Intersection homology with general perversities

Greg Friedman
Texas Christian University

February 15, 2010

To Bruce Williams, for his 60th birthday

Abstract

We study intersection homology with general perversities that assign integers to stratum components with none of the classical constraints of Goresky and MacPherson. We extend Goresky and MacPherson's axiomatic treatment of Deligne sheaves, and use these to obtain Poincaré and Lefschetz duality results for these general perversities. We also produce versions of both the sheaf-theoretic and the piecewise linear chain-theoretic intersection pairings that carry no restrictions on the input perversities.

2000 Mathematics Subject Classification: Primary: 55N33, 57N80; Secondary: 55N45, 55N30, 57P10

Keywords: intersection homology, perversity, intersection product, Poincaré duality, pseudomanifold, Deligne sheaf

Contents

1 Introduction

Intersection homology theory is an important tool for the topological study of stratified spaces, which include algebraic and analytic varieties and certain quotients of manifolds by group actions. The original motivation for its development was to extend an appropriate version of Poincaré duality to such spaces, and other related properties have followed, including versions of the Kähler package for singular varieties. The theory breaks into families indexed by a parameter, the *perversity*, which is often limited to a fairly strict range of possibilities. However, as intersection homology with more general perversities has become an increasingly indispensable tool, it is imperative to recast as many as possible of the foundational properties of intersection homology theory in this light. We here construct a version of the Deligne sheaf machinery in this constext, as well as revisiting the geometric PL chain intersection pairing of Goresky and MacPherson.

More precisely, recall that a perversity parameter for intersection homology is a function that assigns a number to each stratum (or stratum component) of a stratified pseudomanifold X. In the original work of Goresky and MacPherson [28], these perversities were assumed to satisfy very restrictive conditions: A *Goresky-MacPherson perversity* $\bar{p}$ assigns the same number to all stratum components of codimension k, so it can be thought of as a function $\bar{p} : \mathbb{Z}^{\geq 2} \to \mathbb{Z}$, and it must satisfy

$$\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1 \text{ and } \bar{p}(2) = 0.$$

These conditions were necessary in order to prove some of the earliest important properties of intersection homology groups $I^{\bar{p}}H_*(X)$, including the simultaneous possession of Poincaré duality and topological invariance (independence of the stratification of X).

A number of authors have considered variants of intersection homology that allow more general notions of perversity, including Beilinson, Bernstein, and Deligne [5]; MacPherson [40]; King [38]; Cappell and Shaneson [11]; Habegger and Saper [32]; the author [20, 22, 24]; Saralegi [48]; and Hausel, Mazzeo, and Hunsicker [34, 36, 35]. In many of these works, perversities are still required to satisfy at least some of the prior conditions, though completely arbitrary perversities appear as far back as 1982 in the work of Beilinson, Bernstein, and Deligne on perverse sheaves, and they occur more recently in work of the author [24] and Saralegi [48]. These general perversities are functions

$$\bar{p} : \{\text{Connected components of singular strata of } X\} \to \mathbb{Z}$$

without any restrictions whatsoever. These papers also enlarge the class of pseudomanifolds considered by allowing strata of codimension one. While the topological invariance of intersection homology is lost for these more general perversities, all other standard properties of intersection homology remain true, including Poincaré duality, at least with the proper choice of intersection homology theory.

While general perversities appeared within the realm of perverse sheaves from the beginning, let us describe in a bit more detail some of the recent applications of general perversities within intersection homology proper:

In [48], Saralegi proved a de Rham theorem for these general perversities, i.e. that integration induces a linear duality isomorphism between the cohomology of $\bar{p}$-perverse liftable intersection differential forms and chain-theoretic perversity $\bar{t}-\bar{p}$ "relative" intersection homology with real coefficients (see [48] for precise details). Non-traditional perversities also appear in an analytic setting in the works of Hausel, Hunsicker, and Mazzeo [34, 36, 35], in which they demonstrate that groups of L^2 harmonic forms on a manifold with fibered boundary can be identified with cohomology spaces associated to intersection cohomology groups of varying perversities for a canonical compactification of the manifold.

In [24], we used general perversity intersection homology as a critical tool in proving a Künneth theorem for intersection homology. We showed that for any two given perversities $\bar{p}$, $\bar{q}$ on two stratified pseudomanifolds X, Y, there is a family of general perversities (generically denoted Q) on the product pseudomanifold $X \times Y$ such that the intersection chain complex $I^QC_*(X \times Y)$ is quasi-isomorphic to $I^{\bar{p}}C_*(X) \otimes I^{\bar{q}}C_*(Y)$. Even when $\bar{p}$ and $\bar{q}$ are Goresky-MacPherson perversities, Q in general will not be, and so general perversities play a critical role even in the effort to study more classical forms of intersection homology. Without this tool, Cohen, Goresky, and Ji [16] were able to obtain only a much more limited intersection homology Künneth theorem. Together with James McClure and Scott Wilson, the author is currently working on a variety of applications of this Künneth result, including an effort to understand an intersection *co*homology theory founded on cochains and cup products. Such an effort was not possible previously because the usual front face/back face approach to cup products (see [44, Section 48]) is not well-behaved on intersection chains. However, the author's general perversity Künneth theorem provides a way to define the cup product via the Hom dual of the composition

$$I^{\bar{r}}H^c_*(X) \to I^QH^c_*(X \times X) \xrightarrow{\cong} I^{\bar{p}}H^c_*(X) \otimes I^{\bar{q}}H^c_*(X),$$

where the first map is induced by the geometric diagonal inclusion and the righthand quasi-isomorphism is the Künneth theorem of [24]. See [24, 27] for more details.

Thus general perversities have become increasingly useful, and, as such, it is desirable to have a consistent theory of general perversity intersection homology, unified across the various existing flavors of intersection homology theory. Recall that Goresky and MacPherson originally defined intersection homology using simplicial chains on piecewise linear spaces, but in [29] they formulated an equivalent purely sheaf-theoretic approach. Sheaf theory has the advantages of applying on more general spaces and of possessing a powerful toolbox, though perhaps the disadvantage of being further removed from geometric intuition. The geometric chain approach was extended by King [38] to include singular simplices, as well as non-Goresky-MacPherson perversities. However, it was shown in [20] that, when considering King's perversities, singular chains are not quite the right theory to match with the tools coming from sheaf theory. Thus a modification, the stratified coefficient system, was introduced in [20], and this is the version of general perversity singular intersection homology pursued in [20] and [24]; this approach turns out to be equivalent to the "relative" intersection homology introduced by Saralegi in [48].

Our first goal here is to demonstrate how to generalizes the Deligne sheaf construction of Goresky-MacPherson [29] in order to provide hypercohomology groups that agree with the generalized intersection homology of the author and Saralegi. We also generalize the axiomatic characterization that made the Deligne sheaf so useful in classical intersection homology and use this to prove duality and pairing results. We further demonstrate that this generalized Deligne construction is equivalent to a certain intermediate extension in the language of perverse sheaves.

At the other end of the spectrum, we expand also the geometric ramifications of general perversities by extending some of the earliest Goresky-MacPherson results in the realm of piecewise-linear chain complexes by extending the PL intersection pairing to arbitrary pairs of perversities.

While some of our sheaf theoretic results may be implicit (or, in some cases, explicit) in developments from the category perverse sheaves, such an approach involves employment of considerably more abstract categorical formalisms. We hope that our more geometrically explicit constructions will be more approachable and better adapted to future applications. We also hope that, ultimately, having several approaches to general perversity intersection homology, coming with trade-offs between their varying levels of geometrical explicitness and their abstract generality, will provide scientific utility.

We proceed as follows.

Section 2 contains background and notation.

In Section 3, we observe that the classical Deligne sheaf construction is insufficient to provide a complete sheaf-theoretic approach to intersection homology with general perversities. To remedy this problem, we introduce a generalization of the sheaf truncation functor to replace the classical sheaf truncation functor in the Deligne construction. We then show that general perversity intersection homology possesses an axiomatic characterization founded upon the properties of this generalized Deligne sheaf.

In Section 4, we show that our generalized Deligne sheaves are constructible and that this implies a general perversity version of the Poincaré-Verdier-Goresky-MacPherson duality theorem on stratified pseudomanifolds. The full statement of the theorem can be found below as Theorem 4.3. We here state the special case over an orientable pseudomanifold without boundary and for a constant coefficient sheaf whose stalks are the field F. In the statement, $\mathcal{Q}^*$ denotes our generalized Deligne sheaf, $\mathcal{D}$ denotes the Verdier dualizing functor, $U_1 = X - X^{n-1}$, and $\bar{t}$ is the top perversity, $\bar{t}(Z) = \operatorname{codim}(Z) - 2$.

Theorem 1.1. *Let X be an orientable n-dimensional stratified pseudomanifold, and let $\bar{p}$ and $\bar{q}$ be general perversities such that $\bar{p} + \bar{q} = \bar{t}$. Then $\mathcal{D}_X \mathcal{Q}^*_{\bar{p}}[-n] \cong \mathcal{Q}^*_{\bar{q}}(\mathcal{D}_{U_1} F[-n])$ in the derived category of sheaves on X.*

This implies a more familiar-looking statement, which forms the third item of Corollary 4.4:

Corollary 1.2. *If X is closed and orientable and $\bar{p}, \bar{q}$ are general perversities with $\bar{p}+\bar{q}=\bar{t}$, then*

$$I^{\bar{q}}H_{n-i}(X;\mathbb{Q}_0) \cong \mathrm{Hom}(I^{\bar{p}}H_i(X;\mathbb{Q}_0),\mathbb{Q}).$$

This includes the original duality result of Goresky and MacPherson as a special case. The subscript 0 on the $\mathbb{Q}$ coefficients reflects the use of a stratified coefficient system; see Section 2.

While the preceding statements might be somewhat expected considering the theorems of Goresky and MacPherson, their generalization to our current setting does have some nice consequences, including a simple proof of the following Lefschetz duality theorem for pseudomanifolds with boundary.

Corollary 1.3. *If X is a compact and orientable stratified pseudomanifold with boundary and $\bar{p}, \bar{q}$ are general perversities with $\bar{p}+\bar{q}=\bar{t}$, then*

$$I^{\bar{q}}H_{n-i}(X;\mathbb{Q}_0) \cong \mathrm{Hom}(I^{\bar{p}}H_i(X,\partial X;\mathbb{Q}_0),\mathbb{Q}).$$

This corollary follows from Corollary 4.4 and the discussion of Lefschetz duality that follows it in Section 4.[1]

In Section 4.3, we explore the sheaf-theoretic intersection homology pairings. The following theorem appears below as Theorem 4.6 and is much more general than the classical result (see [29, 7]), in which no pairing morphism m can exist at all if $\bar{p}+\bar{q}\not\leq\bar{t}$.

Theorem 1.4. *Given a pairing of local systems $m_1:\mathcal{E}\otimes\mathcal{F}\to\mathcal{G}$ on $X-X^{n-1}$ and general perversities such that $\bar{p}(Z)+\bar{q}(Z)\leq\bar{r}(Z)$ for all singular strata Z, then in the bounded derived category $D^b(X)$, there is a unique morphism $m:\mathcal{Q}^*_{\bar{p}}(\mathcal{E}^*)\overset{L}{\otimes}\mathcal{Q}^*_{\bar{q}}(\mathcal{F})\to\mathcal{Q}^*_{\bar{r}}(\mathcal{G})$ that restricts to m_1 on $X-X^{n-1}$. Furthermore, if $\bar{r}\leq\bar{t}$ and $\mathcal{G}=\mathcal{O}$, the orientation sheaf of $X-X^{n-1}$, then there exists a pairing $m:\mathcal{Q}^*_{\bar{p}}(\mathcal{E}^*)\overset{L}{\otimes}\mathcal{Q}^*_{\bar{q}}(\mathcal{F})\to\mathfrak{D}^*_X[-n]$ that restricts to m_1 on $X-X^{n-1}$, where $\mathfrak{D}^*_X[-n]$ is the shift of the Verdier dualizing complex on X.*

Finally, in Section 5, we return to the geometry of simplicial chains on PL spaces. In Theorem 5.4 we provide a generalization of the original Goresky-MacPherson intersection pairing. We state here a nice special case, assuming X is compact and orientable:

[1] This intersection homology Lefschetz duality is currently being utilized by the author and Hunsicker in their study of intersection homology versions of Novikov additivity and Wall non-additivity for perverse signatures [26]. These are the signatures of the nondegenerate intersection pairing on $\mathrm{im}(I^{\bar{p}}H_{2n}(X^{4n})\to I^{\bar{q}}H_{2n}(X^{4n},\partial X^{4n}))$, when $\bar{p}\leq\bar{q}$; the well-known Witt-space signature is a special case. Signatures arising through intersection homology and the closely related analytic L^2-cohomology and L^2 Hodge theory have been the subject of intense study since the beginnings of the subject. They have been studied topologically by researchers including Goresky and MacPherson [28, 29], Siegel [49], and various combinations of Banagl, Cappell, Libgober, Maxim, Shaneson, and Weinberger, whose papers on the topic include, among many others, [3, 1, 4, 8, 10, 9]. There has also been much interest in the analytic study of these intersection homology signatures as they arise in L^2-cohomology and L^2 Hodge theory and as they may relate to duality in string theory, such as through Sen's conjecture on the dimension of spaces of self-dual harmonic forms on monopole moduli spaces. Results in these areas and closely related topics include those of Müller [43]; Dai [17]; Cheeger and Dai [15]; Hausel, Hunsicker, and Mazzeo [34, 36, 35]; Saper [46, 45]; Saper and Stern [47]; and Carron [12, 14, 13]; and work on analytic symmetric signatures is currently being pursued by Albin, Leichtmann, Mazzeo and Piazza.

Theorem 1.5. *Suppose $x \in I^{\bar{p}}C_i(X;\mathbb{Z}_0)$, $y \in I^{\bar{q}}C_j(X;\mathbb{Z}_0)$ are such that the pairs $(|x|,|y|)$, $(|\partial x|,|y|)$, and $(|x|,|\partial y|)$ are in stratified general position. Then there is a well-defined intersection $x \pitchfork y \in I^{\bar{r}}C_{i+j-n}(X;\mathbb{Z}_0)$ for any $\bar{r} \geq \bar{p}+\bar{q}$.*

Once again, such a general pairing does not exist in classical intersection homology, in which it is not possible to intersect a $\bar{p}$-allowable chain and a $\bar{q}$-allowable chain unless $\bar{p}+\bar{q} \leq \bar{t}$.

We also show in Section 5 that the results of [25] on the existence of partial DGA structures on intersection chain complexes generalize to include general perversities.

Some of the arguments we present here adhere in general form to well-known paths in intersection homology theory or to our other recent work. However, there are several novelties that require special attention and detailed work. These include the introduction of our generalized sheaf truncation functor, as well as the details of the PL chain pairing, for which it is necessary to work with what can be best described as relative homology analogues of more classical "absolute homology" arguments. In our exposition, we hope to have found a middle road that does not repeat too much that can be found elsewhere in the literature but that is sufficiently detailed to allow the reader to appreciate the new results and modified techniques that arise in the study and application of general perversity intersection homology.

Remark 1.6. A detailed expository survey of intersection homology with general perversities, including an overview of some of the present results, can be found in [19].

Acknowledgment. I sincerely thank an anonymous referee for pointing out the appearance of general perversities in the theory of perverse sheaves.

2 Background

We begin with a brief review of definitions, referring the reader to sources such as [28, 29, 7, 39, 2, 38, 22, 19] for more thorough background. We encourage the experts also to skim this section, as we allow a few unconventional generalizations.

2.1 Pseudomanifolds and intersection homology basics

Pseudomanifolds. Let $c(Z)$ denote the open cone on the space Z, and let $c(\emptyset)$ be a point.

A *stratified paracompact Hausdorff space* Y (see [29] or [11]) is defined by a filtration

$$Y = Y^n \supset Y^{n-1} \supset Y^{n-2} \supset \cdots \supset Y^0 \supset Y^{-1} = \emptyset$$

such that for each point $y \in Y^i - Y^{i-1}$, there exists a *distinguished neighborhood* U of y such that there is a *compact* Hausdorff space L, a filtration of L

$$L = L^{n-i-1} \supset \cdots \supset L^0 \supset L^{-1} = \emptyset,$$

and a homeomorphism

$$\phi : \mathbb{R}^i \times c(L) \to U$$

that takes $\mathbb{R}^i \times c(L^{j-1})$ onto $Y^{i+j} \cap U$. The Y^i are called *skeleta*. We denote $Y_i = Y^i - Y^{i-1}$; this is an i-manifold that may be empty. We refer to the connected components of the various Y_i as *strata*[2]. If a stratum Z is a subset of $Y_n = Y - Y^{n-1}$ it is called a *regular stratum*; otherwise it is called a *singular stratum*. L is called a *link*. The *depth* of a stratified space is the number of distinct skeleta it possesses minus one.

Usually, a stratified *(topological) pseudomanifold* of dimension n is defined to be a stratified paracompact Hausdorff space X such that $X^{n-1} = X^{n-2}$, $X - X^{n-2}$ is a manifold of dimension n dense in X, and each link L is, inductively, a stratified pseudomanifold; a space is a (topological) pseudomanifold if it can be given the structure of a stratified pseudomanifold for some choice of filtration. In this paper, we will also allow for the possibility that $X^{n-1} \neq X^{n-2}$. When we do assume $X^{n-1} = X^{n-2}$, intersection homology with Goresky-MacPherson perversities is known to be a topological invariant; in particular, it is invariant under choice of stratification (see [29], [7], [38]). Examples of pseudomanifolds include complex algebraic and analytic varieties (see [7, Section IV]).

A stratified pseudomanifold is orientable (respectively oriented) if $X - X^{n-2}$ is.

We refer to the link L in the distinguished neighborhood U of y as the link of y or of the stratum containing y; it is, in general, not uniquely determined up to homeomorphism, though if X is a pseudomanifold it is unique up to, for example, stratum preserving homotopy equivalence (see, e.g., [21]), which is sufficient for the intersection homology type of the link of a stratum to be determined uniquely. Thus there is no harm, in general, of referring to "the link" of a stratum instead of "a link" of a stratum.

A piecewise linear (PL) pseudomanifold is a pseudomanifold with a PL structure compatible with the filtration, meaning that each skeleton is a PL subspace, and such that each link is a PL pseudomanifold and the distinguished neighborhood homeomorphisms $U \cong \mathbb{R}^{n-k} \times cL$ are PL homeomorphisms.

We will assume all pseudomanifolds X to have a fixed given stratification $\mathfrak{X}$.

Intersection homology. We will work mostly with singular chain intersection homology theory, which was introduced in [38] with finite chains (compact supports) and generalized in [22] to include locally-finite but infinite chains (closed supports). Singular intersection homology can be defined on any filtered space, but we limit ourselves to stratified pseudomanifolds.

