Some extremely brief notes on the Leray spectral sequence

Intro. As a motivating example, consider the long exact homology sequence. We know that if we have a short exact sequence of chain complexes

$$0 \to C_* \to D_* \to E_* \to 0,$$

then this gives rise to a long exact sequence of homology groups

$$\to H_i(C_*) \to H_i(D_*) \to H_i(E_*) \xrightarrow{\partial_*} H_{i-1}(C_*) \to .$$

While we may learn the proof of this, including the definition of the boundary map ∂_* in beginning algebraic topology and while this may be useful to know from time to time, we are often happy just to know that there is an exact homology sequence. Also, in many cases the long exact sequence might not be very useful because the interactions from one group to the next might be fairly complicated. However, in especially nice circumstances, we will have reason to know that certain homology groups vanish, or we might be able to compute some of the maps here or there, and then we can derive some consequences. For example, if we have reason to know that $H_*(D_*) = 0$, then we learn that $H_*(E_*) \cong H_{*-1}(C_*)$.

Similarly, to preserve sanity, it is usually worth treating spectral sequences as "black boxes" without looking much under the hood. While the exact details do sometimes come in handy to specialists, spectral sequences can be remarkably useful even without knowing much about how they work.

What spectral sequences look like. In long exact sequence above, we have homology groups $H_i(C_*)$ (of course there are other ways to get exact sequences). So there is one index - the degree of the homology groups. (In a sense, long exact sequences coming from short exact sequences of chain complexes can be thought to have also a $\mathbb{Z}_3$ grading that indexes the three chain complexes). Spectral sequences have three indices. The objects look like $E^r_{p,q}$, which can be groups, modules, etc. (there is also a cohomological version with cohomological indexing, but we'll stick with homology in these notes). We usually picture these as living at the lattice points (p, q) in a plane labeled r, which we think of representing the rth stage of a certain process. Usually we only consider $r \geq 1$ or perhaps $r \geq r_0$ for some fixed r_0. At stage r, there are also maps $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$. These satisfy $d_r^2 = 0$ when the composition makes sense. At each stage, the group $E^{r+1}_{p,q}$ is the homology of the previous stage, i.e.

$$E^{r+1}_{p,q} = \ker(d_r : E^r_{p,q} \to E^r_{p-r,q+r-1})/\mathrm{im}(d_r : E^r_{p+r,q-r+1} \to E^r_{p-r,q+r-1}).$$

Finding d_{r+1} in terms of what has come before is much more complicated, and this is often one of the biggest difficulties in trying to use spectral sequences. We won't go into this here.

In reasonable situations, eventually things settle down and for each fixed (p, q), the groups $E^r_{p,q}$ become fixed for all r past a certain point. This happens, for example, in "first quadrant" spectral sequences for which all terms vanish unless $p, q \geq 0$. In this case, eventually all arrows that would come into or go out of a fixed (p, q) eventually become trivial and so the homology stabilizes. When this happens, we start referring to the groups as $E^\infty_{p,q}$.

The general mantra of spectral sequences is that we'd like to have terms at the E^1 or E^2 stage that we somehow understand and wind up with E^∞ that relate to something we'd like to compute (though it is possible sometimes to run the knowledge the other way and deduce things about the E^1s from the E^∞s we wind up with). How to recover the thing we want to compute from E^∞ is a little complicated. The word hiding the difficulty at that point is "abutment." We say that the spectral sequence abuts to F_* if we can, roughly speaking, recover F_* from the $E^\infty_{p,q}$. Before going into more detail about how this is done, we give some examples of the kinds of things spectral sequences are good for.

Examples of spectral sequences. In a sense there is really only one spectral sequence, just as there is only one concept of a long exact sequence (although each object may originate in a variety of settings), but there are many different named uses. Here are a few examples.

1. Leray-Serre spectral sequence. Given a fibration $F \hookrightarrow X \to B$ with trivial monodromy on the homology of the fibers, $E^2_{p,q} = H_p(B; H_q(F))$ and the spectral sequence abuts to $H_*(X)$. (If there is monodromy, this still works but replacing $E^2_{p,q} = H_p(B; H_q(F))$ with $E^2_{p,q} = H_p(B; \mathcal{H}_q(F))$, where $\mathcal{H}_q(F)$ is a local system on B). There is also a cohomology version.

