Notes on Proofs v. 1.1

by Greg Friedman

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We have to reinvent the wheel every once in a while; not because we need a lot of wheels but because we need a lot of inventors. - Bruce Joyce

Contents

1 Introduction - updated September 17, 2004

These notes, originally developed for use by students of Yale University's Math 230 Course, are intended to say something useful about how to construct and write proofs. They are in an early stage of development. I hope to add topics as times goes on, reflecting a broader diversity of material. So check back often for updates. The earlier sections deal more with generally strategies, but as the notes go on there's more about particular strategies for particular situations. Section 7 contains information that is actually more basic than that in the earlier sections and is intended as something of an appendix about basic mathematical language.

Any feedback would be much appreciated.

1.1 The Nature of Mathematics, by Dick Beals - included August 21, 2005

The nature of mathematics. What makes mathematics intellectually respectable is the careful use of ideas and logical reasoning. What makes it exciting and beautiful is the way in which ideas that are fundamentally simple can be interwoven and elaborated to reach conclusions that are based on long chains of reasoning, yet still are reliable. This is part of what gives mathematics what a physicist has called its *unreasonable effectiveness*.

These features of mathematics are not always evident in the way it is taught. A course that teaches mathematics as a series of recipes for solving specific problems is not about mathematics at all, it is about becoming a technician.

1.2 What's a proof? - Added August 21, 2005

Your job in writing a proof is to make an incontrovertible argument demonstrating the truth of a mathematical statement. Unfortunately, what constitutes "incontrovertible" is open to interpretation. Nonetheless, your goal is to be as convincing as possible using the logical argument and reasoning. In a college math course, you should imagine that you are trying to convince your professor or TA that the statement is true and provide sufficient argument to substantiate your claim (and preferably little else). Imagine that you are having an argument with your professors and that they are going to be picky about your reasoning skills. Does each statement you make logically follow from the previous one or from an outside source, such as a textbook, that you have clearly cited? Are you sure that you haven't stated anything untrue? Has your argument covered all cases? Have you written clearly and to the point? Will someone reading your argument be convinced that you have demonstrated what you set out to demonstrate? If so, you have created a proof.

1.3 What is not in these notes - updated September 17, 2004

What these notes will not do is give you a set of instructions for how to prove every possible theorem. If such a set of instructions existed, I'd be out of a job, and Gödel would have been a much happier man. Even in an introductory course, there is no fixed group of ideas that will get you through everything. That being said, experience leads one to approach certain proofs in certain ways. Sometimes the overarching idea is straightforward, but there are details that need to be checked. Other times, even getting started requires some clever insight. So do not expect that reading these notes will imbue you with a level of insight that will conquer all proofs. Alas, such is not to be.

1.4 READ PROOFS, READ PROOFS, READ PROOFS - updated September 17, 2004

While these notes might help you get used to constructing proofs, there are absolutely no substitutes for exposure and experience. You will never get the hang of proofs unless you read them and re-read them. *It is essential to be able to recognize tight logical reasoning*, and the only way to get used to that is by seeing it over and over. Read the textbook (or other math books!) to get experience with sound reasoning. Then practice picking apart shoddy reasoning in other courses and in the real world. You should also read proofs multiple times at different levels. I usually start by reading a proof through at a very low level, making sure that I understand how every line leads into the next. You should almost pretend that you're checking to make sure there isn't a mistake in going from one line to the next (sometimes this isn't pretend - everyone makes mistakes). Double check how each equation comes out of the one before it. This ensures that I understand all of the little details, but it does nothing for understanding the big picture. After you've read at the level of specific detail, re-read the proof and try to get a better idea of the larger structure. Why does the proof head off in this direction? How is the main point of the first paragraph used in the third paragraph? What was the original prover thinking when he wrote this proof? One almost never starts writing a proof from the details. Each time you reread a proof, you should see new connections between the ideas and get a fuller understanding of THE BIG PICTURE.

1.5 Write proofs - updated September 17, 2004

It is also essential to get practice. Some of this will come from homework problems. You should also try to do some of the proof-based exercises in the book that aren't assigned. Don't get frustrated if it doesn't all *click* immediately. Rome wasn't built in a day.

2 General strategies - updated September 17, 2004

Even though there is no magical proof algorithm, there are some basic skills that will help you approach a proof and some other things that should be kept in mind at all times. Let's run through some of these.

2.1 Categorize - updated September 2, 2004

This may be obvious, but it helps to decide at the outset roughly what kind of proof you're up against. Is the statement you're trying to prove a question about sets? Is it about geometry? Is it an analysis problem? Does it look like it has something to do with calculus? Each of these disciplines has its own set of tools and strategies, and while none of them live in a vacuum, if you're asked to prove something about a property of derivatives, your book on knot theory may not be the best first place to turn. Like everything else about proofs, it will take experience to begin to recognize the categories of proofs, but once you learn to make these kinds of classifications, it should help you to get started.