A *Goresky-MacPherson perversity* (or *GM perversity*) $\bar{p}$ is a function $\bar{p} : \mathbb{Z}^{\geq 2} \to \mathbb{Z}$ such that $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$ and $\bar{p}(2) = 0$. The elements of $\mathbb{Z}^{\geq 2}$ correspond to the sets X_k. Here the domain is $\mathbb{Z}^{\geq 2}$ because Goresky and MacPherson did not allow codimension 1 strata. King [38] introduced *loose perversities*, which are completely arbitrary functions $\bar{p} : \mathbb{Z}^{\geq 0} \to \mathbb{Z}$. We will go even further, following Saralegi [48], and define a *general perversity*

[2]This definition agrees with some sources, but is slightly different from others, including our own past work, which would refer to Y_i as the stratum and what we call strata as "stratum components."

on a stratified pseudomanifold X to be any function[3] $\bar{p} : \{\text{singular strata of } X\} \to \mathbb{Z}$.

Given a stratified pseudomanifold X, a general perversity $\bar{p}$, and an abelian group G, one defines the *intersection chain complex* $I^{\bar{p}}C^c_*(X;G)$ as a subcomplex of $C^c_*(X;G)$, the complex of compactly supported singular chains on X, as follows: An i-simplex $\sigma : \Delta^i \to X$ in $C^c_i(X)$ is *allowable* if

$$\sigma^{-1}(Z) \subset \{i - \operatorname{codim}(Z) + \bar{p}(Z) \text{ skeleton of } \Delta^i\}$$

for any singular stratum Z of X. The chain $\xi \in C^c_i(X;G)$ is allowable if each simplex with non-zero coefficient in ξ or in $\partial\xi$ is allowable. $I^{\bar{p}}C^c_*(X;G)$ is the complex of allowable chains. $I^{\bar{p}}C^\infty_*(X;G)$ is defined similarly as the complex of allowable chains in $C^\infty_*(X;G)$, the complex of locally-finite singular chains. Chains in $C^\infty_*(X;G)$ may be composed of an infinite number of simplices (with their coefficients), but for each such chain ξ, each point in X must have a neighborhood that intersects only a finite number of simplices (with non-zero coefficients) in ξ. See [22] for more details.

The associated homology theories are denoted $I^{\bar{p}}H^c_*(X;G)$ and $I^{\bar{p}}H^\infty_*(X;G)$ and called *intersection homology* with, respectively, compact or closed supports.

N.B. We will often omit the decorations c or ∞ when these theories are equivalent or when our statements apply to either case.

Relative intersection homology is defined similarly, in the obvious way, though we note that the filtration on a subspace will always be that inherited from the larger space by restriction, and, in the closed support case, all chains are required to be locally-finite in the larger space.

If $\bar{p}$ is a GM perversity and X has no strata of codimension one, then it is also possible to define intersection chains with coefficients in a local system of groups $\mathcal{G}$ that is defined only on $X - X^{n-2}$. See [29, 22, 19] for more details.

Sheaves. Recall from [22] that one can define a sheaf complex[4] $\mathcal{I}^{\bar{p}}\mathcal{S}^*$ on the n-dimensional stratified pseudomanifold X as the sheafification of the presheaf $U \to I^{\bar{p}}C^\infty_{n-*}(X, X - \bar{U}; G)$ or, equivalently, of the presheaf $U \to I^{\bar{p}}C^c_{n-*}(X, X - \bar{U}; G)$. It is shown in [22] that the former presheaf is conjunctive for coverings and has no non-trivial global sections with empty support. Furthermore, the sheaf $\mathcal{I}^{\bar{p}}\mathcal{S}^*$ is homotopically fine. As a consequence, the hypercohomology $\mathbb{H}^i(X;\mathcal{I}^{\bar{p}}\mathcal{S}^*)$ is isomorphic to $I^{\bar{p}}H^\infty_{n-i}(X;G)$, and $\mathbb{H}^i_c(X;\mathcal{I}^{\bar{p}}\mathcal{S}^*) \cong I^{\bar{p}}H^c_{n-i}(X;G)$.

By [29], if $\bar{p}$ is a GM perversity and X is a pseudomanifold, then $\mathcal{I}^{\bar{p}}\mathcal{S}^*$ is quasi-isomorphic to the *Deligne sheaf*, and hence they are interchangeable in the derived category of sheaves.

[3] Technically, our definition of a general perversity is not strictly more general than King's because he allows loose perversities to be defined on regular strata and takes this into account in defining intersection chains. In practice, however, if Z is a regular stratum, the only meaningful distinction for intersection homology is between the cases $\bar{p}(Z) \geq 0$ and $\bar{p}(Z) \leq -1$. The latter condition would force chains to avoid the regular stratum altogether, which is usually not worth considering (see the discussion of this point in [19, Remark 6.1]), and it makes some later statements technically simpler to avoid imposing repeatedly the condition $\bar{p}(Z) \geq 0$ for regular strata. The most elegant solution, used by Saralegi in [48], seems to be to define $\bar{p}$ only on the singular strata and to build this restriction into the definition of intersection chains.

[4] We tend to leave the coefficients out of the sheaf notation for simplicity; the coefficients will always be known from context.

To define the Deligne sheaf, let $U_k = X - X^{n-k}$, and let $i_k : U_k \hookrightarrow U_{k+1}$ be the inclusion. Ri_{k*} denotes right derived functor of the push forward i_{k*} and $\tau_{\leq m}$ is the truncation functor that acts on the complex A^* (in any appropriate category) as

$$(\tau_{\leq m}A^*)^i = \begin{cases} 0, & i > m, \\ \ker(d : A^m \to A^{m+1}), & i = m, \\ A^i, & i \leq m. \end{cases}$$

Recall that $H^i(\tau_{\leq m}A^*) = 0$ for $i > m$, and $H^i(\tau_{\leq m}A^*) = H^i(A^*)$ for $i \leq m$. Let G_{U_1} be the constant sheaf G on the open manifold U_1. Then the Deligne sheaf $\mathcal{P}^*_{\bar{p}}$ is defined inductively as $\tau_{\leq \bar{p}(n)}Ri_{n*}\dots\tau_{\leq \bar{p}(1)}Ri_{1*}G_{U_1}$.

It is again possible to begin with a local system of groups $\mathcal{G}$ that is defined only on $X - X^{n-1}$; one simply replaces G_{U_1} with $\mathcal{G}$.

2.2 Stratified coefficient systems

Motivation. When working with perversities $\bar{p}$ for which $\bar{p}(Z) > \mathrm{codim}(Z) - 2$ for some stratum Z (we call such perversities *superperversities*[5]), it is useful to make a technical adjustment to the definition given above of chain-theoretic intersection homology. This adjustment was introduced in [22] under the guise of using a "stratified coefficient system" $\mathcal{G}_0$ and independently by Saralegi [48] in the form of a certain relative intersection homology; it is shown in [24] that the two approaches yield isomorphic chain complexes. The motivation for stratified coefficients in [22] was the desire to construct a singular chain version of intersection homology with superperversities that agrees with the Deligne sheaf version from [11], while Saralegi's purpose was to prove a de Rham theorem for general perversities. An added benefit is that stratified coefficients let us start with a local coefficient system on $X - X^{n-1}$ for any general perversity, which would not otherwise be possible.

Before providing the definition, we note that the main technical point necessitating stratified coefficients is the need to generalize the intersection homology cone formula so as to obtain the following proposition. This proposition combines [22, Proposition 2.18] with the isomorphism $I^{\bar{p}}H^c_i(cL, L \times \mathbb{R}; \mathcal{G}_0) \cong I^{\bar{p}}H^{\infty}_i(cL; \mathcal{G}_0)$, which follows from [22, Lemma 2.12]. In [22], these propositions are not proven in quite the generality stated here, but the proofs generalize immediately; see also [19].

Proposition 2.1. *Let L be an $n-1$ dimensional filtered space with stratified coefficient system $\mathcal{G}_0$. Let v be the vertex of cL, let cL be filtered such that $(cL)^0 = v$ and $(cL)^i = c(L^{i-1}) - v$ for $i > 0$. Then*[6]

$$I^{\bar{p}}H^c_i(cL; \mathcal{G}_0) \cong \begin{cases} 0, & i \geq n - 1 - \bar{p}(\{v\}), \\ I^{\bar{p}}H^c_i(L; \mathcal{G}_0), & i < n - 1 - \bar{p}(\{v\}). \end{cases}$$

[5] The term "superpervisity" was first used by the author in [22, 20] to refer to the perversities considered by Cappell and Shaneson in [11].

[6] On the second line, we should technically use $I^{\bar{p}}H_{i-1}(L; \mathcal{G}_0|_L)$, but we will leave such restrictions tacit throughout.

If L is compact, then

$$I^{\bar p}H_i^c(cL, L\times\mathbb{R};\mathcal{G}_0)\cong I^{\bar p}H_i^\infty(cL;\mathcal{G}_0)\cong\begin{cases} I^{\bar p}H_{i-1}(L;\mathcal{G}_0), & i\geq n-\bar p(\{v\}),\\ 0, & i< n-\bar p(\{v\}).\end{cases}$$

This proposition is familiar for GM-perversity intersection homology, but it would not be true for general perversities without the stratified coefficient systems. The reader should compare with [38, Proposition 5]; the small but important difference is entirely in degree 0.

While this seems like a minor point, this proposition in its usual form for GM perversities is a key player in proving all of the major results of the theory, and this continues to be borne out for general perversities in [22, 24, 48] and in the results that follow. This seems ample evidence that stratified coefficient systems (or, equivalently, Saralegi's relative groups) are worth using. Furthermore, when $\bar p$ is a Goresky-MacPherson perversity (or, more generally, if $\bar p(Z)\leq \operatorname{codim}(Z)-2$ for each singular stratum Z), then $I^{\bar p}C_*(X;\mathcal{G}_0)\cong I^{\bar p}C_*(X;\mathcal{G})$ as observed in [20, Proposition 2.1][7]. Thus intersection chains with stratified coefficients are a legitimate extension of the traditional setting.

Remark 2.2. The intersection homology sheaves constructed by Habegger and Saper [32] reflect singular intersection homology for which $\bar p(Z)\geq \operatorname{codim}(Z)-1$ may occur but stratified coefficients are not used. Thus the hypercohomology of their sheaves is not the same as the superperverse intersection homology found in [11, 22, 48]. Habegger and Saper also continue to require that $\bar p(k)\leq \bar p(k+1)\leq \bar p(k)+1$. They find a duality theorem in this setting, but it is technically more complicated than the Goresky-MacPherson duality and that produced here. See [32, 19] for more details.

Definition of stratified coefficients. Suppose given a filtered space X and local system of coefficients $\mathcal{G}$ on $X-X^{n-1}$. Then the stratified coefficient system $\mathcal{G}_0$ is defined to consist of the pair of coefficient systems given by $\mathcal{G}$ on $X-X^{n-1}$ and the constant 0 system on X^{n-1}, i.e. we think of $\mathcal{G}_0$ as consisting of a locally constant fiber bundle $\mathcal{G}_{X-X^{n-1}}$ over $X-X^{n-1}$ with fiber G (with the discrete topology) together with the trivial bundle on X^{n-1} with stalk 0. Then a coefficient n of a singular simplex σ can be described by a lift of $\sigma|_{\sigma^{-1}(X-X^{n-1})}$ to $\mathcal{G}_{X-X^{n-1}}$ together with the trivial "lift" of $\sigma|_{\sigma^{-1}(X^{n-1})}$ to the 0 system on X^{n-1}. A coefficient of a simplex σ is considered to be the 0 coefficient if it maps each point of Δ to the 0 section of one of the coefficient systems. If $n\sigma$ is a simplex σ with its coefficient n, its boundary is given by the usual formula $\partial(n\sigma)=\sum_j(-1)^j(n\circ \mathfrak{i}_j)(\sigma\circ\mathfrak{i}_j)$, where $\mathfrak{i}_j:\Delta^{i-1}\to\Delta^i$ is the jth face inclusion map. In this way we can form a chain complex $C_*(X;\mathcal{G}_0)$.

The basic idea behind the definition is essentially that when we consider allowability of chains with respect to a perversity, simplices with support entirely in X^{n-1} should vanish and thus not be counted for allowability considerations. Thus the intersection homology chain complexes $I^{\bar p}C_*^c(X;\mathcal{G}_0)$ and $I^{\bar p}C_*^\infty(X;\mathcal{G}_0)$ are defined just as $I^{\bar p}C_*^c(X;G)$ and $I^{\bar p}C_*^\infty(X;G)$ are, but replacing the coefficients of simplices with coefficients in $\mathcal{G}_0$. Allowability of a simplex is thus determined as above, but all simplices with support in X^{n-1} vanish due

[7]The proposition there has the hypotheses that $X^{n-1}=X^{n-2}$, but this hypothesis in not necessary if for each $Z\subset X_{n-1}$, $\bar p(Z)\leq\operatorname{codim}(Z)-2=-1$, a condition which was not allowed in [20].

to the 0 coefficient they must carry, and so they are automatically allowable. This yields nothing new when considering the allowability of the i-simplices of an i-chain ξ, but now any simplices of $\partial\xi$ with support in X^{n-1} vanish, which may come into play in determining the allowability of ξ as a chain. More details can be found in [22].

Relationship with Saralegi's relative chains. Let G be a constant coefficient system. As noted in [22], use of the coefficient system G_0, in which X^{n-1} carries a formal 0 coefficient system, is *not* the same as attempting to take relative intersection homology $I^{\bar p}H_*(X, X^{n-1}; G)$. For one thing, if $\bar p(Z) \leq \text{codim}(Z) - 1$, there is no such chain submodule as $I^{\bar p}C_*(X^{n-1}; G)$ because no allowable chains are contained entirely within X^{n-1}. However, Saralegi's idea in [48] for an intersection chain complex satisfying Proposition 2.1 is a correct refinement of this idea. He defined a chain complex $S^{\bar p}C_*(X; X_{\bar t-\bar p})$ as

$$S^{\bar p}C^c_*(X, X_{\bar t-\bar p}; G) = \frac{(A^{\bar p}C_*(X;G) + AC^{\bar p+1}_*(X_{\bar t-\bar p};G)) \cap \partial^{-1}\left(A^{\bar p}C_{*-1}(X;G) + AC^{\bar p+1}_{*-1}(X_{\bar t-\bar p};G)\right)}{AC^{\bar p+1}_*(X_{\bar t-\bar p};G) \cap \partial^{-1}AC^{\bar p+1}_{*-1}(X_{\bar t-\bar p};G)},$$

where $\bar t$ is the top perversity, $\bar t(Z) = \text{codim}(Z) - 2$, $A^{\bar p}C_i(X)$ is generated by the $\bar p$-allowable i-simplices of X, $X_{\bar t-\bar p}$ is the closure of the union of the singular strata Z of X such that $\bar t(Z) - \bar p(Z) < 0$, and $A^{\bar p+1}C_i(X_{\bar t-\bar p})$ is generated by the $\bar t - \bar p - 1$ allowable i-simplices with support in $X_{\bar t-\bar p}$. Once again, we see that the idea is to have a complex made up of allowable chains but to kill simplices that lie within the strata where the perversity is too high.

In [24], we proved that $I^{\bar p}C^c_*(X; G_0)$ is quasi-isomorphic to $S^{\bar p}C^c_*(X, X_{\bar t-\bar p}; G)$. More precisely, the proof there uses the extra assumption that the coefficient of any simplex in a chain of $I^{\bar p}C_*(X; G_0)$ must lift to the same coefficient at all points of $\sigma^{-1}(X - X^{n-1})$, and with this assumption, $I^{\bar p}C^c_*(X; G_0)$ and $S^{\bar p}C^c_*(X, X_{\bar t-\bar p}; G)$ are actually isomorphic. However, it is also noted in [24] that this variant of the definition of $I^{\bar p}C^c_*(X; G_0)$ yields a chain complex that is quasi-isomorphic with the chain complex we have been using here. Furthermore, this extra assumption will automatically be true if $\bar p$ is an "efficient perversity," a notion that is defined in the next subsection, where we also show that $\bar p$ can always be replaced with an efficient perversity without changing the intersection homology groups.

Throughout this paper, we will use the stratified coefficient formulation, which we think is slightly simpler to use in most of our arguments, largely because it avoids quotient groups. Also, it is not quite clear how to extend Saralegi's approach to local coefficient systems.

Basic Properties. As shown in [22], even with general perversities and $\mathcal{G}_0$ coefficients, many of the basic properties of $I^{\bar p}H_*(X; \mathcal{G}_0)$ established in [38] and [22] hold with little or no change to the proofs, such as stratum-preserving homotopy equivalence, excision, the Künneth theorem for which one term is an unstratified manifold, Mayer-Vietoris sequences, etc.

It also remains true with general perversities and stratified coefficients that we can define a sheaf $\mathcal{I}^{\bar p}\mathcal{S}^*$ as the sheafification of the presheaf $U \to I^{\bar p}C^\infty_{n-*}(X, X - \bar U; \mathcal{G}_0)$ or, equivalently, of the presheaf $U \to I^{\bar p}C^c_{n-*}(X, X - \bar U; \mathcal{G}_0)$. Again, the former presheaf is conjunctive for coverings and has no non-trivial global sections with empty support, and the sheaf $\mathcal{I}^{\bar p}\mathcal{S}^*$ is homotopically fine. Thus, the hypercohomology $\mathbb{H}^i(X;\mathcal{I}^{\bar p}\mathcal{S}^*)$ is isomorphic to $I^{\bar p}H^{\infty}_{n-i}(X;\mathcal{G}_0)$, and $\mathbb{H}^i_c(X;\mathcal{I}^{\bar p}\mathcal{S}^*) \cong I^{\bar p}H^c_{n-i}(X;\mathcal{G}_0)$. For simplicity of notation, we generally write $\mathcal{I}^{\bar p}\mathcal{S}^*$, without noting explicitly the coefficients.

N.B. Throughout this paper, we will always assume stratified coefficients are in use unless explicitly noted otherwise.

2.3 Efficient perversities

In this section, we show that many perversities provide redundant information. In other words, there is no need to consider perversities that are *too* general.

Definition 2.3. We will say that a general perversity $\bar p$ is *efficient* if, for each singular stratum Z, $-1 \le \bar p(Z) \le \operatorname{codim}(Z) - 1$.

Given a general perversity $\bar p$, we can associate an efficient perversity $\check p$ as follows: Define $\check p$ by

$$\check p(Z) = \begin{cases} \operatorname{codim}(Z) - 1, & \text{if } \bar p(Z) \ge \operatorname{codim}(Z) - 1, \\ \bar p(Z), & \text{if } 0 \le \bar p(Z) \le \operatorname{codim}(Z) - 2, \\ -1, & \text{if } \bar p(Z) \le -1. \end{cases}$$

We now show that we can effectively eliminate consideration of perversities that are not efficient when studying intersection homology.