2. Atiyah-Hirzebruch spectral sequence. If F_* is a generalized homology theory (such as K-homology, bordism, etc.), then there is a spectral sequence with $E^2_{p,q} = H_p(X; F_q(pt))$ that abuts to $F_*(X)$. There is also a generalized cohomology version.

3. Hypercohomology spectral sequences. If $\mathcal{S}^*$ is a complex of sheaves on X, the hypercohomology sequence with $E_2^{p,q} = H^p(X; \mathcal{H}^*(\mathcal{S}^*))$ that abuts to $\mathbb{H}^*(X; \mathcal{S}^*)$, where $\mathcal{H}^*(\mathcal{S}^*)$ is the derived cohomology sheaf and $\mathbb{H}^*$ is hypercohomolgy.

4. A special example of the hypercohomology spectral sequence is the Hodge-De Rham spectral sequence on a complex manifold: $E_2^{p,q} = H^p(X; \Omega^q)$, where Ω^q is the complex of holomorphic q-forms, which abuts to $H^*(X; \mathbb{C})$.

5. Adams spectral sequence. $E^2_{t,q} = \mathrm{Ext}^t_A(H^q(X); \mathbb{Z}_p)$, which abuts to $\pi^s_*(X)$/non-p-torsion, where A is the mod-p Steenrod algebra.

6. Spectral sequences also crop up when trying to generalize the universal coefficient or Künneth theorems to rings that are not PIDs.

A little more detail. One of the most common ways to generate spectral sequences algebraically (and some of the above examples can be seen this way) is to start off with a chain complex C_* and introduce a filtration F^*C_*. Each F^pC_* is a subcomplex of C_*. There are increasing and decreasing filtrations. We will focus on the increasing ones for which $F^{p-1}C_* \subset F^pC_*$ for all p. To make things work out one also often assumes that the filtration is bounded, i.e. $F^pC_* = 0$ for sufficiently small p and $F^pC_* = C_*$ for sufficiently large p. It is also common to assume that C_* is a bounded complex.

Notice that this filtration induces a filtration on homology by $F^pH_*(C_*) = \mathrm{im}(H_*(F^pC_*) \to H_*(C_*))$.

With such niceness assumptions as above (and even sometimes without them), the general theory then says that there is a spectral sequence with $E^1_{p,q} = H_{p+q}(F^pC_*/F^{p-1}C_*)$ and d_1 given by the boundary map of the long exact sequence of the triple $(F^pC_*, F^{p-1}C_*, F^{p-2}C_*)$. The E^∞ terms of this spectral sequence are isomorphic to $F^pH_{p+q}(C_*)/F^{p-1}H_{p+q}(C_*)$.

Since $F^pH_*(C_*) = H_*(C_*)$ for large enough p and it is 0 for small enough p, this tells us that to reconstruct $H_*(C_*)$ given the E^∞ is just a collection of extension problems. However, if C_* is a complex of vector spaces, then the extension problems are trivial and $H_i(C_*) \cong \oplus_{p+q=i}E^\infty_{p,q}$.

The Leray-Serre spectral sequence. Suppose we have a filtration (or a fiber bunder) $F \hookrightarrow X \xrightarrow{\pi} B$. We assume B is a finite dimensional CW complex. We filter the singular chain complex $C_*(X)$ by letting $F^p(C_*(X)) = C_*(\pi^{-1}(B^p))$, where B^p is the p-skeleton of B. Then $F^pC_*(X)/F^{p-1}(C_*(X))$ is the quotient chain complex $C_*(\pi^{-1}(B^p))/C_*(\pi^{-1}(B^{p-1})) \cong C_*(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$. By excision, the homology of this complex is the direct sum

$$\oplus_{e^p}H_*(\pi^{-1}(e), \pi^{-1}(\partial e^p))$$

over the p-cells e^p of B. But since e^p is contractible, the fibration over it is trivial and so homotopy equivalent to $e^p \times F$. So, $H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p)) \cong H_*(e^p \times F, \partial e^p \times F) \cong H_*(D^p \times F, S^{p-1} \times F)$, which by the Künneth theorem is just $H_{*-p}(F)$, which we can also interpret as $H_p(D^p, S^{p-1}; H_{*-p}(F))$.