2.2 Context - updated September 17, 2004

This applies more to course-work than it does to proving things out in the real world, but make sure to use some common sense regarding assigned problems. If you're assigned a proof exercise that's in Section 1.5 of the book, chances are that the main idea for how to do that proof is somewhere in Section 1.5. This isn't 100% infallible advice, but if you're stuck, it should give you a place to start looking for ideas. Along the same line, if you're asked to prove a minor variation of something that's already been proven in the book or in class, don't look to reinvent the wheel. Ask yourself how what you're trying to prove is different from what you've already seen proven, then look at that existing proof and isolate the point where it stops applying to your problem. Then see how to modify the old proof so that it fits your new problem.

2.3 How to proceed - updated September 17, 2004

2.3.1 Know the words

If you're asked to prove that the real part of an analytic function is harmonic, you better making sure you know the meanings of11real part", "analytic", and "harmonic", and a review of what a "function" is might not hurt either. You can't prove something unless you know what it is you're proving and what all of the players do. That being said, just writing out all of the definitions is not a proof either. I've seen it done. Don't try it. I'm not *that* sloppy a grader.

2.3.2 Know what you're trying to prove

Know what your goal is, and don't lose focus. If you're asked to show that a function has a certain property, make sure you know what that property is and work back from there. Don't just start throwing words aimlessly down on paper. You have a target; aim for it.

2.3.3 Know what you're starting with

Any statement that you'll try to prove starts with a hypothesis. Read that hypothesis twice. Know what it says. You might also want to write down all of the things that the hypotheses obviously imply. Don't get too carried away, though; we've got a job to do.

2.3.4 Build a bridge

So now that you know where you're going to start and end, your goal is to build that bridge. You don't always have to build in one direction. Sometimes it makes more sense to start from the hypotheses and work towards the conclusion. Sometimes it makes more sense to work backwards, at least in your initial thinking. In the end, of course, you're going to have to take your reader from the hypotheses to the conclusion, but that doesn't mean you have to think about it in that order. Your main job now is to try to remember or invent the ideas that will connect that starting cluster of ideas (the hypotheses and their obvious consequences) to the ending cluster of ideas (the conclusion and the things that obviously imply it). Where do these bridge ideas come from? Well, look in the book. What theorems do you know that involve one or more of the ideas already in play? Can you jump from theorem to theorem? What can you ascertain simply by your own reasoning? There's a whole cobweb out there, and you have to find the series of strands that get you from the beginning to the end.

EXAMPLE (updated September 17,2004):

Theorem 1. *Suppose that $f : A \to B$ and $g : B \to C$ are function and that the composition $g \circ f$ is surjective (onto). Prove that g is surjective (onto).*

Proof.

1. WHAT ARE WE TRYING TO PROVE? We want to show that given any element $c \in C$, there is some element $b \in B$ such that $g(b) = c$. Why? *Because this is the definition of "surjective" as applied to g.*

2. WHAT DO WE KNOW? We know that $g \circ f$ is surjective. This tells us that for any $c \in C$, there is an $a \in A$ such that $g(f(a)) = c$.

3. BUILD THE BRIDGE. Given that there is some $a \in A$ such that $g(f(a)) = c$, we want to find some $b \in B$ such that $g(b) = c$. But $f(a) \in B$, and $g(f(a)) = c$, so we can take $b = f(a)$ in order to find the b we want. Since our choice of c was arbitrary, we

see that we can apply this process to find such a b for any c. So we have shown that for any c there is some b such that $g(b) = c$.

Note: I was being wordy in the preceding discussion to give you an idea of the thought process. When it comes to actually writing the proof, I would write something like this: Let c be an element of C. Since $g \circ f$ is surjective, there exists an element $a \in A$ such that $g(f(a)) = c$. Thus there is an element $b \in B$, namely $f(a)$, such that $g(b) = c$. So we have shown that for any $c \in C$ there exists a $b \in B$ such that $g(b) = c$, i.e. g is surjective. $\square$

2.4 DRAW PICTURES - updated September 17, 2004

This topic is so important it gets its own subsection. I will say this every day. DRAW PICTURES. Our geometric intuition is often a lot stronger than our pure abstract reasoning intuition. A picture is NOT A PROOF, but it can be damn helpful in getting the ideas rolling towards a correct proof. You will often find that all of the key ideas in a proof come from thinking about the pictures, and then the only thing left in writing the proof is translating the pictures into precise mathematical language. This last part may sound hard, but I guarantee you that it's a lot easier than trying to think in the precise mathematical language from beginning to end. So draw pictures. Even of the things that don't seem like they should warrant pictures. And if it's too complicated to draw, close your eyes and try to picture it in your mind. I promise this will help. Even for the five-dimensional things.

3 Writing your proofs - Updated September 17, 2004

Proofs should be written as clearly and concisely as possible. Your ultimate goal should be to convince your reader (and perhaps yourself) that something is true. Make sure to muster your arguments in advance and to write them down in a clear, logical format. If you use a particular definition, it may help to state that definition. If you invoke a previous theorem from the course, you should be clear about which one and what it states. When doing so, it is preferable if you state more than just the number of the theorem from the book. While it's acceptable to write "the proof now follows from Theorem 4.3," it's much better to write, "the proof now follows from Theorem 4.3, which says that ..."