Lemma 2.4. *Let $\bar p$ be a general perversity and X a pseudomanifold, possibly with codimension one strata. Let $\check p$ be the associated efficient perversity. Then $I^{\bar p}H_*(X;\mathcal{G}_0) \cong I^{\check p}H_*(X;\mathcal{G}_0)$.*

Corollary 2.5. *Let X^- denote the union of the singular strata Z of X such that $\bar p(Z) \le -1$. Then $I^{\bar p}H_*(X;\mathcal{G}_0) \cong I^{\bar p}H_*(X - X^-;\mathcal{G}_0)$.*

Proof. This follows directly from the lemma and from the definition of the intersection chain complex by replacing $\bar p$ with a perversity such that $\bar p(Z) \ll 0$ for each $Z \subset X^-$. □

Proof of Lemma 2.4. We proceed in two steps. First, let $\tilde p$ be the perversity defined so that $\tilde p(Z) = -1$ if $\bar p(Z) \le -1$ and $\tilde p(Z) = \bar p(Z)$ otherwise. Then there is a natural inclusion $I^{\bar p}C_*(X;\mathcal{G}_0) \hookrightarrow I^{\tilde p}C_*(X;\mathcal{G}_0)$ (for either closed or compact supports). This induces a map of sheaves $\mathcal{I}^{\bar p}\mathcal{S}^* \to \mathcal{I}^{\tilde p}\mathcal{S}^*$, which we claim is a quasi-isomorphism. We induct on depth, noting that we clearly have a quasi-isomorphism on depth 0 pseudomanifolds, which have no singular strata. So suppose that we have shown the claim through depth $j-1$ and that $x \in X$, where X has depth j. It suffices to compute locally, and by the standard computations (e.g. [38, 22]),

$$\begin{aligned} H^{n-*}(\mathcal{I}^{\bar p}\mathcal{S}^*_x) &\cong I^{\bar p}H^{\infty}_*(\mathbb{R}^{n-k} \times cL;\mathcal{G}_0) \\ &\cong I^{\bar p}H^{\infty}_{*-n-k}(cL;\mathcal{G}_0), \end{aligned}$$

and similarly for $\tilde p$. In fact, the map of cohomology stalks of sheaves comes down to the map induced by inclusion $I^{\bar p}H^{\infty}_*(cL;\mathcal{G}_0) \to I^{\tilde p}H^{\infty}_*(cL;\mathcal{G}_0)$. But by the cone formula and the induction hypothesis, since $\text{depth}(L) < j$, this is an isomorphism: if $\bar{p}(Z) \leq -1$, each of these groups is 0 for all $*$, using that cL is k dimensional and so $IH_*(cL; \mathcal{G}_0)$ vanishes for $* > k$ automatically (an argument for this can be found below within the proof of Proposition 3.7). If $\bar{p}(Z) \geq 0$, then for $i \geq k - \bar{p}(Z) = k - \tilde{p}(Z)$, $I^{\bar{p}}H_i^\infty(cL; \mathcal{G}_0) \cong I^{\bar{p}}H_{i-1}(L; \mathcal{G}_0) \cong I^{\tilde{p}}H_{i-1}(L; \mathcal{G}_0) \cong I^{\tilde{p}}H_i^\infty(cL; \mathcal{G}_0)$, where the middle isomorphism follows from the induction hypothesis.

Similarly, using the "opposite extreme" of the cone formula, there is a quasi-isomorphism $I^{\hat{p}}C_*(X; \mathcal{G}_0) \hookrightarrow I^{\bar{p}}C_*(X; \mathcal{G}_0)$, where $\hat{p}(Z) = \bar{p}(Z)$ when $\bar{p}(Z) \leq \text{codim}(Z) - 2$ and $\hat{p}(X) = \text{codim}(Z) - 1$ otherwise.

The lemma now follows by noting that $\check{p}$ can be obtained from $\bar{p}$ by first forming $\tilde{p}$ and then applying the $\hat{p}$ construction to $\tilde{p}$ or vice versa. $\square$

Lemma 2.6. *If $\bar{p}$ is an efficient perversity, than so is $\bar{t} - \bar{p}$, where $\bar{t}$ is the top perversity, $\bar{t}(Z) = \text{codim}(Z) - 2$.*

Proof. Since $-1 \leq \bar{p}(Z) \leq \text{codim}(Z) - 1$, $\bar{t}(Z) + 1 \geq \bar{t}(Z) - \bar{p}(Z) \geq \bar{t}(Z) - \text{codim}(Z) + 1$, or $\text{codim}(Z) - 1 \geq \bar{t}(Z) - \bar{p}(Z) \geq -1$. $\square$

Efficient perversities and interiors of simplices. Efficient perversities have a nice feature that makes them technically better behaved than the more general perversities. If $\bar{p}$ is a perversity for which $\bar{p}(Z) \geq \text{codim}(Z)$ for some singular stratum Z, then any i-simplex σ will be $\bar{p}$-allowable with respect to Z. In particular, Z will be allowed to intersect the image under σ of the interior of Δ^i. As such, $\sigma^{-1}(X - X^{n-1})$ could potentially have an infinite number of connected components, and a coefficient of σ might lift each component to a different branch of $\mathcal{G}$, even if $\mathcal{G}$ is a constant system. This could potentially lead to some pathologies, especially when considering intersection chains from the sheaf point of view. However, if $\bar{p}$ is efficient, then for a $\bar{p}$-allowable σ we must have $\sigma^{-1}(X - X^{n-1})$ within the $i-1$ skeleton of Δ^i. Hence assigning a coefficient lift value to one point of the interior of Δ^i determines the coefficient value at all points (on $\sigma^{-1}(X - X^{n-1})$ by the unique extension of the lift and on $\sigma^{-1}(X^{n-1})$, where it is 0). This is technically much simpler and makes the complex of chains in some sense smaller.

In [24], the complex $I^{\bar{p}}C_*(X; \mathcal{G}_0)$ was defined with the assumption that this "unique coefficient" property holds, meaning that a coefficient should be determined by its lift at a single point. However, as noted in [24, Appendix], even for inefficient perversities, this does not change the intersection homology. Having introduced efficient perversities here, we are instead free to assume all perversities are efficient, without loss of any information (at least at the level of quasi-isomorphism), and this provides a reasonable way to avoid the issue entirely.

3 A generalized Deligne sheaf

We now turn to sheaf theoretic versions of general perversity intersection homology.

The need for a generalization of the Deligne construction. We first provide an example showing that general perversity intersection homology is not necessarily given by hypercohomology of the classical Deligne sheaf, even if $\bar{p}(Z) = \bar{p}(Z')$ when $\operatorname{codim}(Z) = \operatorname{codim}(Z')$.

Proposition 3.1. *Let $\bar{p}$ be a general perversity and X an n-dimensional pseudomanifold. Then neither $I^{\bar{p}}H_*(X;\mathbb{Z})$ nor $I^{\bar{p}}H_*(X;\mathbb{Z}_0)$ is necessarily isomorphic to $\mathbb{H}^{n-*}(X;\mathcal{P}^*)$, where $\mathcal{P}^*$ is the perversity-$\bar{p}$ Deligne sheaf.*

Proof. We provide an example. We will use the more traditional notation for perversities taking codimensions as inputs.

Let $X = S^2T^2$, the doubly suspended torus, stratified as such (so that X^0 consists of two points, $X^1 - X^0$ consists of two arcs, and $X^3 = X^2 = X^1$), and let $\bar{p}$ be the general perversity given by $\bar{p}(0) = \bar{p}(1) = \bar{p}(2) = 0$, $\bar{p}(3) = 3$, $\bar{p}(4) = 0$. We first compute $I^{\bar{p}}H_*(X)$ from the singular chain definition of intersection homology.

Since ST^2 has dimension 3 and $\bar{p}(X_0) = 0$, we have by the intersection homology suspension formula (see [38, Proposition 5]),

$$I^{\bar{p}}H_*(X) \cong \begin{cases} I^{\bar{p}}H_{*-1}(ST^2), & * > 3, \\ 0, & * = 3 - \bar{p}(4) = 3, \\ I^{\bar{p}}H_*(ST^2), & * < 3, \end{cases}$$

and, using Proposition 2.1 and a Mayer-Vietoris sequence, the same formula holds with $\mathbb{Z}$ replaced by $\mathbb{Z}_0$.

For $I^{\bar{p}}H_*(ST^2)$, since $\bar{p}(X_1) = 3$, all chains are allowable, and

$$I^{\bar{p}}H_*(ST^2;\mathbb{Z}) \cong H_*(ST^2) \cong \begin{cases} \mathbb{Z}, & * = 3, \\ \mathbb{Z}^2, & * = 2, \\ 0, & * = 1, \\ \mathbb{Z} & * = 0. \end{cases}$$

If we replace the $\mathbb{Z}$ coefficients with $\mathbb{Z}_0$ coefficients, the only change is to replace the $\mathbb{Z}$ in degree 0 with a 0.

Putting these together, we have

$$I^{\bar{p}}H_*(S^2T^2) \cong H_*(S^2T^2) \cong \begin{cases} \mathbb{Z}, & * = 4, \\ 0, & * = 3, \\ \mathbb{Z}^2, & * = 2, \\ 0, & * = 1, \\ \mathbb{Z}, & * = 0, \end{cases}$$

and, again, if we replace the $\mathbb{Z}$ coefficients with $\mathbb{Z}_0$ coefficients, the only change is to replace the $\mathbb{Z}$ in degree 0 with a 0.

We claim, on the other hand, that $\mathbb{H}^*(X;\mathcal{P}^*) \cong H^*(X;\mathbb{Z})$. Thus

$$\mathbb{H}^{4-*}(X;\mathcal{P}^*) \cong H^{4-*}(X) \cong \begin{cases} \mathbb{Z}, & *=4, \\ 0, & *=3, \\ 0, & *=2, \\ \mathbb{Z}^2, & *=1, \\ \mathbb{Z}, & *=0, \end{cases}$$

by the standard suspension formula for cohomology. This provides the desired contradiction.

To verify the claim, we first show that $\mathcal{P}^* \cong \tau_{\leq 0} Rj_*\mathbb{Z}_{X_4}$, where $j = i_4 \circ i_3$ and $\cong$ denotes quasi-isomorphism (since we work in the derived category of sheaf complexes). Recall that, by definition, $\mathcal{P}^* = \tau_{\leq 0} R_{i_4 *} \tau_{\leq 3} R_{i_3 *}\mathbb{Z}_{X_4}$. Now if $x \in X_1 = X_{4-3}$, then due to the pushforward, the cohomology of the stalk of $R_{i_3 *}\mathbb{Z}_{X_4}$ is that of the link T^2 (since $H^*((R_{i_3 *}\mathbb{Z}_{X_4})_x) \cong \varinjlim_{x\in U} \mathbb{H}^*(U - U\cap X^1;\mathbb{Z}) \cong H^*(U - U \cap X^1;\mathbb{Z}) \cong H^*(T^2)$, where for the last isomorphism we have used that $U - U\cap X^1$ is homotopy equivalent to T^2 since U can be assumed to be a distinguished neighborhood $\mathbb{R}^1 \times cT^2$). Thus the truncation $\tau_{\leq 3}$ occurs at a sufficiently large dimension that $\tau_{\leq 3} R_{i_3 *}\mathbb{Z}_{X_4}$ is quasi-isomorphic to $R_{i_3 *}\mathbb{Z}_{X_4}$. Then $\tau_{\leq 0} R_{i_4 *}\tau_{\leq 3} R_{i_3 *}\mathbb{Z}_{X_4} \cong \tau_{\leq 0} R_{i_4 *} R_{i_3 *}\mathbb{Z}_{X_4} \cong \tau_{\leq 0} Rj_*\mathbb{Z}_{X_4}$.

Now, we notice that X can also be stratified as a pseudomanifold with two strata $X \supset \tilde{X}^1$, where $\tilde{X}^1 = X^1 \cup X^0 \cong S^1$, since the double suspension of a space is the same thing as the join of that space with S^1. Using this stratification, $\tau_{\leq 0} Rj_*\mathbb{Z}_{X_4}$ is the perversity-$\bar{0}$ Deligne sheaf, so $\mathbb{H}^*(X;\tau_{\leq 0} Rj_*\mathbb{Z}_{X_4}) \cong I^{\bar{0}}H_{4-*}(X)$. Since X is a normal pseudomanifold, this is just $H^*(X;\mathbb{Z})$ (see [33, Section I.4.1]). $\square$

A Deligne sheaf for general perversities. In this section, we define a generalization of the sheaf truncation functor that, when substituted into the Deligne sheaf formula, yields a sheaf quasi-isomorphic to $\mathcal{I}^{\bar{p}}\mathcal{S}^*$ even if $\bar{p}$ is a general perversity. This truncation functor is a further generalization of the "truncations over a closed subset" functor presented in [29, Section 1.14] and attributed to Deligne; that functor is used in [29, Section 9] to study extensions of Verdier dual pairings in the context of classical perversity intersection homology.

Definition 3.2. Let $\mathcal{A}^*$ be a sheaf complex on X, and let $\mathfrak{F}$ be a locally-finite collection of subsets of X. Let $|\mathfrak{F}| = \cup_{V\in\mathfrak{F}} V$. Let P be a function $\mathfrak{F} \to \mathbb{Z}$. Define the presheaf $T^{\mathfrak{F}}_{\leq P}\mathcal{A}^*$ as follows. If U is an open set of X, then we let

$$T^{\mathfrak{F}}_{\leq P}\mathcal{A}^*(U) = \begin{cases} \Gamma(U;\mathcal{A}^*), & U\cap|\mathfrak{F}| = \emptyset, \\ \Gamma(U;\tau_{\leq \inf\{P(V) | V\in\mathfrak{F}, U\cap V\neq\emptyset\}}\mathcal{A}^*), & U\cap|\mathfrak{F}|\neq\emptyset. \end{cases}$$

Restriction is well-defined because if $m < n$ there is a natural inclusion $\tau_{\leq m}\mathcal{A}^* \hookrightarrow \tau_{\leq n}\mathcal{A}^*$.

Let the *generalized truncation sheaf* $\tau^{\mathfrak{F}}_{\leq P}\mathcal{A}^*$ be the sheafification of $T^{\mathfrak{F}}_{\leq P}\mathcal{A}^*$.

For maps $f : \mathcal{A}^* \to \mathcal{B}^*$ of sheaf complexes over X, we can define $\tau^{\mathfrak{F}}_{\leq P} f$ in the obvious way. In fact, $T^{\mathfrak{F}}_{\leq P} f$ is well-defined by applying the ordinary truncation functors on the appropriate subsets, and we obtain $\tau^{\mathfrak{F}}_{\leq P} f$ again by passing to limits in the sheafification process.

The following lemma contains the key facts we will need about the generalized truncation; they all follow immediately from the definition and the properties of the usual truncation $\tau_{\leq m}$.

Lemma 3.3. *1. $\tau_{\leq P}^{\mathfrak{F}}$ is a functor of sheaf complexes on X.*

2. *There is an inclusion of sheaves $\tau_{\leq P}^{\mathfrak{F}}\mathcal{A}^* \hookrightarrow \mathcal{A}^*$.*

3. *Suppose $\mathfrak{F}$ has the property that for each $V \in \mathfrak{F}$ and each $x \in V$, there is a neighborhood U of x such that $U \cap V' = \emptyset$ for each $V' \in \mathfrak{F}$ such that $V' \neq V$. Then $(\tau_{\leq P}^{\mathfrak{F}}\mathcal{A}^*)|_{X-|\mathfrak{F}|} = \mathcal{A}^*|_{X-|\mathfrak{F}|}$ and for each $V \in F$, $(\tau_{\leq P}^{\mathfrak{F}}\mathcal{A}^*)|_V = (\tau_{\leq P(V)}\mathcal{A}^*)|_V = \tau_{\leq P}(\mathcal{A}^*|_V)$.*

Remark 3.4. It follows from the last statement of the lemma that if $\mathfrak{F} = \{X\}$, then $\tau_{\leq P}^{\mathfrak{F}}\mathcal{A}^* = \tau_{\leq P(X)}\mathcal{A}^*$, which is a truncation in the usual sense.

Remark 3.5. $T_{\leq P}^{\mathfrak{F}}\mathcal{A}^*$ will not necessarily be a sheaf, so the sheafification in the definition is necessary. For example, let $X = \{v, w\}$ be the two point discrete set, and let each stalk of $\mathcal{A}^*$ be the the chain complex that is trivial except for the isomorphism $\mathbb{Z} \to \mathbb{Z}$ from degree 0 to degree 1. Now, let $\mathfrak{F} = \{\{w\}\}$ and let $P(\{w\}) = 0$. Consider $T_{\leq P}^{\mathfrak{F}}\mathcal{A}^*$. We have $T_{\leq P}^{\mathfrak{F}}\mathcal{A}^*(\{v\}) = \mathcal{A}^*(\{v\})$, but $T_{\leq P}^{\mathfrak{F}}\mathcal{A}^*(\{w\}) = T_{\leq P}^{\mathfrak{F}}\mathcal{A}^*(X) = 0$, since, on each of the latter sets, the kernel of $d : \mathcal{A}^0 \to \mathcal{A}^1$ is trivial. But this means that the trivial section of $T_{\leq P}^{\mathfrak{F}}\mathcal{A}^*$ over $\{w\}$ and the section that is 1 in degree 0 over $\{v\}$ cannot be pieced together into a global presheaf section. So $T_{\leq P}^{\mathfrak{F}}\mathcal{A}^*$ is not conjunctive and therefore not a sheaf.

Now, let X be an n-pseudomanifold, and recall the standard notation: We let X^k be the k-skeleton, $X_k = X^k - X^{k-1}$, $U_k = X - X_{n-k}$, and $i_k : U_k \hookrightarrow U_{k+1}$. Notice that $U_{k+1} = U_k \cup X_{n-k}$. Also, we will write $\tau_{\leq P}^{X_k}$, allowing X_k to stand for the set of connected components of X_k. We continue to allow the possibility that our pseudomanifolds possesses a codimension 1 stratum.

Given a fixed general perversity $\bar{p}$ and a local coefficient system $\mathcal{G}$ on $X - X^{n-1}$, let

$$\mathcal{Q}_{\bar{p}}^*(\mathcal{G}) = \tau_{\leq \bar{p}}^{X_0} Ri_{n*} \dots \tau_{\leq \bar{p}}^{X_{n-1}} Ri_{1*}\mathcal{G}.$$

This is the generalization of the Deligne sheaf using our generalized truncations instead of the usual truncations. We will often omit $\mathcal{G}$ from the notation and refer only to $\mathcal{Q}_{\bar{p}}^*$ or $\mathcal{Q}^*$ when the context is sufficient. We can now state and prove our main theorem of this section:

Theorem 3.6. *Given a general perversity $\bar{p}$ and a system of coefficients $\mathcal{G}$ on $U_1 = X - X^{n-1}$, $\mathcal{Q}_{\bar{p}}^*(\mathcal{G}_0 \otimes \mathcal{O}_0)$ is quasi-isomorphic to $\mathcal{I}^{\bar{p}}\mathcal{S}^*(\mathcal{G}_0)$, where $\mathcal{O}$ is the orientation sheaf on $X - X^{n-1}$ and $\mathcal{I}^{\bar{p}}\mathcal{S}^*(\mathcal{G}_0)$ is the sheaf of $\bar{p}$-perversity intersection chains with coefficients in $\mathcal{G}_0$.*

The proof consists of two propositions. The first tells us that $\mathcal{I}^{\bar{p}}\mathcal{S}^*$ satisfies the Deligne sheaf axioms with the perversity being replaced by a general perversity; the second tells us that any sheaf satisfying the axioms is quasi-isomorphic to the generalized Deligne sheaf.