So for the spectral sequence, we have $E^1_{p,q} = H_{p+q}(F^pC_*/F^{p-1}C_*) \cong \oplus_{e^p}H_p(D^p, S^{p-1}; H_q(F))$. The map d_1 takes this to $\oplus_{e^{p-1}}H_{p-1}(D^{p-1}, S^{p-2}; H_q(F))$ by the boundary map of the long exact sequence of the triple (B^p, B^{p-1}, B^{p-2}). But this is precisely a description of the boundary map of the CW-chain complex of B with coefficients in $H_q(F)$ (note, however, that these coefficients can be twisted by the monodromy of the fibration if it is non-trivial). Thus taking homology to get to $E^2_{p,q}$ yields $E^2_{p,q} = H_p(B; \mathcal{H}_q(F))$.

The spectral sequence abuts to a filtration of $C_*(X)$.

Some easy examples of applications. The easiest applications come when most of the terms $E^2_{p,q}$ vanish. For example, consider a fibration $S^m \hookrightarrow X \to S^n$. The only nontrivial $E^2_{p,q}$ will be $\mathbb{Z}$s when (p,q) is $(0,0)$, $(0,m)$, $(n,0)$, or (n,m). Unless $(0,m) = (n-r, r-1)$ for some r, the E^2 stage will be equal to the E^∞ stage, and we learn that $H_*(X)$ is trivial except for $\mathbb{Z}$s in dimensions $0, m, n, m+n$ (or a $\mathbb{Z}^2$ in dimension m if $m = n$). If $(0,m) = (n-r, r+1)$ for some r, then there are other possibilities. For example, in the Hopf fibration of S^3, we have a map $d_2 : E^2_{2,0} \to E^2_{0,1}$, and since we know the homology of S^3 already, we must have d_2 is an isomorphism (which is not necessarily easy to show directly). For $S^2 \times S^1$, we must have $d_2 = 0$.

Note in general that if $H_*(B) = 0$ for $* > p$ and $H_*(F) = 0$ for $* > q$, then $H_*(X) = 0$ for $* > p+q$ and $H_{p+q}(X) = H_p(B; \mathcal{H}_q(F))$.

Euler characteristics. Another nice application is that if B is simply connected (or there is no fiber monodromy on homology) and all homology groups are finite rank, then $\chi(X) = \chi(B)\chi(F)$. Proof: Work with rational coefficients. For E^2, look at $\chi_2 = \sum_{p,q}(-1)^{p+q}b_2^{p,q}$, where $b_r^{p,q}$ is the Betti number of $E_{p,q}^r$. For fixed q, we have $\sum_p(-1)^{p+q}b_2^{p,q} = (-1)^q\chi(B)b_F^q$, where b_F^q is the qth Betti number of F. So $\chi_2 = \chi(B)\chi(F)$. Next, we notice that χ_r is independent of r. Let $d_{p,q}^r$ be the boundary map with $E_{p,q}^r$ the domain. Then

$$\begin{aligned} E_{p,q}^r &\cong \ker(d_{p,q}^r) \oplus \operatorname{im}(d_{p,q}^r) \\ &\cong \operatorname{im}(d_{p+r,q-r+1}^r) \oplus E_{r+1}^{p,q} \oplus \operatorname{im}(d_{p,q}^r) \end{aligned}$$

So

$$\begin{aligned} \chi_r &= \sum_{p,q}(-1)^{p+q}b_r^{p,q} \\ &\cong \sum_{p,q}(-1)^{p+q}(\dim(\operatorname{im}(d_{p+r,q-r+1}^r)) + b_{r+1}^{p,q} + \dim(\operatorname{im}(d_{p,q}^r))) \\ &\cong \chi_{r+1} + \sum_{p,q}(-1)^{p+q}(\dim(\operatorname{im}(d_{p+r,q-r+1}^r)) + \dim(\operatorname{im}(d_{p,q}^r))) \\ &\cong \chi_{r+1}, \end{aligned}$$

since it is easy to check that the terms on the right cancel in pairs.

On the other hand, at the abutment stage,

$$\begin{aligned} \chi(X) &= \sum_i(-1)^i \dim(H_i(X)) \\ &= \sum_i(-1)^i \sum_{p+q=i} \dim(E_{p,q}^\infty) \\ &= \sum_{p,q}(-1)^{p+q}b_\infty^{p,q} \\ &= \chi_\infty = \chi_r = \chi(B)\chi(F). \end{aligned}$$