You should try to avoid writing down facts that are not relevant to the proof. A STANDARD FRESHMAN MISTAKE is to bury the proof in definitions, attempting to show the grader that you know what the words all mean. While it is important to know your definitions, you should only invoke the ones that are relevant, and definitions alone rarely suffice. You must make all of the connections between ideas for your reader. IT IS UNACCEPTABLE TO ATTEMPT TO HIDE BEHIND AN AVALANCHE OF STATEMENTS THAT YOU DO NOT TIE TOGETHER. Eschew obfuscation.

4 How to show that something is not true - Added October 10, 2005

Before we really dig in to proof techniques, it's worth talking about how to show that something is *not* true, which in mathematics and be just as important as showing that something is true. Remember that when you prove a mathematical statement, you argue that the statement *always* holds. So what's the opposite of "always"? It's "not always" (don't confuse math with common English where the answer *might* be "never"). So to show that a statement is not true, you only have to show that it's not always true. In other words, you have to show there are cases where it doesn't hold. Or, most simply, you have to come up with a single counterexample, a single case where the statement doesn hold.

This is all best illustrated with a very simple example. Suppose you're curious about whether the following statement is true: $(x+y)^2 = x^2+y^2$. Of course we know that this isn't true (at least for real numbers!) since the correct formula is $(x+y)^2 = x^2+2xy+y^2$. But suppose you didn't know that and your teacher asked you to show that $(x+y)^2 = x^2+y^2$ is not correct as a formula for real numbers. All you have to say is the following: "Suppose $x = 1$ and $y = 1$. Then $(x+y)^2 = (1+1)^2 = 4$, but $x^2+y^2 = 1^2+1^2 = 2$. So $(x+y)^2 = x^2+y^2$ is not true."

Note that this *does not* mean that $(x+y)^2 = x^2+y^2$ is *never* true. For example this equation *will* hold if, say, $x = y = 0$. But that's only a special case. In general, the equation is not true. So in a general proof about properties of real number, you can't use it.

So in a certain sense, it's "easier" to show that something isn't true than that something is true. To show that something is true, you have to come up with a proof. To show that something isn't true, you just have to find *one* counterexample. Of course in actual practice finding such a counterexample is not always easy, especially if the statement actually turns out to be true!

5 Some standard proof formats

5.1 If and only if - Updated September 17, 2004

These are the proofs that ask you to show that Statement A is true if and only if Statement B is true. Of course proving such a thing could fall into any of the following categories, none, or more than one, but I include these in their own section to remind you that in an "if and only if" proof YOU HAVE TO DO TWO THINGS. You have to show that Statement A implies Statement B AND that Statement B implies Statement A. It is often the case that one of these things is much simpler to do than the other, but *DO NOT FORGET TO DO BOTH.*

5.2 Checking definitions - Updated September 5, 2004

This happens rarely, but sometimes a proof is just a matter of checking that something satisfies a property straight from the definition of that property.

Example: Prove that every integer is a rational number. Proof: By definition, a number is a rational number if and only if it is a quotient of integers (a fraction). Every integer is the quotient of itself with 1 (if $z \in \mathbb{Z}$, then $z = z/1$). Thus every integer is a rational number.

See also Example 1.

The general approach to these proofs is often straightforward (I have to show that every x satisfies property B), though in practice the details may become tricky. For example, if you are asked to show that a function is continuous, in principle you simply have to check that the definition for continuity is satisfied by the function. In reality, however, this may be difficult to do directly, and the more useful approach may be to apply other theorems about continuity that have already been developed (e.g. that sums and products of continuous functions are continuous).

5.3 Checking cases - Added September 17, 2004

Sometimes a proof is just a matter of checking a number of cases (although actually checking each case might be difficult). Consider the following example:

Theorem 2. *For any integer $z \in \mathbb{Z}$, z^2 is congruent to 0 or 1 mod 4 (i.e. its remainder is 0 or 1 upon division by 4).*

Proof. Any integer z is congruent to either 0, 1, 2, or 3 mod 4, so it suffices to check these cases:

1. $z \equiv 0 \mod 4$. In this case, z is a multiple of 4, and thus so is its square. So $z^2 \equiv 0 \mod 4$.

2. $z \equiv 1 \mod 4$. In this case, $z = 4n+1$ for some n, so $z^2 = 16n^2+8n+1 = 4(4n^2+2n)+1$. Thus $z^2 \equiv 1 \mod 4$.

3. $z \equiv 2 \mod 4$. In this case, $z = 4n + 2$, so $z^2 = 16n^2 + 16n + 4$, which is a multiple of 4. So $z^2 \equiv 0 \mod 4$.