The first proposition is mostly analogous to the standard case. We consider the following axioms $(AX1)_{\bar{p},\mathfrak{X},\mathcal{G}}$ for a sheaf $\mathcal{S}^*$, where $\mathfrak{X}$ refers to the fixed stratification on X:

1. $\mathcal{S}^*$ is bounded, $\mathcal{S}^i = 0$ for $i < 0$, and $\mathcal{S}^*|_{U_1} = \mathcal{G}$.

2. If $x \in Z$ for a stratum Z, then $H^i(\mathcal{S}^*_x) = 0$ for $i > \bar{p}(Z)$.

3. For $x \in Z \subset X_k$, the attachment map $\alpha_k : \mathcal{S}^*_{k+1} \to R_{i_k *}\mathcal{S}^*_k$ is a quasi-isomorphism at x up to $\bar{p}(Z)$.

Proposition 3.7. *For a general perversity $\bar{p}$, $\mathcal{I}^{\bar{p}}\mathcal{S}^*(\mathcal{G}_0)$ satisfies the axioms $(AX1)_{\bar{p},\mathfrak{X},\mathcal{G}\otimes\mathcal{O}}$.*

Proof. Most of these properties follow, essentially, from the work in [22], especially the cone formula, Proposition 2.1.

$\mathcal{I}^{\bar{p}}\mathcal{S}^i(\mathcal{G}_0) = 0$ for $i > n$, since these sheaves would be defined by singular chains in $I^{\bar{p}}C_{n-i}$, which is clearly trivial for $i > n$.

The condition $\mathcal{I}^{\bar{p}}\mathcal{S}^i = 0$ for $i < 0$ in property (1) is meant in the sense of the derived category so that only the cohomology has to vanish in this range. That this happens can be seen by induction on depth: If the depth of X is 0, X is a manifold, and for any $x \in X$, $H^*(\mathcal{I}^{\bar{p}}\mathcal{S}^*_x; \mathcal{G}_0) \cong H^\infty_{n-*}(\mathbb{R}^n; \mathcal{G})$, which is certainly 0 for $* < 0$. Now, assuming the condition of depth $d-1$, if X has depth d and $x \in X_{n-k}$, then $H^i(\mathcal{I}^{\bar{p}}\mathcal{S}^*(\mathcal{G}_0)_x) \cong I^{\bar{p}}H^\infty_{n-i}(\mathbb{R}^{n-k} \times cL; \mathcal{G}_0)$, which, depending on the perversity and i, is either 0 or $I^{\bar{p}}H_{k-1-i}(L; \mathcal{G}_0)$. By the induction hypothesis, the intersection sheaf $\mathcal{I}^{\bar{p}}\mathcal{S}^*_L(\mathcal{G}_0)$ on L is quasi-isomorphic to 0 in negative dimensions, which also implies by the hypercohomology spectral sequence that $I^{\bar{p}}H_{k-1-i}(L; \mathcal{G}_0) \cong \mathbb{H}^i(L; \mathcal{I}^{\bar{p}}\mathcal{S}^*_L)$ is 0 for $i < 0$. Thus $H^*(\mathcal{I}^{\bar{p}}\mathcal{S}^*(\mathcal{G}_0)_x) = 0$ for $* < 0$, and we are finished by induction.

Using excision, the restriction to U_1 is quasi-isomorphic to the sheaf complex of ordinary singular chains with coefficients in $\mathcal{G}$, which in turn is quasi-isomorphic to $\mathcal{G} \otimes \mathcal{O}$ since U_1 is an n-manifold (note that with our indexing conventions, degree 0 of a sheaf complex corresponds to degree n of the singular chain complex).

Finally, for $x \in Z \subset X_{n-k}$, $H^i(\mathcal{I}^{\bar{p}}S^*_x) \cong \varinjlim_{x\in U} I^{\bar{p}}H^\infty_{n-i}(U; \mathcal{G}_0) \cong I^{\bar{p}}H^\infty_{n-i}(\mathbb{R}^{n-k} \times cL; \mathcal{G}_0)$. Properties 2 and 3 now hold by applying the Künneth formula for products with $\mathbb{R}^{n-k}$ and Proposition 2.1 (the relation between the cone formula and the attaching map is explained more fully in [22]). $\square$

The second proposition makes use of our new generalized truncation.

Proposition 3.8. *For a general perversity $\bar{p}$, any sheaf complex satisfying the axioms $(AX1)_{\bar{p},\mathfrak{X},\mathcal{G}}$ is quasi-isomorphic to $\mathcal{Q}^*_{\bar{p}}(\mathcal{G})$.*

Proof. Substituting the appropriate $\tau^{\mathfrak{F}}_{\leq P}$ for $\tau_{\leq m}$, the proof is analogous to that in the usual case for GM perversities; see, e.g., [7, Theorem 2.5]. We run through the argument for completeness. Let $\mathcal{S}^*$ be any sheaf complex satisfying the axioms. We let $\mathcal{S}^*_k$ denote $\mathcal{S}^*|_{U_k}$.

We proceed by induction by showing that $\mathcal{S}^*_{k+1} \cong \tau^{X_{n-k}}_{\leq \bar{p}} Ri_{k*}\mathcal{S}^*_k$.

On U_1, by property (1), we know that $\mathcal{S}^*|_{U_1} \cong \mathcal{G} \cong \mathcal{Q}^*|_{U_1}$. This last quasi-isomorphism is easy to check by repeated application of the third property of Lemma 3.3 and the fact that $i^*Ri_*\mathcal{A}^*$ and $\mathcal{A}^*$ are always quasi-isomorphic when i is an inclusion of an open subset.

Using Lemma 3.3, we form the diagram

$$\begin{tikzcd}
\tau^{X_{n-k}}_{\leq \bar{p}} \mathcal{S}^*_{k+1} \arrow[r, "\tau^{X_{n-k}}_{\leq \bar{p}} \alpha_k"] \arrow[d] & \tau^{X_{n-k}}_{\leq \bar{p}} Ri_{k*}\mathcal{S}^*_k \arrow[d] \\
\mathcal{S}^*_{k+1} \arrow[r, "\alpha_k"] & Ri_{k*}\mathcal{S}^*_k.
\end{tikzcd}$$

The lefthand vertical map is the identity over U_k and a quasi-isomorphism over X_{n-k}, using the third property of Lemma 3.3 and the assumption that $\mathcal{S}^*$ satisfies condition (2) of the axioms. Furthermore, using property (3) of the axioms, $\tau^{X_{n-k}}_{\leq \bar{p}} \alpha_k$ is also a quasi-isomorphism: on U_k, it restricts to the identity (in the derived category), and on the stratum $Z \subset X_{n-k}$ it restricts to $\tau_{\leq \bar{p}(Z)}(\mathcal{S}^*_{k+1}|_Z) \to \tau_{\leq \bar{p}(Z)}((Ri_{k*}\mathcal{S}^*_k)|_Z)$, which again is a quasi-isomorphism because α_k is a quasi-isomorphism over Z for $* \leq \bar{p}(Z)$ by assumption. Thus, utilizing the left side and the top of the diagram, we obtain a quasi-isomorphism between $\mathcal{S}^*_{k+1}$ and $\tau^{X_{n-k}}_{\leq \bar{p}} Ri_{k*}\mathcal{S}^*_k$.

The proposition follows by induction. $\square$

Together, the propositions prove Theorem 3.6. $\square$

Proposition 3.8 and Theorem 3.6 together yield the following corollary, which is a version of Lemma 2.4 in the sheaf world. This corollary says that any generalized Deligne sheaf is equivalent to an efficient-perversity Deligne sheaf.

Corollary 3.9. *For any general perversity $\bar{p}$ and coefficient system $\mathcal{G}$ on $X - X^{n-1}$, $\mathcal{Q}^*_{\bar{p}}(\mathcal{G})$ is quasi-isomorphic to $\mathcal{Q}^*_{\check{p}}(\mathcal{G})$, where $\check{p}$ is the efficient perversity associated to $\bar{p}$.*

3.1 Comparison with intermediate extensions

The machinery of perverse sheaves developed axiomatically by Beilinson, Bernstein, and Deligne in [5] also contains a method for creating sheaf complexes that satisfy the intersection homology axioms $AX1$. For background on perverse sheaves, we refer the reader to [5], [37, Chapter X], [2, Chapter 7], [6], and [18, Chapter 5].

Let $U \subset X$ be an open subset of X that is a union of strata, let $i : U \hookrightarrow X$ be the inclusion, and let $\mathcal{S}^*$ be a P-perverse sheaf on U for some general perversity P. Then there is defined axiomatically in [5] the "intermediate extension functor" $i_{!*}$ such that $i_{!*}\mathcal{S}^*$ is the unique extension in the category of P-perverse sheaves of $\mathcal{S}^*$ to X (meaning that the restriction of $i_{!*}\mathcal{S}^*$ to U is quasi-isomorphic to $\mathcal{S}^*$) such that for each stratum $Z \subset X - U$ and inclusion $j : Z \hookrightarrow X$,we have $\mathcal{H}^k(j^*i_{!*}\mathcal{S}^*) = 0$ for $k \geq P(Z)$ and $\mathcal{H}^k(j^!i_{!*}\mathcal{S}^*) = 0$ for $k \leq P(Z)$. We refer the reader to [5, Section 1.4] or [18, Section 5.2] for the precise definition of the functor $i_{!*}$.

In particular, suppose we let $U = X - X^{n-1}$, that $\mathcal{S}^*$ is just the local system $\mathcal{G}$, and that $\bar{p}$ is a general perversity on X. The sheaf $\mathcal{G}$ is certainly P-perverse on U with respect to the perversity $P(U) = 0$. Now let $P(Z) = \bar{p}(Z) + 1$. It follows that for each singular stratum inclusion $j : Z \hookrightarrow X$, we have $\mathcal{H}^k(j^*i_{!*}\mathcal{G}) = 0$ for $k > \bar{p}(Z)$ and $\mathcal{H}^k(j^!i_{!*}\mathcal{G}) = 0$

for $k \leq \bar{p}(Z) + 1$. In the presence of the first condition, the second condition is equivalent to the attachment map at the stratum Z being an isomorphism up through degree $\bar{p}(Z)$; see [7, page 87]. But according to the axioms $AX1$, these conditions are satisfied by the perversity $\bar{p}$ Deligne sheaf, which is also easily seen to be P-perverse. Thus, since $i_{!*}\mathcal{G}$ is the unique extension of $\mathcal{G}$ with these properties, $i_{!*}\mathcal{G}$ is none other than the Deligne sheaf (up to quasi-isomorphism). Thus we can also think of the generalized Deligne process as a means to provide a concrete realization of $i_{!*}\mathcal{G}$ (which is defined quite abstractly and axiomatically in [5]).

4 Constructibility and duality

Having established the equivalence in the realm of perverse sheaves between generalized Deligne sheaves and intermediate extensions of coefficient systems, constructibility and duality results follow (at least with field coefficients) from the abstract perverse sheaf machinery of [5]. However, we will provide (at least a sketch of) more direct proofs in the spirit of Goresky-MacPherson [29] and Borel [7].

4.1 Constructibility

We refer to [7, Section V.3] for the definitions of $\mathfrak{X}$-clc ($\mathfrak{X}$-cohomologically locally constant), $\mathfrak{X}$-cc ($\mathfrak{X}$-cohomologically constructible), and cc (cohomologically constructible), where $\mathfrak{X}$ refers to the stratification of X. These concepts are used briefly in the proof of Theorem 4.3, below, so the following proposition is worth having, but overall these concepts play a minor role here, so we choose not to go into too much detail. We assume that the coefficient system $\mathcal{G}$ on $X - X^{n-1}$ is a system of finitely-generated R-modules for a Noetherian ring R of finite cohomological dimension.

Proposition 4.1. *Any sheaf satisfying $AX1_{\bar{p},\mathfrak{X},\mathcal{G}}$ with respect to the general perversity $\bar{p}$ is constructible. More particularly, it is $\mathfrak{X}$-clc and $\mathfrak{X}$-cc with respect to the given stratification $\mathfrak{X}$ of X, and it is cc.*

Sketch of proof. This follows by induction over the strata just as for the traditional case (e.g. [7, Proposition 3.12]), which relies nowhere on the particular form of the perversity. We refer the reader there for thorough details. We simply note that, just as in that proof, it suffices for us, by Proposition 3.7 and 3.8, to consider only the Deligne sheaves $\mathcal{Q}^*$, and then it follows by general sheaf theory and the properties of the pseudomanifold X that the functor Ri_{k*} preserves the property of being $\mathfrak{X}$-cc [7, Corollary V.3.11.iii]. Similarly, it is not hard to see that being $\mathfrak{X}$-cc is preserved by our generalized truncation functors, using Lemma 3.3. $\square$

4.2 Duality

In this section, we show that the generalized Deligne sheaves with complementary general perversities are Verdier dual to each other. This implies Goresky-MacPherson duality on intersection homology for complementary general perversities. In fact, we will consider the more general duality with coefficients in a principal ideal domain considered by Goresky and Siegel [30]. For this, we need a definition.

Definition 4.2. Let R be a PID, and let $\mathcal{E}$ be a local coefficient system on $X - X^{n-1}$ of finitely-generated free modules over R. We say that the pseudomanifold X is *locally ($\bar p$,$\mathcal{E}$)-torsion free* if for all singular strata Z and each $x \in Z$, $I^{\bar p}H^c_{\operatorname{codim}(Z)-2-\bar p(Z)}(L_x; \mathcal{E}_0)$ is R-torsion free, where L_x is the link of x in X.

This definition is a direct analogue of the definition of locally $\bar p$-torsion free in Goresky-Siegel [30]. Note that any X is automatically locally ($\bar p$,$\mathcal{E}$)-torsion free if R is a field.

Let $\mathcal{Q}^*_{\bar p}(\mathcal{E})$ be the generalized Deligne sheaf of perversity $\bar p$ with coefficients $\mathcal{E}$. Let $\bar t$ be the top perversity, $\bar t(Z) = \operatorname{codim}(Z) - 2$. Let $\mathcal{D}_Y$ denote the Verdier dualizing functor over the space Y - see [7, 29, 2] for details. Recall also that $A^*[n]$ denotes the shifted complex such that $(A^*[n])^i = A^{i+n}$.

Theorem 4.3. *Let $\bar p$ and $\bar q$ be general perversities such that $\bar p + \bar q = \bar t$, and let X be an n-dimensional stratified pseudomanifold that is locally $(\bar p, \mathcal{E})$-torsion free. Let $U_1 = X - X^{n-1}$. Then $\mathcal{D}_X \mathcal{Q}^*_{\bar p}(\mathcal{E})[-n] \cong \mathcal{Q}^*_{\bar q}(\mathcal{D}_{U_1}\mathcal{E}[-n])$ in the derived category of sheaves on X.*

This is the usual duality statement in intersection homology (see [7, Theorem V.9.8]). The principal novelty is the generality of the perversities and the acceptability of codimension one strata.

Proof. The standard proof for GM perversities, given in [29] and [7] proceeds by showing that $\mathcal{D}_X \mathcal{Q}^*_{\bar p}(\mathcal{E})[-n]$ satisfies the axioms $AX2_{\bar q, \mathfrak{X}, \mathcal{D}_{U_1}\mathcal{E}[-n]}$, which, when $\bar p$ and $\bar q$ are GM perversities, are equivalent to $AX1_{\bar q, \mathfrak{X}, \mathcal{D}_{U_1}\mathcal{E}[-n]}$. However, when $\bar p$ and $\bar q$ are general perversities, the axioms $AX2$ are no longer well-formulated. In particular, when $\bar p$ is a GM perversity, $AX2$ utilizes the notion $\bar p^{-1}(j) = \min\{c \mid \bar p(c) \geq j\}$ and is formulated in terms of conditions such as "$\dim \operatorname{supp}(\mathcal{H}^j\mathcal{S}^*) \leq n - \bar p^{-1}(j)$ for all $j > 0$." But when we allow $\bar p$ to take different values on different strata of the same dimension, it is no longer clear that such an equivalent formulation of these axioms is possible. Thus instead we will show that $\mathcal{D}_X \mathcal{Q}^*_{\bar p}(\mathcal{E})[-n]$ satisfies the axioms $AX1_{\bar q, \mathfrak{X}, \mathcal{D}_{U_1}\mathcal{E}[-n]}$, which suffice by Propositions 3.8. This will be similar to the proof of our main theorem of [23], though simpler since the context there called for much more general spaces.

Let $\tilde{\mathcal{E}} = \mathcal{E} \otimes \mathcal{O}$, where $\mathcal{O}$ is the orientation sheaf on $X - X^{n-1}$. Using Lemma 2.4 and Corollary 3.9 and noting that if $\bar p + \bar q = \bar t$, then $\check p + \check q = \bar t$, where $\check p, \check q$ are the efficient perversities associated to $\bar p, \bar q$, it suffices to prove the theorem when $\bar p$ and $\bar q$ are efficient. By Propositions 3.7 and 3.8, we are free to interpret $\mathcal{Q}^*_{\bar p}(\mathcal{E})$ as either the generalized Deligne sheaf or the sheaf of intersection chains $\mathcal{I}^{\bar p}\mathcal{S}^*(\tilde{\mathcal{E}}_0)$, since we have seen that these sheaves are quasi-isomorphic, making them equivalent in the derived category in which we work. It will be useful to be able to use both points of view. Since $\mathcal{Q}^*_{\bar p}(\mathcal{E})$ is $\mathfrak{X}$-cc and cc by Proposition 4.1, $\mathcal{D}_X \mathcal{Q}^*_{\bar p}(\mathcal{E})$ is also $\mathfrak{X}$-cc and cc by [7, Corollary V.8.7 and Proposition V.3.10.e]. In particular, $\mathcal{D}_X \mathcal{Q}^*_{\bar p}(\mathcal{E})$ is $\mathfrak{X}$-clc, which is part of the definition of $\mathfrak{X}$-cc (see [7, Section V.3]).

As we work in the derive category, we are free to replace the condition $\mathcal{S}^i = 0$ for $i < 0$ in axiom (1) with the condition $\mathcal{H}^*(\mathcal{S}^*) = 0$ for $i < 0$ - see [7, Remark V.2.7.b].

The coefficient part of axiom (1) follows by the usual arguments: Let $U_1 = X - X^{n-1}$, and let $i : U_1 \hookrightarrow X$ be the inclusion. Since this is an inclusion of an open set $i^* = i^!$ [2, p. 62], and, so $i^*\mathcal{D}_X = i^!\mathcal{D}_X = \mathcal{D}_{U_1} i^*$ [2, Proposition 3.4.5]. Thus the restriction of $\mathcal{D}_X \mathcal{Q}^*_{\bar{p}}(\mathcal{E})[-n]$ to U_1 is $(\mathcal{D}_{U_1} i^* \mathcal{Q}^*_{\bar{p}}(\mathcal{E}))[-n]$, which is quasi-isomorphic to $(\mathcal{D}_{U_1}\mathcal{E})[-n]$, using axiom (1) for $\mathcal{Q}^*_{\bar{p}}(\mathcal{E})$.

