann. of Math **84** (1966), no. 2, 537–554.

[8] ______, *An algebraic classification of some knots of codimension two*, Comment. Math. Helv. **45** (1970), 185–198.

[9] E. Raptis, O. Talelli, and D. Varsos, *On the Hopficity of certain HNN-extensions with base a Baumslag-Solitar group*, Algebra Colloq. **9** (2002), 39–48.

[10] Dale Rolfsen, *Knots and links*, Publish or Perish, Inc., Berkeley, CA, 1976.

[11] John Stallings, *On topologically unknotted spheres*, Ann. of Math. (2) **77** (1963), 490–503.

[12] J.H.C. Whitehead, *On equivalent sets of elements in a free group*, Ann. of Math. (2) **37** (1936), 782–800.

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