4. $z \equiv 3 \mod 4$. In this case $z = 4n + 3$, so $z^2 = 16n^2 + 24n + 9 = 4(4n^2 + 6n + 2) + 1$. So $z^2 \equiv 1 \mod 4$.

$\square$

Note that it is important not only to check all of the cases, but to argue why all possible cases have been considered!

5.4 Proofs by contradiction - Updated September 17, 2004

Some people dislike this method as it is somewhat indirect. In a proof by contradiction, we assume that the statement we are trying to prove *is not true* and then argue until we get a logical contradiction. This demonstrates that it is impossible for the statement to be untrue, so it must be true!

Here is a famous example:

Theorem. The square root of two, $\sqrt{2}$, is irrational. Proof. Assume that this statement is false, i.e. that $\sqrt{2}$ *is* rational. That would imply that $\sqrt{2} = p/q$ for some integers p and q, and we are free to assume that the fraction is reduced (p and q have no common divisors). If this is true, then clearly $p^2 = 2q^2$. So p^2 is an even number. But the only way for p^2 to be even is if p is even (since the product of two odd number is odd). So $p = 2t$ for some other integer t, and $2q^2 = p^2 = 4t^2$. But this implies that $q^2 = 2t^2$, and by making the same argument again, we see that q must also be even. But now we have arrived at a contradiction, since p and q were to have no common divisors, which is impossible if they are both even. Since our logical was impeccable, the only problem must be that we assumed $\sqrt{2}$ to be rational. Hence it must not be.

CAUTION: When doing proofs by contradiction, make sure that the contradiction comes only from your assumption that the theorem is false, NOT from some mistake that you made along the way!

5.5 Proofs by the contrapositive - updated September 17, 2004

These are somewhat similar to proofs by contradiction though of a different flavor. Suppose you have to show that Statement A implies Statement B. In other words, you want to show that if Statement A is true, then Statement B is also true. The contrapositive statement to "A implies B" is "(not B) implies (not A)". In other words, to prove the contrapositive, you show that if Statement B is false, then so is Statement A.

An implication (A implies B) is true if and only if its contrapositive (not B implies not A) is true. This is not hard to see: suppose the contrapositive is true, and that A is true. Well then, if B were false, the contrapositive would say that A is false. But this is not the case so B must be true. Hence if the contrapositive holds, so does the initial statement that A implies B. I leave it to the reader to show that if the original implication holds then so does its contrapositive.

Example: If f is a differentiable function and a is a local minimum for f then $f'(a) = 0$. One can prove this directly using the definition of the derivative and the definition of a local minimum. However, one also could proceeded by showing that at any point b such that $f'(b) \neq 0$, b cannot be a minimum. In the end, this approach for this problem would also come down to the definition of the derivative, but in general trying the contrapositive might lead to new approaches to problems.

5.6 Proofs by induction - Updated September 5, 2004

Proof by induction is used when you want to simultaneously prove an entire sequence of statements that are indexed by natural numbers. For example, suppose you want to show that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for any $n \geq 1$. This is really an entire sequence of statements, one for each n. Abstractly, we sometimes speak of trying to prove the statements $I(n)$ for some range of number n. In the example, $I(n)$ is the given statement ($\sum_{i=1}^{n} = \frac{n(n+1)}{2}$), and we want to prove $I(n)$ for all $n \in \mathbb{N}$.

The basic plan for induction is the following

1. Prove the base case or cases: In other words, show that $I(n)$ is true for the lowest value of n. Sometimes it might be necessary to prove $I(n)$ for the first few values of n, depending on the specific problem.

2. Assume an induction hypothesis: This means that we now *assume* that the statement is true for some fixed but arbitrary value of n, say $N-1$. This hardly seems allowable, but I'll explain below. Note that when I say arbitrary, I mean arbitrary. We don't assume here that the statements are true for n up to 5; we assume they are true up to $N-1$. Just like that. N could be anything.

3. The induction step: Show that the induction hypothesis implies the next case. In other words, if we assume by induction hypothesis that $I(n)$ is true for $N-1$, then in this step we show that $I(N)$ is true, *under that assumption*.

4. That's it. We're done. If you've completed the above process, you've shown that $I(n)$ is true for all n.

So what's going on here, and why can we make that induction hypothesis? The easiest way to explain is to work backwards. I claim that if you've followed the above steps, then $I(n)$ is true for all n. So let's pick a case, say $I(7)$. Why is $I(7)$ true. Well, we showed in Step 3 that $I(7)$ will be true if $I(6)$ is true. Similarly, *that same step*, show that $I(6)$ is true if $I(5)$. And so on. Eventually we get down to, say, $I(2)$ will be true if $I(1)$ was true. But if $I(1)$ was the base case that we treated in the first step, we know that it's true! So the $I(2)$ is true, and $I(3)$ is true...and $I(7)$ is true.

Let's show how this works for our example:

1. Prove the base cases or cases: In our example $I(1)$ says that $\sum_{k=1}^{1} k = \frac{1(1+1)}{2}$. But $\sum_{k=1}^{1} k = 1$, and $\frac{1(1+1)}{2} = 1$. So this is true. We have established the base case.