Next, let $x \in Z \subset X^{n-k} - X^{n-k-1}$ and consider $H^*(\mathcal{D}^*_X(\mathcal{I}^{\bar{p}}\mathcal{S}^*(\tilde{\mathcal{E}}_0))_x[-n])$, which is isomorphic to $\varinjlim_{x\in U} \mathbb{H}^{*-n}(U; \mathcal{D}^*_x(\mathcal{I}^{\bar{p}}\mathcal{S}^*(\tilde{\mathcal{E}}_0)))$. For any sheaf complex $\mathcal{A}^*$ over R in the derived category $D^b(X)$ and any open set $U \in X$, we have an exact sequence

$$0 \to \mathrm{Ext}(\mathbb{H}^{i+1}_c(U; \mathcal{A}^*), R) \to \mathbb{H}^{-i}(U; \mathcal{D}_X\mathcal{A}^*) \to \mathrm{Hom}(\mathbb{H}^i_c(U; \mathcal{A}^*), R) \to 0 \tag{1}$$

(see [2, Section 3.4]). Thus taking $\mathcal{A}^* = \mathcal{I}^{\bar{p}}\mathcal{S}^*(\tilde{\mathcal{E}}_0)$ and shifting indices, there is an exact sequence

$$0 \to \mathrm{Ext}(IH^c_{*-1}(U; \tilde{\mathcal{E}}_0), R) \to \mathbb{H}^{*-n}(U; \mathcal{D}^*(\mathcal{I}^{\bar{p}}\mathcal{S}^*(\tilde{\mathcal{E}}_0))) \to \mathrm{Hom}(IH^c_*(U; \tilde{\mathcal{E}}_0), R) \to 0.$$

Since the distinguished neighborhoods of any point in a pseudomanifold constitute a cofinal system of neighborhoods, we may assume that $U \cong \mathbb{R}^{n-k} \times cL^{k-1}$. Now, we may employ the the Künneth formula with $\mathbb{R}^{n-k}$ and the cone formula of Proposition 2.1 to conclude that $IH^c_*(U; \mathcal{E}) = 0$ for $* > k - 2 - \bar{p}(Z) = k - 2 - (k - 2 - \bar{q}(Z)) = \bar{q}(Z)$. Furthermore, since X is locally ($\bar{p}$,$\mathcal{E}$)-torsion free, $\mathrm{Ext}(IH^c_{k-2-\bar{p}(Z)}(U; \mathcal{E}), R)$ is also 0, so $\mathbb{H}^{*-n}(U; \mathcal{D}^*(\mathcal{I}^{\bar{p}}\mathcal{S}^*(\tilde{\mathcal{E}}_0))) = 0$ for $* > \bar{q}(Z)$. It is also clear that these groups must be 0 for $* < 0$. This completes the demonstration of axioms (1) and (2).

Finally, we must verify the attaching axiom, axiom (3). If $x \in Z$, this axiom is equivalent to the condition that $H^i(\mathcal{S}^*)_x \cong \varinjlim_{x\in U} \mathbb{H}^i(U - U \cap Z; \mathcal{S}^*)$ for all $i \le \bar{p}(Z)$ (see [7, V.1.7], [2, Section 4.1.4]), where U runs over all open neighborhoods of x. Again, we can limit ourselves to a cofinal system of distinguished neighborhoods, and it suffices to find then isomorphisms $\mathbb{H}^j(U; \mathcal{D}^*(\mathcal{I}^{\bar{p}}\mathcal{S}^*(\tilde{\mathcal{E}}_0))[-n]) \to \mathbb{H}^j(U - U \cap Z; \mathcal{D}^*(\mathcal{I}^{\bar{p}}\mathcal{S}^*(\tilde{\mathcal{E}}_0))[-n])$ that are functorial in that they commute with further restrictions $U \to V$. By [23, Appendix], there is a map of short exact sequences

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Ext}(\mathbb{H}_c^{n-j+1}(U;\mathcal{I}^{\bar p}\mathcal{S}(\tilde{\mathcal{E}}_0)),R) \arrow[r] \arrow[d] & \mathrm{Ext}(\mathbb{H}_c^{n-j+1}(U-U\cap Z;\mathcal{I}^{\bar p}\mathcal{S}^*(\tilde{\mathcal{E}}_0)),R) \arrow[d] \\
\mathbb{H}^j(U;\mathcal{D}_X^*(\mathcal{I}^{\bar p}\mathcal{S}^*(\tilde{\mathcal{E}}_0))[-n]) \arrow[r] \arrow[d] & \mathbb{H}^j(U-U\cap Z;\mathcal{D}_X^*(\mathcal{I}^{\bar p}\mathcal{S}^*(\tilde{\mathcal{E}}_0))[-n]) \arrow[d] \\
\mathrm{Hom}(\mathbb{H}_c^{n-j}(U;\mathcal{I}^{\bar p}\mathcal{S}(\tilde{\mathcal{E}}_0)),R) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathbb{H}_c^{n-j}(U-U\cap Z;\mathcal{I}^{\bar p}\mathcal{S}^*(\tilde{\mathcal{E}}_0)),R) \arrow[d] \\
0 & 0
\end{tikzcd},
$$

where the maps of the top and bottom terms are induced by the inclusion maps $I^{\bar p}H_j^c(U-U\cap X_{n-k};\tilde{\mathcal{E}}_0)\to I^{\bar p}H_j^c(U;\tilde{\mathcal{E}}_0)$. Again since $U\cong\mathbb{R}^{n-k}\times cL$, we can use the Künneth theorem with $\mathbb{R}^{n-k}$ to obtain that $\mathbb{H}_c^{n-*}(U-U\cap Z;\mathcal{I}^{\bar p}\mathcal{S}(\tilde{\mathcal{E}}_0))\cong I^{\bar p}H_*^c(U-U\cap Z;\tilde{\mathcal{E}}_0)$ is isomorphic to $I^{\bar p}H_*^c(L;\tilde{\mathcal{E}}_0)$ and $\mathbb{H}_c^{n-*}(U;\mathcal{I}^{\bar p}\mathcal{S}(\tilde{\mathcal{E}}_0))\cong I^{\bar p}H_*^c(U;\tilde{\mathcal{E}}_0)\cong I^{\bar p}H_*^c(cL;\tilde{\mathcal{E}}_0)$. By the cone formula, the inclusion $I^{\bar p}H_*^c(L;\tilde{\mathcal{E}}_0)\to I^{\bar p}H_*^c(cL;\tilde{\mathcal{E}}_0)$ is an isomorphism for $*<k-1-\bar p(Z)$. Thus, by the five lemma, $\mathbb{H}^j(U;\mathcal{D}_X(\mathcal{I}^{\bar p}\mathcal{S}(\tilde{\mathcal{E}}_0))[-n])\to\mathbb{H}^j(U-U\cap Z;\mathcal{D}_X(\mathcal{I}^{\bar p}\mathcal{S}^*(\tilde{\mathcal{E}}_0))[-n])$ is an isomorphism for $j\le\bar q(Z)$. Since this computation is functorial with respect to restrictions, we obtain the desired isomorphism in the limits.

Thus $\mathcal{D}^*(\mathcal{I}^{\bar p}\mathcal{S}^*(\tilde{\mathcal{E}}))[-n]$ satisfies the axioms $AX1_{\bar q,X}(\mathcal{D}^*_{U_1}(\mathcal{E})[-n])$, which completes the proof of the theorem. $\square$

Corollary 4.4. *Let X be a locally ($\bar p$,$\mathcal{E}$)-torsion free n-dimensional stratified pseudomanifold, possibly with codimension one strata, where $\mathcal{E}$ is a local coefficient system on $X-X^{n-1}$ of finitely-generated free modules over the principal ideal domain R. Let $\mathcal{O}$ be the orientation sheaf of the n-manifold $X-X^{n-1}$. Let $\bar p$ and $\bar q$ be dual efficient perversities ($\bar p(Z)+\bar q(Z)=$ codim$(Z)-2$). Let TH_* and FH_* denote, respectively, the R-torsion subgroup and R-torsion free quotient group of IH_*, and let $Q(R)$ denote the field of fractions of R. Then,*

1. *Suppose that* $\mathrm{Hom}(T^{\bar p}H^c_{i-1}(X;\mathcal{E}_0),Q(R)/R)$ *is a torsion R-module (in particular, if $T^{\bar p}H^c_{i-1}(X;\mathcal{E}_0)$ is finitely generated). Then*

$$\mathrm{Hom}(F^{\bar p}H_i^c(X;\mathcal{E}_0),R)\cong F^{\bar q}H^\infty_{n-i}(X;Hom(\mathcal{E},R_{X-X^{n-1}})_0\otimes\mathcal{O}_0)$$

and

$$\mathrm{Hom}(T^{\bar{p}}H^c_{i-1}(X;\mathcal{E}_0), Q(R)/R) \cong T^{\bar{q}}H^{\infty}_{n-i}(X; Hom(\mathcal{E}, R_{X-X^{n-1}})_0 \otimes \mathcal{O}_0).$$

2. *If $\mathcal{E}$ is a local coefficient system on $X - X^{n-1}$ of finitely-generated $\mathbb{F}$-modules for a field $\mathbb{F}$, then*
$$\mathrm{Hom}(I^{\bar{p}}H^c_{n-i}(X;\mathcal{E}_0); \mathbb{F}) \cong I^{\bar{q}}H^{\infty}_{i}(X; Hom(\mathcal{E}, \mathbb{F}_{X-X^{n-1}})_0 \otimes \mathcal{O}_0).$$

3. *When X is compact and orientable, we obtain as a special case the simpler, but more familiar, special case*
$$\mathrm{Hom}(I^{\bar{p}}H_i(X;\mathbb{Q}_0), \mathbb{Q}) \cong I^{\bar{q}}H_{n-i}(X;\mathbb{Q}_0).$$
If, in addition, X is locally $(\bar{p},\mathbb{Z})$-torsion free, we have
$$\mathrm{Hom}(F^{\bar{p}}H_i(X;\mathbb{Z}_0), \mathbb{Z}) \cong F^{\bar{q}}H_{n-i}(X;\mathbb{Z}_0) \qquad \text{and} \qquad \mathrm{Hom}(T^{\bar{p}}H_i(X;\mathbb{Z}_0), \mathbb{Q}/\mathbb{Z}) \cong T^{\bar{q}}H_{n-i}(X;\mathbb{Z}_0).$$

Proof. These statements follow direction from the theorem, using the universal coefficient short exact sequence (1). See [7, 30, 23] for more details. □

Intersection Lefschetz duality. These duality theorems imply easy proofs of Lefschetz-type duality theorems on the intersection homology of pseudomanifolds with boundary. The proof is reminiscent of the standard proof that intersection homology duality implies classical Lefschetz duality on manifolds.

Let X be a compact oriented n-pseudomanifold with collared boundary ∂X (we leave the non-compact and/or non-orientable cases to the reader). In other words, $X - \partial X$ is a pseudomanifold as defined above, ∂X is an $n-1$ pseudomanifold, and there is a neighborhood of ∂X in X stratified homeomorphic to $\partial X \times [0,1)$, where $[0,1)$ is unstratified. We do not consider ∂X to be a codimension one stratum of X, but rather we let it inherit its stratification from that of X. This is the traditional convention for discussing pseudomanifold bordism; see e.g. Siegel [49]. Then our duality theorems can be interpreted to give a duality between $I^{\bar{p}}H_*(X)$ and $I^{\bar{q}}H_*(X,\partial X)$ (where "duality" is interpreted as providing a set of results akin to those in Corollary 4.4, i.e. a set of intersection pairings and linking pairings between the relevant torsion subgroups and torsion-free quotients across the appropriate dimensions, provided X is appropriately locally torsion-free).

To see this, let $\hat{X} = X \cup_{\partial X} \bar{c}\partial X$, the space obtained by adjoining to X a cone on the boundary (or, equivalently, pinching the boundary to a point). Let v denote the vertex of the cone point. Let $\bar{p}$, $\bar{q}$ be dual general perversities on X, and let $\bar{p}_-$, $\bar{q}_+$ be the dual perversities on $\hat{X}$ such that $\bar{p}_-(Z) = \bar{p}(Z)$ and $\bar{q}_+(Z) = \bar{q}(Z)$ for each stratum Z of X, $\bar{p}_-(\{v\}) = -2$, and $\bar{q}_+(\{v\}) = n$. Our duality theorems provide a duality, in the above sense, between $I^{\bar{p}_-}H_*(\hat{X})$ and $I^{\bar{q}_+}H_*(\hat{X})$. But now we simply observe that $I^{\bar{p}_-}H_*(\hat{X}) \cong I^{\bar{p}_-}H_*(\hat{X} - \{v\}) \cong I^{\bar{p}}H_*(X)$, because the perversity condition at v ensures that no singular simplex of a relevant dimension may intersect v. On the other hand, since $I^{\bar{q}_+}H_*(c\partial X) = 0$ by Proposition 2.1, $I^{\bar{q}_+}H_*(\hat{X}) \cong I^{\bar{q}_+}H_*(\hat{X}, \bar{c}\partial X)$ by the long exact sequence of the pair, but $I^{\bar{q}_+}H_*(\hat{X}, c\partial X) \cong I^{\bar{q}_+}H_*(X,\partial X) \cong I^{\bar{q}}H_*(X,\partial X)$ by excision.

Classical Lefschetz duality results via codimension one strata. Another interesting feature of our extended duality theorems is that they provide a proof of the Lefschetz duality theorem for manifolds with boundary M *without* needing to employ the added vertex trick of the preceding paragraphs. This is not a direct feature of traditional intersection homology theory. More generally, we obtain duality results of Lefschetz type for pseudomanifolds with codimension one strata; these are not available in traditional intersection homology.

Let us see how this works for a manifold with boundary. Let M be a compact oriented n-manifold with collared boundary, and let $(M, \partial M)$ provide the stratification. For simplicity, assume that ∂M is connected, though we will loosen this condition in a moment. Then there are only two efficient perversity possibilities for ∂M, say $\bar{p}(\partial M) = 0$ and $\bar{q}(\partial M) = -1$, and these perversities are dual because $-1 = \bar{p}(\partial M) + \bar{q}(\partial M) = \bar{t}(\partial M) = \operatorname{codim}(\partial M) - 2 = -1$.

Now we claim that the dual intersection homology groups $I^{\bar{q}}H_*(M; R_0)$ and $I^{\bar{p}}H_*(M; R_0)$ are respectively isomorphic to $H_*(M; R)$ and $H_*(M, \partial M; R)$.

First, we note that, by the proof of Lemma 2.4, $I^{\bar{q}}C_*(M; R_0)$ is quasi-isomorphic to $I^{\bar{q}'}C_*(M; R_0)$, where $\bar{q}'(\partial M) = m$ for any negative integer m. It follows then easily from the definition of allowability that $I^{\bar{q}}C_*(M; R_0)$ is quasi-isomorphic to $C^c_*(M - \partial M; R)$, which is quasi-isomorphic to $C_*(M; R)$ by homotopy equivalence. Secondly, it also follows Lemma 2.4 that $I^{\bar{p}}C_*(M; R_0)$ is quasi-isomorphic to $I^{\bar{p}'}C_*(M; R_0)$, where $\bar{p}'(\partial M)$ is any non-negative integer. In particular, taking $\bar{p}'(\partial M) = 1$, all chains become allowable, and we see that $I^{\bar{p}'}C_*(M; R_0) \cong C_*(M; R_0) \cong C_*(M, \partial M; R)$. This last isomorphism is discussed further in Section 5.1.

Hence the duality of intersection homology becomes the dual pairing between $H_*(M; R)$ and $H_*(M, \partial M; R)$.

It is not hard to generalize this reasoning to see that we can also obtain the Lefschetz pairing in its stronger form: Suppose M is a compact oriented n-manifold with boundary $M = M_1 \cup M_2$, where each M_i is a union of connected components of the boundary. Suppose that $\bar{p}(Z) = 0$ if Z is a connected component of M_1 and $\bar{p}(Z) = -1$ if Z is a connected component of M_2. Let $\bar{q} = \bar{t} - \bar{p}$. Then the duality between $I^{\bar{p}}H_i(M; R_0)$ and $I^{\bar{q}}H_j(M; R_0)$ translates into the duality between $H_*(M, M_1; R)$ and $H_*(M; M_2; R)$.

Remark 4.5. An anonymous referee has points pointed out an alternative proof of Lefschetz duality for intersection homology. The author is unaware of this proof in the literature, so it is reproduced here for the interested reader:[8]

> Let $(X^n, \partial X)$ be a compact oriented pseudomanifold with boundary, $i : X - \partial X \subset X$, $j : \partial X \subset X$. For $\mathcal{S}^*$ a sheaf complex on X, the distinguished triangle
>
> $$i_! i^* \mathcal{S}^* \to \mathcal{S}^* \to Rj_* j^* \mathcal{S}^* \xrightarrow{+1}$$
>
> induces on hypercohomology the exact sequence of the pair $(X, \partial X)$ with coefficients in $\mathcal{S}^*$:
>
> $$\cdots \mathbb{H}^k(X, \partial X; \mathcal{S}^*) \to \mathbb{H}^k(X; \mathcal{S}^*) \to \mathbb{H}^k(\partial X; \mathcal{S}^*) \to \mathbb{H}^{k+1}(X, \partial X; \mathcal{S}^*) \cdots .$$

[8] We have made some changes to notation for consistency with the present paper, but otherwise we quote directly.

Let $\mathcal{I}^{\bar{p}}C^*$ and $\mathcal{I}^{\bar{q}}\mathcal{C}^*$ be complementary [perversity] intersection chain sheaves on the interior of $X - \partial X$. Setting $\mathcal{S}^*_{\bar{p}} = Ri_*\mathcal{I}^{\bar{p}}C^*$ and $\mathcal{S}^*_{\bar{q}} = Ri_*\mathcal{I}^{\bar{q}}C^*$, we get

$$\begin{aligned}\mathbb{H}^k(X, \partial X; \mathcal{S}^*_{\bar{p}}) &= \mathbb{H}^k(i_! i^* \mathcal{S}^*_{\bar{p}}) \\ &= \mathbb{H}^k(i_! \mathcal{I}^{\bar{p}}C^*) \\ &= \mathbb{H}^k(i_! DI^{\bar{q}}C^*[-n]) \\ &= \mathbb{H}^{k-n}(D(i_*\mathcal{I}^{\bar{q}}C^*)) \\ &= \mathrm{Hom}(\mathbb{H}^{n-k}(X; \mathcal{S}^*_{\bar{q}}), \mathbb{Q}),\end{aligned}$$

which is the desired Poincaré-Lefschetz result.

4.3 Sheaf pairings

In this section, we present the following theorems.