2. Assume an induction hypothesis: Here we just say that we assume that $\sum_{k=1}^{N-1} k = \frac{(N-1)(N-1+1)}{2} = \frac{(N-1)N}{2}$ is true. That's it.

3. The induction step: Okay, now show $I(N)$, using our assumption about $I(N-1)$: We

have

$$\begin{aligned}\sum_{k=1}^{N} k &= (\sum_{k=1}^{N-1} k) + N \\ &= \frac{(N-1)N}{2} + N \\ &= \frac{N^2 - N}{2} + \frac{2N}{2} \\ &= \frac{N^2 + N}{2} \\ &= \frac{N(N+1)}{2}\end{aligned}$$

Note that from the first line to the second line, we used the induction hypothesis to replace $\sum_{k=1}^{N-1} k$ with $\frac{(N-1)N}{2}$.

4. That's it. We're done. The theorem now follows by induction.

One also sometimes uses "generalized induction". In this case, instead of just assuming $I(N-1)$ is true in the induction step, we instead assume that $I(n)$ is true for all $n \leq N-1$ and use this to proof that $I(N)$ is true. This also works, essentially for the same reasons.

NOTE: Don't be too fixed about notation. For example it is often notationally more convenient in the induction step to assume $I(N)$ and use it to show $I(N+1)$. The notation is slightly different, but clearly the idea is the same. Similarly, it may also be necessary to prove a few base cases in order to get the induction going.

6 Specific Tips

6.1 Proving that two sets are equal - updated September 17, 2004

If you are asked to show that $A = B$, where A and B are two sets, the simplest approach is usually to show that $A \subset B$ and $B \subset A$. In other words, show that every element of A is also in B and that every element in B is also in A. This implies that the elements of A and B are the same, i.e. that the sets are equal. Don't forget to do both parts!

6.2 Delta-epsilon arguments - Added September 17, 2004

These are perhaps the most challenging arguments for students to become acquainted with. For one thing, there are at least *two* quantifiers: a "for all" and a "there exists" (see Section 7.7 below for more about quantifiers). Let's look again at the definition of continuity of a function f at a:

Definition 1. The function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at a if *for all* $\epsilon > 0$ *there exists* a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

Let's just look at the first part of this: "*for all* $\epsilon > 0$ *there exists* a $\delta > 0$ such that..." This says that if we choose *any* $\epsilon > 0$ then there must be *some* $\delta > 0$ that makes the following statement true. Note that δ MAY DEPEND UPON ϵ, but the ϵ is free to be arbitrary. In this specific definition, the concept is as follows: we want to know that if x is close to a then $f(x)$ is close to $f(a)$. The above definition provides a rigorous mathematical definition of what we mean by close. Sometimes this is called a challenge procedure: if you challenge me to make $f(x)$ "ϵ-close" to $f(a)$ then I tell you that it can be done so long as we stick to points that are "δ-close" to a.

This is just a sample delta-epsilon definition, and of course there's nothing to prove about a definition - it is what it is. The actual delta-epsilon arguments you will have to make come about in trying to apply the definitions. So let's look at the following theorem, whose proof is, in principle, just a matter of checking the definition, though of course the actual checking itself requires a little ingenuity.

Theorem 3. *Suppose that f and g are both functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ and that both are continuous at a. Then the function $f + g$ is continuous at a.*

Proof. Well, we need to check that $f + g$ satisfies the definition of continuity at a: we must show that for any $\epsilon > 0$ there is some $\delta > 0$ such that $|(f + g)(x) - (f + g)(a)| < \epsilon$ whenever $|x - a| < \delta$. So, let us pick an arbitrary fixed $\epsilon > 0$. If we can find a δ that works for this ϵ, we will be done: even though ϵ is fixed for the moment, it was fixed arbitrarily, so whatever argument we give would work for any ϵ.

Okay, so now that ϵ is fixed, how do we go from here? Well, we want to show something about $|(f + g)(x) - (f + g)(a)| = |f(x) + g(x) - f(a) - g(a)|$. Since we already know something about the continuity of f and g at a, we expect that to come in, so let's rewrite this as $|f(x) - f(a) + g(x) - g(a)|$, which looks a little more like it has to do with the continuity formulas for f and g. In fact, if we want to use those formulas, we should try to compare this to something involving $|f(x) - f(a)|$ and $|g(x) - g(a)|$. For this we can use the triangle inequality to write $|f(x) + g(x) - f(a) - g(a)| \leq |f(x) - f(a)| + |g(x) - g(a)|$. This is looking good, since continuity of f and g should allow us to conclude that the right hand side of this is small. In fact, since we want to make $|f(x) + g(x) - f(a) - g(a)|$ less than ϵ, it will be good enough to show that each of $|f(x) - f(a)|$ and $|g(x) - g(a)|$ is less than $\epsilon/2$ (right?).

Now using the definition of continuity at a applied to each of f and g, we know that there exists some $\delta_f > 0$ such that $|f(x) - f(a)| < \epsilon/2$ and $\delta_g > 0$ such that $|g(x) - g(a)| < \epsilon/2$. (Note: you may be thinking, "Hey! The definition of continuity for f and g tells us something about ϵ, not $\epsilon/2$", but remember: the ϵ in each definition is arbitrary; it just represents any number > 0. We know from the definitions that there are appropriate δs for *any* number > 0, including $\epsilon/2$ for the current fixed ϵ, so the definition applies and allows us to conclude there exist the δs we want.)

Okay, so now we have two deltas, δ_f and δ_g, each one giving us a radius around a such that the corresponding function (f or g) do what we want near a. But in order to get $|(f + g)(x) - (f + g)(a)| < \epsilon$, we need BOTH of these to hold. So we take $\delta = \min(\delta_f, \delta_g)$, the minimum of the two. So if $|x - a| < \delta$, then $|x - a|$ is smaller than both δ_f and δ_g. So if

$|x-a| < \delta$, then $|f(x)-f(a)| < \epsilon/2$ and $|g(x)-g(a)| < \epsilon/2$. So by our triangule inequality argument, $|(f+g)(x)-(f+g)(a)| < \epsilon$. Thus we have found a δ that works for this ϵ, and we are done! $\square$

There is another truth that is often handy when dealing with delta-epsilon proofs (this is proven in Hubbard and Hubbard's *Vector Calculus, Linear Algebra, and Differential Forms*, and probably in many other analysis books): In choosing δ, it actually suffices to find a δ that yields any function of ϵ that goes to 0 as ϵ goes to 0. To explain what this means, consider again the previous example. We chose δ_f and δ_g so that $|f(x) - f(a)|$ and $|g(x) - g(a)|$ would be less than $\epsilon/2$. Suppose, instead, that we had chosen δ_f and δ_g only so that these distances would be $< \epsilon$. Then if we define $\delta = \min(\delta_f, \delta_g)$ again, we will only be able to conclude that for $|x-a| < \delta$ then $|(f+g)(x)-(f+g)(a)| < 2\epsilon$. Our first thought is "Oh no! It didn't work!" But this nice theorem says that it actually did: since $2\epsilon \to 0$ as $\epsilon \to 0$, THIS IS SUFFICIENT. This nice fact often comes in handy, as it is often much trickier to get the calculations to come out exactly to ϵ. The theorem tells us that we could make this happen with some extra work, but we don't need to.

7 The Basic Structures of Mathematics

7.1 Statements: the language of mathematics - Added August 21, 2005

You may at some point in your mathematical studies have come across what is called "propositional logic" or "symbolic logic". This often involves a lot of p's and q's, symbols like $p \Rightarrow q \wedge r$, and "truth tables". Familiarity with the advanced intricacies of this language is not really necessary for writing proofs, and mathematicians rarely reduce their proofs to such tight formalisms. Proofs are not written by constructing truth tables and making sure that the T's wind up in the right places. Nonetheless, some basic understanding of the rudiments of propositional logic is essential to being able to read and write proofs and to understanding mathematics. We will tackle the most important points here.

7.2 Statements

A statement is simply that: a statement. Here are some examples:

- Lemons are yellow.
- Pears are silver.
- $2+2=4$.
- $2+2=5$.
- If f is a continuous function and the domain of f is compact, then f is uniformly continous.

So, a statement is simply a declarative sentence. It may be true or false. Needless to say a math class will be most concerned with statements like the last three.

To keep things simple, one sometimes uses symbols to stand for whole statements. We might declare that the statement "Lemons are yellow" will be represented by the symbol P and the statement "Pears are silver" will be represented by the letter Q. We can save some space this way: instead of writing "If lemons are yellow, then pears are silver", we can write "If P then Q". In practice, this is rarely done in math proofs. The symbols are most convenient in meta-situations like the current one in which we want to talk about general statements without really specifying them. So below we'll say things like "Suppose P is some statement", and then go from there.

Note also that statements can be built up from simpler statements. So P, Q, and "If P, then Q" are all statements, and if we wished, we could give "If P, then Q" its own name, say R.

7.3 Not! -Added 8/21/05

Every statement also has an opposite. In math, the opposite of P is usually called "not P" and denoted $-P$. The opposite of $2+2=4$ is $2+2 \neq 4$. The opposite of "Lemons are yellow" is "Lemons are not yellow". The opposite of a true statement is false, and the opposite of a false statement is true.

Everybody got that?

7.4 Implications - Added 8/21/05

Math is really all about implications - statements of the form "If P, then Q", where P and Q are themselves statements (see how handy that P and Q can be?). For example, "If lemons are yellow, then pears are silver". You may have seen truth tables about implications - when is the implication true or false based on whether P and Q are themselves true or false. Forget all that for now. The statement "If P, then Q" means that if P is true then Q is true. Your job is usually either to believe that, if it comes to you as a theorem, or to prove it.