Theorem 4.6. *Given a pairing of local systems $m_1 : \mathcal{E} \otimes \mathcal{F} \to \mathcal{G}$ on $X - X^{n-1}$ and general perversities such that $\bar{p}(Z) + \bar{q}(Z) \leq \bar{r}(Z)$ for all singular strata Z, then in the bounded derived category $D^b(X)$, there is a unique morphism $m : \mathcal{Q}^*_{\bar{p}}(\mathcal{E}^*) \overset{L}{\otimes} \mathcal{Q}^*_{\bar{q}}(\mathcal{F}) \to \mathcal{Q}^*_{\bar{r}}(\mathcal{G})$ that restricts to m_1 on $X - X^{n-1}$. Furthermore, if $\bar{r} \leq \bar{t}$ and $\mathcal{G} = \mathcal{O}$, the orientation sheaf of $X - X^{n-1}$, then there exists a pairing $m : \mathcal{Q}^*_{\bar{p}}(\mathcal{E}^*) \overset{L}{\otimes} \mathcal{Q}^*_{\bar{q}}(\mathcal{F}) \to \mathfrak{D}^*_X[-n]$ that restricts to m_1 on $X - X^{n-1}$, where $\mathfrak{D}^*_X[-n]$ is the shift of the Verdier dualizing complex on X.*

Theorem 4.7. *If $\bar{q} = \bar{t} - \bar{p}$, then the pairing on intersection homology induced by the pairings of Theorem 4.6 coincide with those of Corollary 4.4.*

Remark 4.8. Note that we do not require the perversities in these theorems to be efficient, and even if $\bar{p}$ and $\bar{q}$ are efficient, $\bar{p} + \bar{q}$ may not be. Of course in the statements we can always replace, for example, $\mathcal{Q}^*_{\bar{r}}$ by $\mathcal{Q}^*_{\hat{r}}$, using Corollary 3.9.

We will explain in Remark 4.10 why we must have $\bar{r} \leq \bar{t}$ in the second statement of Theorem 4.6.

Once again, the details of the proofs are mostly those that can be found already in Chapter V.9.C of Borel's book [7]. In this case, we will leave it to the reader to verify that Borel's treatment holds in our context. We simply note that the crucial point of generalization is the following modification to [7, Lemma V.9.1].

Lemma 4.9. *Let Y be a topological space and $\mathcal{A}^* \in D(Y)$, the derived category of sheaves on Y. Let $\mathfrak{F}$ be a locally-finite collection of subsets of Y, and let P be a function $\mathfrak{F} \to \mathbb{Z}$. Suppose that the natural inclusion $\tau^{\mathfrak{F}}_{\leq P}\mathcal{A}^* \hookrightarrow A^*$ is a quasi-isomorphism. Then,*

1. *for any $\mathcal{B}^* \in D(Y)$, the natural homomorphism*

$$Mor_{D(Y)}(\mathcal{A}^*, \tau^{\mathfrak{F}}_{\leq P}\mathcal{B}^*) \to Mor_{D(Y)}(\mathcal{A}^*, \mathcal{B}^*)$$

is an isomorphism, and

2. if $U = Y - |\mathfrak{F}|$ is an open subset of Y and $i : U \hookrightarrow Y$ is the inclusion, then for any $\mathcal{B}^* \in D(Y)$,

$$Mor_{D(Y)}(\mathcal{A}^*, \tau^{\mathfrak{F}}_{\leq P} Ri_*\mathcal{B}^*) \to Mor_{D(Y)}(i^*\mathcal{A}^*, \mathcal{B}^*)$$

is an isomorphism.

Proof. The proof is the same as that of [7, Lemma V.9.1], utilizing the properties of $\tau^{\mathfrak{F}}_{\leq P}$ in place of those of the standard truncation. $\square$

As in [7], one used the lemma to extend the relevant morphisms by induction and uniquely from the coefficient pairings on $X - X^{n-1}$ up to the full pairings on X.

Remark 4.10. In order for these extension arguments to work, the hypotheses of Lemma 4.9 must be satisfied in the appropriate contexts. The reason that we must have $\bar{r} \leq \bar{t}$ in the second statement of Theorem 4.6 is that we cannot apply Lemma 4.9 when $\bar{p} + \bar{q} > \bar{t}$ and the desired codomain is $\mathfrak{D}^*_X[-n]$.

In slightly more detail, the idea of the proof of Theorem 4.6 is to show that on a given $U_{k+1} = X - X^{n-k-1}$, $(\mathcal{Q}^*_{\bar{p}} \overset{L}{\otimes} \mathcal{Q}^*_{\bar{q}})|_{U_{k+1}}$ will be quasi-isomorphic to $\tau^{X_{n-k}}_{\leq \bar{r}}(\mathcal{Q}^*_{\bar{p}} \overset{L}{\otimes} \mathcal{Q}^*_{\bar{q}})_{U_{k+1}}$ for any $\bar{r} \geq \bar{p} + \bar{q}$. Of course we also have $\mathcal{Q}^*_{\bar{r}}|_{U_{k+1}} = \tau^{X_{n-k}}_{\leq \bar{r}} R_{i_k*}\mathcal{Q}^*_{\bar{r}}|_{U_k}$ by the definition of $\mathcal{Q}^*$. So an extension argument can be made using the lemma. However, by [7, Lemma V.9.3], on U_{k+1}, $\mathfrak{D}^*_X[-n]|_{U_{k+1}}$ is only quasi-isomorphic to $Ri_{k*}\mathfrak{D}^*_X[-n]_{U_k}$ up to degree $k - 2 = \bar{t}(X_{n-k})$. Hence if $\bar{p} + \bar{q} > \bar{t}$, the extension argument will not work.

At the level of chains, the issue is that for any $\bar{p} \leq \bar{t}$, we know from [22, Proposition 2.1] that $I^{\bar{p}}C_*(X; R_0)$ is isomorphic to $I^{\bar{p}}C_*(X; R)$, which is a subgroup of $C_*(X; R)$, whose homology groups, up to the proper re-indexing, are the cohomology groups of $\mathfrak{D}_X[-n]$, assuming X is oriented. So, there is a well-defined map $I^{\bar{p}}C_*(X; R_0) \to C_*(X; R)$. However, if $\bar{p}(Z) > \bar{t}(Z)$ for any singular stratum Z, then we will not necessarily have an isomorphism from $I^{\bar{p}}C_*(X; R_0)$ to $I^{\bar{p}}C_*(X; R)$, and hence no obvious chain map $I^{\bar{p}}C_*(X; R_0) \to C_*(X; R)$, only a map $I^{\bar{p}}C_*(X; R_0) \to C_*(X; R_0)$.

5 Intersection pairings on piecewise linear pseudomanifolds

In this final section, we look at the intersection pairing of chains on a piecewise linear (PL) pseudomanifold and see how the classical Goresky-MacPherson pairing of [28] and the more general pairings of [25] extend to general perversities.

Throughout this section, X will be a compact PL pseudomanifold, possibly with codimension one strata. In particular, this means that X possesses a family of triangulations compatible with the stratification in the sense that each X^k will be a subcomplex of any triangulation. More details on PL pseudomanifolds can be found in [7, Chapters I,II]. In this section, we will consider only the PL chain complexes.

5.1 Some basics concerning PL chains and stratified coefficients

Recall that the PL chain complex $C_*^c(X)$ is defined to be $\varinjlim C_*^{c,T}(X)$, where $C_*^{c,T}(X)$ is the simplicial chain complex based on the triangulation T and the limit is taken over all compatible triangulations of X. In particular, every PL chain can be described in a specific triangulation. This definition generalizes in the obvious way to closed supports to yield $C_*^\infty(X)$.

We can also extend the PL chain complex to include stratified coefficient systems $\mathcal{G}_0$. In fact, we observe that any simplex σ of a triangulation is either completely contained in X^{n-1}, in which case it inherits an automatic 0 coefficient, or it intersects X^{n-1} only in its boundary, in which case a unique coefficient lift can be assigned to σ by determining what happens at one interior point. Furthermore, the boundary of the simplex with its coefficient is well-defined - any boundary faces in X^{n-1} vanish due to the 0 coefficient strata and the other faces inherit their coefficients from σ. Thus if X is a PL pseudomanifold, it is possible to define the simplicial chain complex $C_*(X;\mathcal{G}_0)$, as well as the subcomplexes $I^{\bar p}C_*(X;\mathcal{G}_0)$ for any general perversity. These are defined precisely as above in Section 2, though the allowability condition for a PL i-simplex σ can be simplified to $\dim(\sigma\cap Z)\le i-\operatorname{codim}(Z)+\bar p(Z)$.

We form the subcomplexes and quotient complexes $C_*(U;\mathcal{G}_0)$ and $C_*(X,U;\mathcal{G}_0)$ in the obvious way. We generally continue to use $\mathcal{G}_0$ also to denote the restriction of this coefficient system to subsets. If ξ is a chain in $C_i(X;\mathcal{G}_0)$ represented by a chain in $C_i^T(X;\mathcal{G}_0)$ for some triangulation T, we let the support $|\xi|$ be the union of all i-simplices σ of T such that the coefficient of σ in ξ is not identically 0.

Notice that in the PL setting we obtain for *any* perversity the nice feature we observed previously in Section 2.3 for efficient perversities: the coefficient of any i-simplex σ is defined by the lift of a single point $\sigma^{-1}(X-X^{n-1})$, which includes the entire interior of Δ^i. A useful consequence of this is that if we use constant coefficients on $X-X^{n-1}$, then we have $C_*(X;G_0)\cong C_*(X,X^{n-1};G)$, where the isomorphism assigns to any $g\sigma\in C_*(X;G_0)$ such that g is not identically 0 the obvious extension of g to the constant system G on X; conversely, any $g\sigma+C_*(X^{n-1};G)\in C_*(X,X^{n-1};G)$ for which $|\sigma|\not\subset X^{n-1}$ determines an element of $C_*(X;G_0)$ by restricting the coefficient lift to $\sigma^{-1}(X-X^{n-1})$. The homomorphisms thus determined are obviously mutual inverses and encompass all generators. Thus, we may think of $I^{\bar p}C_*(X;G_0)\subset C_*(X;G_0)$ as being a subcomplex of $C_*(X,X^{n-1};G)$ when it is useful to do so.

Pursuing this idea slightly further, there is also a natural homomorphism, though it is not a chain map, $\rho: C_*(X;G_0)\to C_*(X;G)$. We define ρ as the composition of the isomorphism $C_*(X;G_0)\cong C_*(X,X^{n-1};G)$ with the standard splitting $C_*(X,X^{n-1};G)\to C_*(X;G)$ that takes $g\sigma+C_*(X^{n-1};G)\in C_*(X,X^{n-1};G)$ such that $|\sigma|\not\subset X^{n-1}$ to $g\sigma\in C_*(X;G)$. This assignment is clearly additive, yielding a homomorphism. We observe that $|\xi|=|\rho(\xi)|$. However, it is not necessarily true that $|\partial\rho(\xi)|=|\rho(\partial\xi)|$; for example, if the boundary of $\rho(\xi)$ is in X^{n-1}, then we might have $\emptyset\ne|\partial\rho(\xi)|\subset X^{n-1}$ but $\rho(\partial\xi)=\rho(0)=0$. Thus ρ will not in general be a chain map.

Also, notice that $|\rho(\xi)|\cap(X-X^{n-1})=|\xi|\cap(X-X^{n-1})$, and $|\partial\rho(\xi)|\cap(X-X^{n-1})=|\rho(\partial\xi)|\cap(X-X^{n-1})=|\partial\xi|\cap(X-X^{n-1})$.

To simplify notation below, we will often denote $\rho(\xi)$ by $\bar{\xi}$.

The correspondence between chains and homology. Given closed PL subspaces $A \supset B$ such that $\dim(A) = i$, $\dim(B) < i$, it is well-known that there is a bijection between $H_i(A, B; G)$ and the set of chains $\xi \in C_*^\infty(X; G)$ such that $|\xi| \subset A$ and $|\partial\xi| \subset B$ (see [31, 28, 42]). We observe here a relative version of this phenomenon that includes stratified local coefficients:

Lemma 5.1. *Suppose that $A \supset B$ are closed PL subspaces of X such that $\dim(A - A \cap X^{n-1}) \le i$ and $\dim(B - B \cap X^{n-1}) \le i-1$. Let $\hat{A}$ denote the closure of $A - A \cap X^{n-1}$ in X (or, equivalently, the union of all simplices σ in a triangulation of A such that $\sigma \cap (X - X^{n-1}) \neq \emptyset$) and similarly for B. Then there is a bijection between $H_i(A, B; \mathcal{G}_0) \cong H_i(\hat{A}, \hat{B}; \mathcal{G}_0)$ and the set of chains $\xi \in C_i(X; \mathcal{G}_0)$ such that $|\xi| \subset \hat{A}$ and $|\partial\xi| \subset \hat{B}$. Furthermore, if G is a constant coefficient system, then there is a further bijection to $H_i(A \cup X^{n-1}, B \cup X^{n-1}; G) \cong H_i(A, B \cup (A \cap X^{n-1}); G) \cong H_i(\hat{A}, \hat{B} \cup (\hat{A} \cap X^{n-1}); G)$.*

Proof. The isomorphism $H_i(A, B; \mathcal{G}_0) \cong H_i(\hat{A}, \hat{B}; \mathcal{G}_0)$ is immediate because simplices supported in X^{n-1} must carry a 0 coefficient. The isomorphisms $H_i(A \cup X^{n-1}, B \cup X^{n-1}; G) \cong H_i(A, B \cup (A \cap X^{n-1}); G) \cong H_i(\hat{A}, \hat{B} \cup (\hat{A} \cap X^{n-1}); G)$ are by excision. Furthermore, for a constant coefficient system G, there is an evident isomorphism $C_*(A \cup X^{n-1}, B \cup X^{n-1}; G) \cong C_*(A, B; G_0)$.

Now we slightly generalize an argument of McClure [42, Lemma 4.1]: $H_i(\hat{A}, \hat{B}; \mathcal{G}_0)$ is the ith homology group of $\mathcal{C}_* = C_*(\hat{A}; \mathcal{G}_0)/C_*(\hat{B}; \mathcal{G}_0)$, which is the cycles of $\mathcal{C}_i$ modulo the boundaries of $\mathcal{C}_i$. But the group of cycles of $\mathcal{C}_i = C_i(\hat{A}; \mathcal{G}_0)/C_i(\hat{B}; \mathcal{G}_0)$ consists precisely of those chains of $C_*(X; \mathcal{G}_0)$ that are supported in $\hat{A}$ and whose boundaries are contained in $\hat{B}$. Note that any piece of the boundary that would be contained in X^{n-1} vanishes due to the coefficient system. But the subgroup of boundaries in $\mathcal{C}_i$ is $\partial C_{i+1}(A; \mathcal{G}_0) + C_i(\hat{B}; \mathcal{G}_0)$, which is zero due to the dimension hypotheses and again the fact that all chains supported in X^{n-1} vanish.

□

Remark 5.2. In the situation of the preceding lemma and assuming constant coefficients G, note that if ξ is a chain in $C_i(X; G_0)$ corresponding to a certain element of $H_i(\hat{A}, \hat{B} \cup (\hat{A} \cap X^{n-1}); G)$, then the element of $C_i(X; G)$ that traditionally corresponds to this homology element is precisely $\rho(\xi) = \bar{\xi}$.

5.2 The Goresky-MacPherson pairing

Now, we generalize to general perversities the Goresky-MacPherson intersection pairing of [28]. For simplicity, we start with constant coefficients and an oriented pseudomanifold. Let X^n be a compact oriented piecewise-linear pseudomanifold, possibly with codimension one strata. $C_*(X)$ and $I^{\bar{p}}C_*(X)$ denote complexes of PL chains over a fixed coefficient ring R.

Let us first recall the intersection pairing for GM perversities $\bar{p}, \bar{q}$ such that $\bar{p} + \bar{q} \le \bar{r} \le \bar{t}$. Suppose $x \in I^{\bar{p}}C_i(X; R)$ and $y \in I^{\bar{q}}C_j(X; R)$ are two chains such that the pairs $(|x|, |y|)$,

$(|\partial x|, |y|)$, and $(|x|, |\partial y|)$ are each pairs of chains in stratified general position. Recall that $|x|$ is the support of x, and that two subspaces A and B of X are in stratified general position if $A \cap Z$ and $B \cap Z$ are in general position in Z for each stratum Z, regular or singular. We take the intersection product of x and y, following the procedure in [28] and generalized in [25].

The chains x and y can be represented by homology classes $H_i(|x|, |\partial x|; R)$ and $H_j(|y|, |\partial y|; R)$ (see [28, 7]). Let $J = |\partial x| \cup |\partial y| \cup X^{n-1}$. Then the Goresky-MacPherson intersection product $x \pitchfork y$ is defined by first applying the following composition (all groups have R coefficients):

$$\begin{array}{c}
H_i(|x|, |\partial x|) \times H_j(|y|, |\partial y|) \\
\big\downarrow \\
H_i(|x|, |x| \cap J) \times H_j(|y|, |y| \cap J) \\
\big\downarrow \\
H_i(|x| \cup J, J) \times H_j(|y| \cup J, J) \\
\big\downarrow {\scriptstyle (-1)^{n(n-i)}(\cdot \cap [X])^{-1} \times (-1)^{n(n-j)}(\cdot \cap [X])^{-1}} \\
H^{n-i}(X - J, X - (|x| \cup J)) \times H^{n-j}(X - J, X - (|y| \cup J)) \\
\big\downarrow {\scriptstyle \cup} \\
H^{2n-i-j}(X - J, X - ((|x| \cap |y|) \cup J)) \\
\big\downarrow {\scriptstyle (-1)^{n(2n-i-j)}(\cdot \cap [X])} \\
H_{i+j-n}((|x| \cap |y|) \cup J, J) \\
\big\downarrow {\scriptstyle \cong} \\
H_{i+j-n}(|x| \cap |y|, |x| \cap |y| \cap J).
\end{array} \tag{2}$$

Here the last isomorphism is by excision, $[X]$ is the fundamental orientation class of X, and $\cap[X]$ is the cap product inducing the Whitehead-Dold-Goresky-MacPherson duality isomorphism (see [28, Appendix]). The signs here were not present in the original Goresky-MacPherson paper [28], but an argument is made in [25] that these signs are useful in order for the duality to be a chain map in the appropriately graded sense.

The last step in the Goresky-MacPherson intersection product is to use the long exact sequence of the triple, excision, and the fact that $\dim(|x| \cap |y| \cap X^{n-1}) \leq i+j-n-2$, which follows from the perversity conditions and the stratified general position, to conclude that $H_{i+j-n}(|x| \cap |y|, |x| \cap |y| \cap |J|) \cong H_{i+j-n}(|x| \cap |y|, (|\partial x| \cap |y|) \cup (|x| \cap |\partial y|))$, which represents a chain, the intersection product $x \pitchfork y$, in $C_{i+j-n}(X; R)$. In particular, the third term in the exact sequence of $(|x| \cap |y|, |x| \cap |y| \cap J, (|\partial x| \cap |y|) \cup (|x| \cap |\partial y|))$ is isomorphic by excision to $H_*(|x| \cap |y| \cap X^{n-1}, ((|\partial x| \cap |y|) \cup (|x| \cap |\partial y|)) \cap X^{n-1})$, but if $\dim(|x| \cap |y| \cap X^{n-1}) \leq i+j-n-2$, this term vanishes for $* > i+j-n-2$, which is enough for the desired isomorphism.