Here are some simple mathematical examples:

- If $x = 2$, then $x^2 = 4$.
- If f is differentiable, then f is continuous.
- If L_1 and L_2 are two nonparallel lines in the plane, then they intersect.

Of course implications can also be false, e.g. "If $x = 2$, then $x^2 = 5$."

Most proofs require you to demonstrate that an implication is true. You'll be asked, "Prove: If P, then Q," or "Prove: If f is differentiable, then f is continuous." In these cases, your assignment is to assume that P is true and then give an argument showing that Q must then also be true, e.g. to assume that f is some differentiable function and to prove that f is continuous.

Note: "P implies Q" means *exactly the same thing* as "if P, then Q."

7.5 Converses, inverses, and contrapositive: mutilations of an implication - 8/21/05

As soon as an implication is written down, there's trouble brewing, because it's human nature to start manipulation statements and sentences, and this can lead to big trouble in mathematics if it's not done carefully. In short, there are three basic ways to mangle the implication "If P, then Q":

- The converse: "If Q, then P."
- The inverse: "If not P, then not Q." ("If -P, then -Q.")
- The contrapositive: "If not Q, then not P." ("If -Q, then -P").

Let's have a look at each of these, both with a mathematical example and a nonmathematical example. For our mathematical example, we'll keep using "If f is differentiable, then f is continous." For our nonmathematical example, let's use "If Bob is a platypus, then Bob has a bill." Certainly, both of these are true implications.

The converse: The converse statements would read "If Bob has a bill, then Bob is a platypus," and "if f is continuous, then f is differentiable." Notice that these are no longer true. Bob could be a duck, and the function $|x|$ is continous but not differentiable. It is a common beginner's mistake when asked to prove an implication to try to do this by proving the converse. BE CAREFUL NOT TO MAKE THAT MISTAKE.

Also, it is not always the case that if an implication is true then its converse is false. Can you come up with an example where P implies Q *and* Q implies P?

The inverse: The inverse statements would read "If Bob is not a platypus, then Bob does not have a Bill," and "if f is not differentiable, then f is not continous." Once again, these are no longer true. Bob could be a duck, and the function $|x|$ is not differentiable but it is continous. It is a common beginner's mistake when asked to prove an implication to try to do this by proving the inverse. BE CAREFUL NOT TO MAKE THAT MISTAKE.

Also, it is not always the case that if an implication is true then its inverse is false. Can you come up with an example where P implies Q *and* not P implies not Q?

The contrapositive: The contrapostive to "If Bob is a platypus, then Bob has a bill." is "If Bob does not have a bill, then Bob is not a platypus." The contrapositive to "If f is differentiable, then f is continous" is "If f is not continuous, then f is not differentiable." You might notice that these statements don't seem quite so badly garbled. In fact, they make sense: If Bob doesn't have a Bill, then how could he be a playpus since platypodes all have bills? This is because a statement and its contrapostive are *equivalent*. That is if one is true then the other is true, and if one is false then the other is false. Let's look at why this is true.

Suppose we have a statement "If P, then Q" (If Bob is a platypus, then Bob has a Bill). This says the if P is true, then so is Q. Again, the contrapositive is "If not Q, then not P" (If Bob does not have a Bill, then Bob is not a platypus). Let's suppose "If P, then Q" is true. So if P is true, then Q is true. Okay, now let's look at the contrapositive. It starts "If not Q", i.e. we assume Q is not true. But if Q is not true, then P can not be true since we know that if P is true then Q is true because we assumed the original statement. Therefore, "If P, then Q" *implies* "If not Q, then not P" ("If Bob is a platypus, then Bob has a bill" implies that "if Bob does not have a bill, then Bob is not a platypus"). Got that?

Similarly, the contrapositive implies the original statement. If we know "If not Q, then not P" is true, and then we must have "if P, then Q": If we are told "if Bob does not have a bill, then Bob is not a platypus," then it follows that "If Bob is a platypus, then Bob has a bill".

The equivalence between a statement and its contrapositive is essential in mathematicas. In fact, many statements are proven by instead showing the contrapositive. Then if the contrapositive is shown to be true, we know the original statement is. This may seem like a silly way to proceed since both things are equivalent, but sometimes looking at a statement through its contrapositive hopes to clarify matters in your mind.

See Section 5.5 for more on proofs using the contrapositive.

Summary: The important thing to take away from this section is that a statement is the same as its contrapositive, but NOT necessarily the same as its converse or inverse. You can prove a statement by proving its contrapositive. But proving a converse or an inverse is NOT sufficient. BE CAREFUL NOT TO MAKE THAT ERROR.

Bonus exercise: Can you show that the inverse of a statement is equivalent to the converse of that statement? In fact, they are the contrapostives of each other!

7.6 AND and OR

The word "and" means pretty much the same thing in math as it does in real life. If I say, "Theorem: If f is continuous AND its domain is a closed iterval, then f is bounded," then both parts of the hypothesis must be true for the conclusion to follow: If f is continuous and, at the same time, its domain is a closed interval, then you can conclude that f is bounded. But if only one of the hypotheses is satisfied, say just that f is continuous but NOT that its domain is a closed interval, then you cannot invoke the theorem to conclude that f is bounded.

Here's an English version: "If Bob comes to the party and Fred comes to the party, then we're going to have a lot of fun." This sentence makes no promises about what will happen if only one of Bob and Fred or if neither of them comes to the party. It only tells us about what happens if both conditions are met.

On the other hand, "or" means something slightly different in math than it does in real life. In math, "P or Q" really means "P or Q or both". "Or" does not imply mutual exclusivity. For example, mathematically "If Bob comes to the party or Fred comes to the party, then we're going to have fun," means that we'll have fun if Bob comes, if Fred comes, or if both Bob and Fred come. A mathematical statement might read, "If $x = 0$ or $y = 0$, there is a solution". In this case, there is a solution in any of these cases: 1. $x = 0$ and $y \neq 0$, 2. $x \neq 0$ and $y = 0$, and 3. $x = 0$ and $y = 0$.

7.7 Quantifiers

Many mathematical statements contain one or more quantifier; these are phrases involving "there exists" or "for all". By themselves, these mean pretty much what you'd expect, but they can be tricky when they occur in combinations. So let's start simple.

7.7.1 "There exists" and "For all"

A typical "there exists" phrase would be something like "There exists a solution to the equation $x^2 = 2$." "There exists" is denoted symbolically by the symbol $\exists$, so this phrase could be written more symbolically as "$\exists x, x^2 = 4''$ or "$\exists x \mid x^2 = 4$." The comma or $|$ is often read out loud as "such that", so that these symbolic sentences would be read as "There exists x such that $x^2 = 4$." There's also room for a little more specificity, such as "$\exists x \in \mathbb{Z} \mid x^2 = 4$", i.e. "there exists an $x \in \mathbb{Z}$ such that $x^2 = 4$," or "there exists an integer such that $x^2 = 4$." The meaning of this phrase is exactly what you think it should be. We're making no claim deeper than that there is some solution to the equation.

"There exists"'s close friend and rival is "for all". "For all" shows up in phrases like "All positive real numbers are greater than -1." In symbols, "for all" looks like $\forall$. So more formally, we would write "$\forall x > 0, x > -1$," i.e. "For all x that are greater than 0, x is greater than -1." Notice that where "there exists" just stipulates that there is one thing satisfying a property, "for all" says that *everything* satisfies the property, at least everything within the stated restrictions.

Here are some more examples:

- $\exists x \in \mathbb{Z} \mid 0 < x < 5$, i.e. "There exists an integer that is greater than 0 and less than 5."
- $\exists x \in \mathbb{Z} \mid 0 < x < 1$, i.e. "There exists an integer that is greater than 0 and less than 1." (remember, statements don't always have to be true).
- $\forall x \in \mathbb{R}, x^2 \geq 0$, i.e. "For all real numbers x, the square of x is nonnegative."
- $\forall x \in \mathbb{Z}, 2x \in \mathbb{Z}$, i.e. "The double of every integer is also an integer."

7.7.2 Mixed quantifiers

Things get more complicated when there is more than one quantifier in a statement. For example "$\forall x > 0, \exists y \mid y^2 = x''$. This statement is read "For all $x > 0$, there exists a y such that $y^2 = x$." In other words, "all positive numbers have a square root." But what if instead I wrote "$\exists x > 0 \mid \forall y, y^2 = x$." This statement says, "There is an $x > 0$ such that for all y, $y^2 = x$", or "There is a posivite number that is the square root of every number," which certainly isn't true. ORDER OF QUANTIFIERS MATTERS.

Let's try some English examples. Try to understand the difference in the following phrases which are all made up of the same components but in different orders. At the beginning, we'll put the symbols $\exists$ and $\forall$ in the order they occur in the phrase, and at the end we'll put a more English friendly interpretation of the mathematically stilted phrase:

- $\forall$, $\exists$: For all kangaroos, there exists a hunter. (Every kanagaroo has a hunter who hunts it.)

- $\exists$, $\forall$: There exists a kangaroo for all hunters. (There's one poor kangaroo being hunted by everyone.)

- $\forall$, $\exists$: For all hunters, there exists a kangaroo. (Each hunter has some kangaroo he gets to hunt.)

- $\exists$, $\forall$: There exists a hunter for all kangaroos. (There is a hunter out there who is hunting all of the kangaroos). *Notice that the math interpretation of this sentence is slightly different from what one would expect in ordinary English.*

Here's everyone's least favorite example of a statement with two quantifiers: "f is continuous at a" means: for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x-a| < \delta$ implies $|f(x)-f(a)| < \epsilon$. A good exercise is to think about that phrase and how it's different from "there exists $\epsilon > 0$ such that for all $\delta > 0$, $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$."