The reason that $\dim(|x| \cap |y| \cap X^{n-1}) \leq i+j-n-2$ is that if $\bar{p}+\bar{q} \leq \bar{r} \leq \bar{t}$, then for each stratum Z, $\dim(|x| \cap Z) \leq i - \operatorname{codim}(Z) + \bar{p}(Z)$ and $\dim(|y| \cap Z) \leq j - \operatorname{codim}(Z) + \bar{q}(Z)$ so that, if they are in stratified general position,

$$\begin{aligned}
\dim(|x| \cap |y| \cap Z) &\leq i - \operatorname{codim}(Z) + \bar{p}(Z) + j - \operatorname{codim}(Z) + \bar{q}(Z) - \dim(Z) \\
&= i + j + \bar{p}(Z) + \bar{q}(Z) - \operatorname{codim}(Z) - (\operatorname{codim}(Z) + \dim(Z)) \\
&\leq i + j + \bar{t}(Z) - \operatorname{codim}(Z) - n \\
&= i + j + \operatorname{codim}(Z) - 2 - \operatorname{codim}(Z) - n \\
&= i + j - 2 - n.
\end{aligned}$$

Furthermore, by the same type of computations, we see that the simplices of $x \pitchfork y$ must be r-allowable. Similarly, if $(|\partial x|, |y|)$ and $(|x|, |\partial y|)$ are in stratified general position, then since $\partial |x \pitchfork y| \subset (|\partial x| \cap |y|) \cup (|x| \cap |\partial y|)$, an analogous argument shows that $\partial(x \pitchfork y)$ is allowable. Thus $x \pitchfork y \in I^{\bar{r}}C_*(X; R)$.

Now, suppose that $\bar{p}$ and $\bar{q}$ are general perversities, that $x \in I^{\bar{p}}C_i(X; R_0)$ and $y \in I^{\bar{p}}C_j(X; R_0)$. In order to apply the pairings, we first translate x, y to elements of $C_*(X; R)$ by considering $\rho(x) = \bar{x}$ and $\rho(y) = \bar{y}$ as defined above. We can then proceed as before to translate $\bar{x}, \bar{y}$ into elements of $H_i(|\bar{x}|, |\partial \bar{x}|; R)$ and $H_j(|\bar{y}|, |\partial \bar{y}|; R)$ and then apply the sequence of maps in diagram (2). The trouble is the last step, since in the general perversity setting, it may no longer be true that $\dim(|\bar{x}| \cap |\bar{y}| \cap X^{n-1}) \leq i+j-n-2$. In addition, $\partial \bar{x}$ may contain simplices in X^{n-1} that do not occur in ∂x due to the stratified coefficient system, which may also *a priori* cause some trouble.

However, letting again $J = |\partial \bar{x}| \cup |\partial \bar{y}| \cup X^{n-1}$, we do still obtain a well-defined homology class in $H_{i+j-n}(|\bar{x}| \cap |\bar{y}|, |\bar{x}| \cap |\bar{y}| \cap J) \cong H_{i+j-n}(|\bar{x}| \cap |\bar{y}|, (|\partial \bar{x}| \cap |\bar{y}|) \cup (|\bar{x}| \cap |\partial \bar{y}|) \cup (|\bar{x}| \cap |\bar{y}| \cap X^{n-1}))$. Now, assume that the pairs $(|x|, |y|)$, $(|\partial x|, |\bar{y}|)$, and $(|\bar{x}|, |\partial y|)$ are all in stratified general position. Then, applying computations such as those above,

$$\begin{aligned}
&\dim((|\bar{x}| \cap |\bar{y}|) \cap (X - X^{n-1})) \\
&\qquad = \dim((|x| \cap |y|) \cap (X - X^{n-1})) \\
&\qquad \leq i + j - n
\end{aligned}$$

and

$$\begin{aligned}
&\dim(((|\partial \bar{x}| \cap |\bar{y}|) \cup (|\bar{x}| \cap |\partial \bar{y}|) \cup (|\bar{x}| \cap |\bar{y}| \cap X^{n-1})) \cap (X - X^{n-1})) \\
&\qquad = \dim(((|\partial \bar{x}| \cap |\bar{y}|) \cup (|\bar{x}| \cap |\partial \bar{y}|)) \cap (X - X^{n-1})) \\
&\qquad = \dim(((|\partial x| \cap |y|) \cup (|x| \cap |\partial y|)) \cap (X - X^{n-1})) \qquad \leq i + j - n - 1.
\end{aligned}$$

So by Lemma 5.1, our homology class corresponds to a chain in $C_{i+j-n}(X; R_0) \cong C_{i+j-n}(X, X^{n-1}; R)$ with support in the closure of $|\bar{x}| \cap |\bar{y}| \cap (X - X^{n-1}) = |x| \cap |y| \cap (X - X^{n-1})$ and with boundary supported in the closure of $((|\partial\bar{x}| \cap |\bar{y}|) \cup (|\bar{x}| \cap |\partial\bar{y}|)) \cap (X - X^{n-1})$. This latter closure is contained in $(|\partial x| \cap |y|) \cup (|x| \cap |\partial y|)$, where the boundary is taken in $C_*(X; R_0)$, while the former is contained in $|x| \cap |y|$.

Now, computations just like those above, applied to the pairs $|\bar{x}| \cap |\bar{y}|$, $|\partial x| \cap |y|$, and $|\partial y| \cap |x|$ allow us to conclude that this chain is in $I^{\bar{r}}C_{i+j-n}(X; R_0)$.

In summary, we have proven the following theorem:

Theorem 5.3. *Let X be a compact oriented PL stratified n-pseudomanifold. Let $\bar{p}, \bar{q}$ be general perversities. Let $x \in I^{\bar{p}}C_*(X; R_0)$, $y \in I^{\bar{q}}C_*(X; R_0)$ be such that the pairs $(|\bar{x}|, |\bar{y}|)$, $(|\overline{\partial x}|, |\bar{y}|)$, and $(|\bar{x}|, |\overline{\partial y}|)$ are in stratified general position. Then the Goresky-MacPherson intersection pairing yields a well-defined chain $x \pitchfork y \in I^{\bar{r}}C_*(X; R_0)$ for any $\bar{r} \geq \bar{p} + \bar{q}$. Furthermore, if $\bar{r} \leq \bar{t}$, then $x \pitchfork y \in I^{\bar{r}}C_*(X; R)$.*

The last statement uses the fact that $I^{\bar{r}}C_*(X; R) \cong I^{\bar{r}}C_*(X; R_0)$ when $\bar{r} \leq \bar{t}$ - see Section 2.2. Furthermore, using this identification, we see that this pairing reduces to the usual Goresky-MacPherson pairing if $\bar{p}, \bar{q}$ are traditional with $\bar{p} + \bar{q} \leq \bar{t}$.

Local Coefficients. Theorem 5.3 can be generalized in the local coefficient/possibly non-orientable case to the following:

Theorem 5.4. *Let X be a compact PL n-pseudomanifold, let $\bar{p}, \bar{q}$ be general perversities, and let $\mathcal{E}$ and $\mathcal{F}$ be local coefficient systems on $X - X^{n-1}$ over a principal ideal domain R with a pairing $\mathcal{E} \otimes \mathcal{F} \to \mathcal{G}$. Let $\mathcal{O}$ be the R orientation coefficient system on $X - X^{n-1}$. Let $x \in I^{\bar{p}}C_*(X; \mathcal{E}_0)$, $y \in I^{\bar{q}}C_*(X; \mathcal{O} \otimes \mathcal{F}_0)$ be such that the pairs $(|x|, |y|)$, $(|\partial x|, |y|)$, and $(|x|, |\partial y|)$ are in stratified general position. Then the Goresky-MacPherson intersection pairing extends to yield a well-defined chain $x \pitchfork y \in I^{\bar{r}}C_*(X; \mathcal{G}_0)$ for any $\bar{r} \geq \bar{p} + \bar{q}$.*

The arguments are mostly the same as those above, however we will need the following version of the Whitehead-Dold-Goresky-MacPherson duality isomorphism.

Lemma 5.5. *Suppose $X^{n-1} \subset D \subset C$ are closed PL subspaces of the compact, not necessarily oriented, pseudomanifold X^n. Let $\mathcal{G}$ be a local coefficient system of R modules on $X - X^{n-1}$, where R is a principal ideal domain. Let $\mathcal{O}$ be the local orientation system with R coefficients on $X - X^{n-1}$. Then there is an isomorphism $H^i(X - D, X - C; \mathcal{G}) \to H_{n-i}(C, D; \mathcal{O}_0 \otimes \mathcal{G}_0)$ that is induced by inclusions and cap product with the fundamental class $[X_R] \in H_n(X; \mathcal{O}_0)$.*

Note that since $X - D \subset X - X^{n-1}$, $H^i(X - D, X - C; \mathcal{G})$ is well-defined with no need for stratified coefficients.

We will turn to the proof of the lemma in a moment. Assuming it for now, we show why Theorem 5.4 holds.

Firstly, since we have local coefficient systems, we cannot represent x and y as elements of the form $H_*(A, B; R)$ but only as elements of the form $H_*(A, B; \Upsilon_0)$, where A, B denote the supports of the relevant chains, $* = i$ or j, and Υ is one of $\mathcal{E}$ or $\mathcal{O} \otimes \mathcal{F}$, as appropriate. However, there are nonetheless obvious inclusion morphisms $H_i(A, B; \Upsilon_0) \to H_i(A \cup J, J; \Upsilon_0)$. Note that in this case we don't even need to utilize $\bar{x}$ or $\bar{y}$ at all, so this also provides an alternative approach to the discussion above.

Now, we can apply 5.5 to each term Lemma in place of the Whitehead-Dold-Goresky-MacPherson duality of [28] to obtain cochains of the form $H^*(X - B, X - A; \mathcal{O} \otimes \Upsilon)$. The cup product, using the pairing $\mathcal{E} \otimes \mathcal{F} \to \mathcal{G}$, gives us an element of $H^{2n-i-j}(X - J, X - (|x| \cap |y| \cup J); \mathcal{O} \otimes \mathcal{G})$, and then applying duality again gives an element of $H_{i+j-n}((|x| \cap |y|) \cup J, J; \mathcal{G}_0)$. The excision isomorphism to $H_{i+j-n}(|x| \cap |y|, |\bar{x}| \cap |\bar{y}| \cap J; \mathcal{G}_0)$ applies as usual, and applying Lemma 5.1, this corresponds to an element of $C_{i+j-n}(X; \mathcal{G}_0)$, which we can again verify to be an element of $I^{\bar{r}} C_{i+j-n}(X; \mathcal{G}_0)$.

This yields Theorem 5.4.

Now we return to Lemma 5.5.

Sketch of proof. Assume X is triangulated as a complex K such that C, D, and each skeleton are subcomplexes. If A is a subcomplex of K, let $\bar{A}$ denote the complement of the open first derived neighborhood of A in the derived triangulation K'. We first note that $H^i(X - D, X - C; \mathcal{G}) \cong H^i(\bar{D}, \bar{C}; \mathcal{G})$ by homotopy equivalences.

Now we use the standard simplicial duality arguments as in, e.g. [41, Chapter 5], which generalize in evident ways to the local coefficient case. In particular, we can think of $C^i(\bar{D}, \bar{C}; \mathcal{G})$ as being generated by the i-cochains $c_{\sigma,g}$ that evaluate to 0 except on a single i-simplex σ of K' supported in $\bar{D}$ but not in $\bar{C}$, which evaluates to an element $g \in \mathcal{G}_x$, where x is a point in σ. Since all such simplices are in the interior of the manifold $X - X^{n-1}$, if we let z be a chain representing the orientation class $[X_R]$, then after passing to the second derived subdivision, $c_{\sigma,g} \cap z$ is represented by the dual block $e(\sigma)$ to σ, carrying the coefficient in $\mathcal{O} \otimes \mathcal{G}_0$ inherited from the evaluation of $c_{\sigma,g}$ and from the coefficient carried by z in a neighborhood of σ (by working within a contractible neighborhood of σ, we can see that this coefficient is well-defined). So $\cap z$ takes the cochains of $C^*(\bar{D}, \bar{C}; \mathcal{G})$ to the dual blocks (with coefficients) of the simplices of $\bar{D}$ not supported in $\bar{C}$, and, as shown in [41], this is a chain isomorphism to the dual block complex $\bar{C}_{n-*}(\bar{\bar{C}}, \bar{\bar{D}}; \mathcal{O} \otimes \mathcal{G}_0)$; we are free to use the stratified coefficients $\mathcal{G}_0$ since we never get close enough to X^{n-1} for the 0 system on it to matter. This chain isomorphism induces an isomorphism $H^i(\bar{D}, \bar{C}; \mathcal{G}) \to H_{n-i}(\bar{\bar{C}}, \bar{\bar{D}}; \mathcal{O} \otimes \mathcal{G}_0)$. If we do not wish to think about dual blocks near the singular set X^{n-1}, we can alternatively think of the pair $(\bar{\bar{C}}, \bar{\bar{D}})$ as being the appropriate dual pair in the manifold double $\bar{D} \cup -\bar{D}$ (with the appropriately extended coefficient systems), but then the resulting relative homology group is isomorphic by excisions to our $H_{n-i}(\bar{\bar{C}}, \bar{\bar{D}}; \mathcal{O} \otimes \mathcal{G}_0)$ in X.

Finally, we note that $\bar{\bar{D}}$ strongly deformation retracts to D, and similarly for $\bar{\bar{C}}$. Just as for ordinary homology, it follows that $H_*(\bar{\bar{D}}; \mathcal{O} \otimes \mathcal{G}_0) \cong H_*(D; \mathcal{O} \otimes \mathcal{G}_0)$ and similarly for C. Thus by the five lemma, $H_{n-i}(\bar{\bar{C}}, \bar{\bar{D}}; \mathcal{O} \otimes \mathcal{G}_0) \cong H_{n-i}(C, D; \mathcal{O} \otimes \mathcal{G}_0)$. □

5.3 The pairing algebra

Having established the basic results on the intersection pairing in the previous section, most of the results on the algebraic properties of intersection pairings developed in [25] go through without much extra effort. We will state the main theorems.

First, we must recall some definitions from [25]. Let $X(k)$ denote the product of k copies of X, endowed with the product stratification. Let $\bar{k} = \{1, \dots, k\}$. If $\mathscr{R} : \bar{k} \to \bar{k}'$ is any map of sets, it induces a map $\mathscr{R}^* : X(k') \to X(k)$ by $\mathscr{R}^*(x_1, \dots, x_{k'}) = (x_{\mathscr{R}(1)}, \dots, x_{\mathscr{R}(k)})$. These are generalizations of the diagonal map $x \to (x, x)$, which is induced by the unique map $\bar{2} \to \bar{1}$.

If A is a PL subset of $X(k)$, we will say that A is in *stratified general position* with respect to $\mathscr{R}^*$ if for each stratum $Z \subset X_{d_1} \times \dots \times X_{d_k}$ of $X(k)$ such that $d_i = d_\ell$ if $\mathscr{R}(i) = \mathscr{R}(\ell)$, we have

$$\dim((\mathscr{R}^*)^{-1}(A \cap Z)) \leq \dim(A \cap Z) + \sum_{i=1}^{k'} d_{\mathscr{R}^{-1}(i)} - \sum_{i=1}^{k} d_i. \tag{3}$$

Note that the condition on the d_is implies that $d_j = d_\ell$ for any $j, \ell \in \mathscr{R}^{-1}(i)$ so that this sum is well-defined.

In other words, A is in stratified general position with respect to $\mathscr{R}^*$ if for each stratum Z of $X(k)$, $A \cap Z$ is in general position with respect to the map of manifolds from the stratum containing $(\mathscr{R}^*)^{-1}(Z)$ to Z. A PL chain is said to be in stratified general position if its support is, and we write $C_*^{\mathscr{R}^*}(X(k))$ for the subcomplex of PL chains D of $C_*(X(k))$ such that both D and ∂D are in stratified general position with respect to $\mathscr{R}^*$. Similarly, we let $C_*^{\mathscr{R}^*}(X(k); R_0)$ be those chains D such that $|D|$ and $|\partial D|$ are in stratified general position.

For a differential graded complex C_*, $S^m C_*$ is the shifted complex with $(S^m C_*)_i = C_{i-m}$ and $\partial_{S^m C_*} = (-1)^m \partial_{C_*}$. For a chain ξ, we define $|S^m \xi| = |\xi|$.

The product ε is the multilinear extension of the product that takes $\sigma_1 \otimes \sigma_2$, where the σ_i are oriented simplices, to a chain with support $|\sigma_1| \times |\sigma_2|$ and with appropriate orientation. This is a direct generalization of the standard simplicial cross product construction (see e.g. [44]); we refer the reader to [42, Section 7] for details. With $\dim(X) = n$, $\bar{\varepsilon}_k : S^{-n}C_*(X) \otimes \dots \otimes S^{-n}C_*(X) \to S^{-kn}C_*(X(k))$ is defined to be $(-1)^{e_2(n,\dots,n)}$ times the composition $S^{-kn}\bar{\varepsilon} \circ \Theta$, where Θ is the appropriately signed chain isomorphism $\Theta : S^{-n}C_*(X) \otimes \dots \otimes S^{-n}C_*(X) \to S^{-nk}(C_*(X(k)))$ (see [25, Remark 3.2]). Here $e_2(n, \dots, n)$ is the elementary symmetric polynomial of degree two on the k symbols $n, \dots, n$, so $e_2(n, \dots, n) = \sum_{i=1}^{k} \sum_{j<i} n^2$. In other words, $\bar{\varepsilon}$ is the composite

$$S^{-n}C_*(X) \otimes \dots \otimes S^{-n}C_*(X) \xrightarrow{\quad\Theta\quad} S^{-nk}(C_*(X) \otimes \dots \otimes C_*(X_k))$$

$$\xrightarrow{\quad(-1)^{e_2} S^{-nk}\varepsilon\quad} S^{-nk}C_*(X(k)).$$

ε and $\bar{\varepsilon}$ are monomorphisms. Furthermore, $\bar{\varepsilon}$ is a degree 0 chain map.

Unfortunately, all of this shifting and application of signs is necessary for the appropriate pairings to be degree 0 chain maps.

We note that this definition also determines a map $\bar{\varepsilon} : S^{-n}C_*(X, X^{n-1})\otimes\cdots\otimes S^{-n}C_*(X, X^{n-1}) \to S^{-nk}C_*(X(k), X(k)^{nk-1})$, or, equivalently $\bar{\varepsilon} : S^{-n}C_*(X; R_0)\otimes\cdots\otimes S^{-n}C_*(X; R_0) \to S^{-nk}C_*(X(k); R_0)$. This is a well-defined monomorphism because $|\bar{\varepsilon}(\sigma_1 \otimes \cdots \otimes \sigma_k)| \subset (X(k))^{nk-1}$ if and only if $|\sigma_i| \subset X^{n-1}$ for some at least one i.

Next, recall the following definition from [25], generalizing that of [42], of the complex of chains in general position.

Definition 5.6. For $k \geq 2$, let the *domain* G_k be the subcomplex of $(S^{-n}C_*(X; R))^{\otimes k}$ consisting of elements D such that both $\bar{\varepsilon}(D)$ and $\bar{\varepsilon}(\partial D)$ are in stratified general position with respect to all generalized diagonal maps, i.e.

$$G_k = \bigcap_{k'<k} \bigcap_{\mathscr{R}:\bar{k}\twoheadrightarrow\bar{k}'} \bar{\varepsilon}^{-1}(S^{-nk}C_*^{\mathscr{R}^*}(X(k))).$$

It is shown in [25] that the inclusion $G_k \hookrightarrow (S^{-n}C_*(X; R))^{\otimes k}$ is a quasi-isomorphism for all $k \geq 1$.

Furthermore, there are the intersection chain versions. If $P = (\bar{p}_1, \ldots, \bar{p}_k)$ is a collection of GM perversities and $G_k^P = G_k \cap (S^{-n}IC_*^{\bar{p}_1}(X) \otimes \cdots \otimes S^{-n}IC_*^{\bar{p}_k}(X))$, then the inclusion $G_k^P \hookrightarrow S^{-n}IC_*^{\bar{p}_1}(X) \otimes \cdots \otimes S^{-n}IC_*^{\bar{p}_k}(X)$ is a quasi-isomorphism.

Here we will need also a relative version.

Definition 5.7. For $k \geq 2$, let the *domain* $G_{k,0}$ be the subcomplex of $(S^{-n}C_*(X, X^{n-1}; G))^{\otimes k}$ consisting of elements D such that both $\bar{\varepsilon}(D)$ and $\bar{\varepsilon}(\partial D)$ are in stratified general position with respect to all generalized diagonal maps, i.e.

$$G_{k,0} = \bigcap_{k'<k} \bigcap_{\mathscr{R}:\bar{k}\twoheadrightarrow\bar{k}'} \bar{\varepsilon}^{-1}(S^{-nk}C_*^{\mathscr{R}^*}(X(k); R_0)).$$

First, we should observe that $G_{k,0}$ is a well-defined chain complex. If $x, y \in G_{k,0}$, then certainly $-x \in G_{k,0}$, as well at $x + y$ because the sum of chains in general position will also be in general position. Furthermore, if $D \in G_{k,0}$, then it is built into the definition that ∂D will also be in stratified general position, and of course $\partial\partial D = 0$; thus $\partial D \in G_{k,0}$. So $G_{k,0}$ is a chain complex.

Similarly, we can define the intersection chain versions:

Definition 5.8. For $k \geq 2$ and a sequence of general perversities $P = (\bar{p}_1, \ldots, \bar{p}_k)$, let the *domain* $G_{k,0}^P$ be the subcomplex of $S^{-n}I^{\bar{p}_1}C_*(X; R_0) \otimes \cdots \otimes S^{-n}I^{\bar{p}_k}C_*(X; R_0)$ consisting of elements D such that both $\bar{\varepsilon}(D)$ and $\bar{\varepsilon}(\partial D)$ are in stratified general position with respect to all generalized diagonal maps, i.e.

$$G_{k,0}^P = G_{k,0} \cap \left(S^{-n}IC_*^{\bar{p}_1}(X) \otimes \cdots \otimes S^{-n}IC_*^{\bar{p}_k}(X)\right).$$

Then we have the following theorem.

Theorem 5.9. *The inclusion* $G_{k,0} \hookrightarrow (S^{-n}C_*(X; R_0))^{\otimes k}$ *is a quasi-isomorphism for all* $k \geq 1$, *as are the inclusions* $G_{k,0}^P \hookrightarrow S^{-n}IC_*^{\bar{p}_1}(X; R_0) \otimes \cdots \otimes S^{-n}IC_*^{\bar{p}_k}(X; R_0)$.

Sketch of proof. Ultimately, the proof is more-or-less the same as that of Theorem 3.5 of [25], which shows that G_k is quasi-isomorphic to $(S^{-n}C_*(X))^{\otimes k}$. The general idea is to push chains by homotopies until all desired general positions are achieved. In fact, since $\overline{\partial \bar{D}} \subset \partial \bar{D}$, it suffices to use the arguments of the proof of [25, Theorem 3.5], which constructs all relevant homotopies and homologies for $\bar{D}$ and $\partial \bar{D}$. These are absolute homologies, but they become relative homologies when considered with coefficients in R_0. □

Umkehr maps. The motivation in [42] for creating the complexes G_k for the manifold M is that G_k serves as the domain for a generalized intersection pairing. Using this domain, the intersection pairing can be defined as a chain map, rather than as an ad hoc construction on pairs of chains in suitable general position. Furthermore, this intersection pairing on G_k is used to show that the intersection pairing extends to the algebraic structure of a Leinster partial commutative differential graded algebra (DGA). Similarly, in [25], the G_k^P are shown to be domains for the intersection pairing of intersection chains, and these pairings are shown to induce partial restricted commutative DGAs.

To extend these results here, we need to consider the appropriate relative form of the general intersection pairings.

The more general intersection homology multi-products come from considering umkehr maps on pseudomanifolds. This was done in [25, Section 4.2]. Here we consider the relative, or stratified coefficient, version, which in some sense is simpler, just as for the Goresky-MacPherson pairing we were able not to concern ourselves with the final excision step.

The umkehr map proceeds as follows.

Suppose $f : X^n \to Y^m$ is a PL map of compact oriented PL stratified pseudomanifolds such that $f^{-1}(Y^{m-1}) \subset X^{n-1}$, where X^{n-1} and Y^{m-1} are the respective singular sets of X and Y. Suppose that $D \in C_i(Y; R_0) = C_i(Y; Y^{m-1})$. Then by Lemma 5.1, D corresponds to the homology class $[D] \in H_i(|\bar{D}| \cup Y^{m-1}, |\overline{\partial \bar{D}}| \cup Y^{m-1}; R) \cong H_i(|\bar{D}|, |\overline{\partial \bar{D}}| \cup (|\bar{D}| \cap Y^{m-1}); R)$. Let $A = |\bar{D}|$, $B = |\overline{\partial \bar{D}}|$, $A' = f^{-1}(A)$, and $B' = f^{-1}(B)$. We consider the following composition of maps, in which all groups have R coefficients.

$$\begin{aligned}
S^{-m}H_i(A \cup Y^{m-1}, B \cup Y^{m-1}) &\xrightarrow[\cong]{(-1)^{m(m-i)}(\cdot \cap [Y])^{-1}} H^{m-i}(Y - (B \cup Y^{m-1}), Y - (A \cup Y^{m-1})) \\
&\xrightarrow{\quad f^* \quad} H^{m-i}(X - (B' \cup X^{n-1}), X - (A' \cup X^{n-1})) \\
&\xrightarrow[\cong]{(-1)^{n(m-i)}(\cdot \cap [X])} S^{-n}H_{i+n-m}(A' \cup X^{n-1}, B' \cup X^{n-1})
\end{aligned}$$

The indicated signed cap products with the respective fundamental classes again represent the Whitehead-Dold-Goresky-MacPherson duality isomorphism - see [28, Appendix]. We also incorporate the sign conventions of [25].

If $\dim(A' - A' \cap X^{n-1}) \leq i + n - m$ and $\dim(B' - B' \cap X^{n-1}) \leq i + n - m - 1$, then by Lemma 5.1, the image of this composition represents a well-defined chain in $S^{-n}C_{i+n-m}(X, X^{n-1}; R) = S^{-n}C_{i+n-m}(X; R_0)$. In this case, we say that D is in general position with respect to f, and we denote the composite map $C_i(Y; R_0) \to C_{i+n-m}(X; R_0)$ by

$f_!$. Furthermore, $f_!$ is a degree 0 chain map as follows from the same arguments as presented in the proof of [25, Lemma 7.3].

In particular now, suppose that $\Delta : X \to X(k)$ is the diagonal embedding $x \to (x, \dots, x)$. Then $\Delta = \mathscr{R}^*$ for the unique surjection $\mathscr{R} : \bar{k} \to \bar{1}$. For some collection of general perversities P, suppose $D \in G^P_{k,0}$. Then by assumption, $\bar{\varepsilon}D$ is in stratified general position, so by the definition of the last section, $\dim(\Delta^{-1}(Z(k) \cap \rho(\bar{\varepsilon}D))) = \dim(Z \cap \Delta^{-1}(\rho(\bar{\varepsilon}D))) \leq \dim(Z(k) \cap \rho(\bar{\varepsilon}D)) + \dim(Z)(1-k)$. In particular, if D is (the shift of) an i-chain and Z is a regular stratum, we get $\dim((X - X^{n-1}) \cap \Delta^{-1}(\rho(\bar{\varepsilon}D))) = \dim(\Delta^{-1}((X - X^{n-1})(k) \cap \rho(\bar{\varepsilon}D))) \leq i + n(1-k)$, and similarly for ∂D. Thus we have a well-defined $\Delta_! : G^P_{k,0} \to S^{-n}C_*(X, X^{n-1}; R) \cong S^{-n}C_*(X; R_0)$, which is the generalized intersection pairing. Furthermore, using analogous computations to the GM perversity case (see [25, Proposition 4.5]), which extend the sort of computations in our discussion of the Goresky-MacPherson pairing above, we see that $\Delta_!(D) \in S^{-n}I^{\bar{r}}C_*(X; R_0)$ for any $\bar{r} \geq \sum_{i=1}^k \bar{p}_i$.

Note that in the special case where $D \in G^P_{2,0}$ and $D = S^{-n}x \otimes S^{-n}y$, the condition that $\bar{\varepsilon}(D)$ be in stratified general position with respect to $\Delta : X \to X \times X$ says precisely that for each stratum Z, $Z \cap \Delta^{-1}(|\varepsilon D|) = \dim(Z \cap |x| \cap |y|) \leq \dim((Z \times Z) \cap (|x| \times |y|)) - \dim(Z) = \dim(|x| \cap Z) + \dim(|y| \cap Z) - \dim(Z)$, which is precisely the condition that $|x|$ and $|y|$ be in stratified general position. Similarly, the condition that $\bar{\varepsilon}(\partial D)$ be in stratified general position with respect to Δ is equivalent to the pairs $(|x|, |\partial y|)$ and $(|\partial x|, |y|)$ each being in general position. So, in this special setting, the conditions for $\Delta_!$ to be well-defined reduce to those for the Goresky-MacPherson pairing, and arguments as those in the proof of [25, Proposition 4.9] show that these pairings are equivalent, up to the appropriate index shifts.

It also follows analogously to the arguments of [25, Section 6] that these geometric intersection chain pairings are compatible with the purely sheaf theoretic pairings of Theorem 4.6.

Further results from [25]. Now that we have established the necessary modifications to the umkehr map and, more generally, understood the role that stratified coefficients and relative chains play in the generalization of the pairing and general position arguments to the general perversity case, the remaining major results of [25] go through with little difficulty. In particular, one can define a category $\mathfrak{GP}_n$ of general perverse chain complexes consisting of functors from the poset category of general n-perversities (in which $\bar{p} \to \bar{q}$ exists uniquely if and only if $\bar{p}(Z) \leq \bar{q}(Z)$ for all singular strata Z) to the category of chain complexes. Loosely perverse chain complexes are denoted $\{C^\star_*\}$ with evaluation at $\bar{p}$ denoted by $\{C^\star_*\}^{\bar{p}} = C^{\bar{p}}_*$. $\mathfrak{GP}_n$ is a symmetric monoidal category with product $\boxtimes$ obtained by setting $(\{D^\star_*\} \boxtimes \{E^\star_*\})^{\bar{r}} = \varinjlim_{\bar{p}+\bar{q} \leq \bar{r}} D^{\bar{p}}_* \otimes E^{\bar{q}}_*$.

It follows as by the proof of [25, Theorem 5.3], with some minor modifications, that the following theorem holds:

Theorem 5.10. *For any compact oriented PL stratified pseudomanifold X, the partially-defined intersection pairing on the perverse chain complex $\{S^{-n}I^\star C_*(X, R_0)\}$ extends to the structure of a partial perverse commutative DGA.*

A partial perverse commutative DGA is what we referred to as a partial restricted commutative DGA in [25, Section 5] with the change that we now allow general perversities. This wording is also more reflective of the fact that we no longer need *restrict* to perversities (or sums of perversities) below $\bar{t}$. We refer the reader to [25, Section 5] for the original definition of this structure and the details of the proof, which readily generalize in light of the work above.

References

[1] Markus Banagl, *The signature of singular spaces and its refinement to generalized homology theory*, To appear in Topology of Stratified Spaces, Mathematical Sciences Research Institute Publications 58, Cambridge University Press, New York, 2010.

[2] ______, *Topological invariants of stratified spaces*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2006.

[3] Markus Banagl, *The signature of partially defined local coefficient systems*, J. Knot Theory Ramifications **17** (2008), no. 12, 1455–1481.

[4] Markus Banagl, Sylvain Cappell, and Julius Shaneson, *Computing twisted signatures and L-classes of stratified spaces*, Math. Ann. **326** (2003), 589–623.

[5] A.A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982), 5–171.

[6] Armand Borel, *Introduction to middle intersection cohomology and perverse sheaves*, Algebraic groups and their generalizations: classical methods (University Park, PA, 1991), Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., Providence, RI, 1994, pp. 25–52. MR MR1278699 (95h:55006)

[7] A. Borel et. al., *Intersection cohomology*, Progress in Mathematics, vol. 50, Birkhauser, Boston, 1984.

[8] Sylvain Cappell, Julius Shaneson, and Shmuel Weinberger, *Classes topologiques caractéristiques pour les actions de groupes sur les espaces singuliers*, C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), no. 5, 293–295.

[9] Sylvain E. Cappell, Anatoly Libgober, Laurentiu G. Maxim, and Julius L. Shaneson, *Hodge genera of algebraic varieties. II*, Math. Ann. **345** (2009), no. 4, 925–972.

[10] Sylvain E. Cappell, Laurentiu G. Maxim, and Julius L. Shaneson, *Hodge genera of algebraic varieties. I*, Comm. Pure Appl. Math. **61** (2008), no. 3, 422–449.

[11] Sylvain E. Cappell and Julius L. Shaneson, *Singular spaces, characteristic classes, and intersection homology*, Annals of Mathematics **134** (1991), 325–374.

[12] G. Carron, *L^2-cohomology of manifolds with flat ends*, Geom. Funct. Anal. **13** (2003), no. 2, 366–395.

[13] Gilles Carron, *Cohomologie L^2 des variétés QALE*, Preprint, 2005.

[14] ———, *Cohomologie L^2 et parabolicité*, J. Geom. Anal. **15** (2005), no. 3, 391–404.

[15] Jeff Cheeger and Xianzhe Dai, *L^2-cohomology of a non-isolated conical singularity and nonmultiplicatity of signatures*, Preprint.

[16] Daniel C. Cohen, Mark Goresky, and Lizhen Ji, *On the Künneth formula for intersection cohomology*, Trans. Amer. Math. Soc. **333** (1992), 63–69.

[17] Xianzhe Dai, *Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence*, J. Amer. Math. Soc. **4** (1991), no. 2, 265–321.

[18] Alexandru Dimca, *Sheave in topology*, Springer-Verlag, Berlin-Heidelberg-New York, 2004.

[19] Greg Friedman, *An introduction to intersection homology with general perversity functions*, To appear in Topology of Stratified Spaces, Mathematical Sciences Research Institute Publications 58, Cambridge University Press, New York, 2010.

[20] ———, *Superperverse intersection cohomology: stratification (in)dependence*, Math. Z. **252** (2006), 49–70.

[21] ———, *Intersection homology of stratified fibrations and neighborhoods*, Adv. Math. **215** (2007), no. 1, 24–65.

[22] ———, *Singular chain intersection homology for traditional and super-perversities*, Trans. Amer. Math. Soc. **359** (2007), 1977–2019.

[23] ———, *Intersection homology and Poincaré duality on homotopically stratified spaces*, Geom. Topol. **13** (2009), 2163–2204.

[24] Greg Friedman, *Intersection homology Künneth theorems*, Math. Ann. **343** (2009), no. 2, 371–395.

[25] Greg Friedman, *On the chain-level intersection pairing for PL pseudomanifolds*, Homology, Homotopy and Applications **11** (2009), 261–314.

[26] Greg Friedman and Eugénie Hunsicker, *Additivity and non-additivity for perverse signatures*, preprint; see http://arxiv.org/abs/0911.3915.

[27] Greg Friedman and James McClure, *Cup and cap products in intersection (co)homology*, in preparation.

[28] Mark Goresky and Robert MacPherson, *Intersection homology theory*, Topology **19** (1980), 135–162.

[29] ______, *Intersection homology II*, Invent. Math. **72** (1983), 77–129.

[30] Mark Goresky and Paul Siegel, *Linking pairings on singular spaces*, Comment. Math. Helvetici **58** (1983), 96–110.

[31] Nathan Habegger, *From PL to sheaf theory (rock to Bach)*, Intersection cohomology (Bern, 1983), Progr. Math., vol. 50, Birkhäuser Boston, Boston, MA, 1984, pp. 23–34.

[32] Nathan Habegger and Leslie Saper, *Intersection cohomology of cs-spaces and Zeeman's filtration*, Invent. Math. **105** (1991), 247–272.

[33] A. Haefliger, *Introduction to piecewise linear intersection homology*, Intersection cohomology (Bern, 1983), Progr. Math., vol. 50, Birkhäuser Boston, Boston, MA, 1984, pp. 1–21.

[34] Tamás Hausel, Eugenie Hunsicker, and Rafe Mazzeo, *Hodge cohomology of gravitational instantons*, Duke Math. J. **122** (2004), no. 3, 485–548.

[35] Eugénie Hunsicker, *Hodge and signature theorems for a family of manifolds with fibre bundle boundary*, Geom. Topol. **11** (2007), 1581–1622.

[36] Eugenie Hunsicker and Rafe Mazzeo, *Harmonic forms on manifolds with edges*, Int. Math. Res. Not. (2005), no. 52, 3229–3272.

[37] Masaki Kashiwara and Pierre Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1994.

[38] Henry C. King, *Topological invariance of intersection homology without sheaves*, Topology Appl. **20** (1985), 149–160.

[39] Frances Kirwan and Jonathan Woolf, *An introduction to intersection homology theory. second edition*, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[40] Robert MacPherson, *Intersection homology and perverse sheaves*, Unpublished Colloquium Lectures.

[41] Charles Richard Francis Maunder, *Algebraic topology*, Cambridge University Press, Cambridge, 1980.

[42] J.E. McClure, *On the chain-level intersection pairing for PL manifolds*, Geom. Topol. **10** (2006), 1391–1424.

[43] Werner Müller, *L^2-index theory, eta invariants and values of L-functions*, Geometric and topological invariants of elliptic operators (Brunswick, ME, 1988), Contemp. Math., vol. 105, Amer. Math. Soc., Providence, RI, 1990, pp. 145–189.

[44] James R. Munkres, *Elements of algebraic topology*, Addison-Wesley, Reading, MA, 1984.

[45] Leslie Saper, *L^2-cohomology of locally symmetric spaces. I*, Pure Appl. Math. Q. **1** (2005), no. 4, part 3, 889–937.

[46] ———, *$\mathscr{L}$-modules and the conjecture of Rapoport and Goresky-MacPherson*, Automorphic forms. I. Astérisque (2005), no. 298, 319–334.

[47] Leslie Saper and Mark Stern, *L_2-cohomology of arithmetic varieties*, Ann. of Math. (2) **132** (1990), no. 1, 1–69.

[48] Martintxo Saralegi-Aranguren, *de Rham intersection cohomology for general perversities*, Illinois J. Math. **49** (2005), no. 3, 737–758 (electronic).

[49] P.H. Siegel, *Witt spaces: a geometric cycle theory for KO-homology at odd primes*, American J. Math. **110** (1934), 571–92.