A multiperversity generalization of intersection homology

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Abstract

We define a generalization of intersection homology, based on considering a set of perversities rather than a single perversity, and explore some of its properties. The question of whether these homology groups are independent of the stratification is left open, however some steps in this direction are made following known proofs of the topological invariance of the classical intersection homology groups.

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1 Introduction

Intersection homology groups $IH_*^{\bar{p}}(X)$ for stratified spaces X were discovered by Goresky and MacPherson in [3, 4]. These groups depend on certain sequences of numbers $\bar{p} = (\bar{p}(0), \bar{p}(1), \dots)$ which are called perversities. These sequences are defined to satisfy the following simple properties: $\bar{p}(0) = \bar{p}(1) = \bar{p}(2) = 0$ and $\bar{p}(i) \le \bar{p}(i+1) \le \bar{p}(i) + 1$. Roughly speaking, these perversities describe how simplices are allowed to intersect the singular part of X. Goresky and MacPherson proved two remarkable basic results for these groups for the case that X is a pseudomanifold. The first is that intersection homology groups are topological invariants and, in particular, do not depend on the stratification of X. From this it follows that for the case that X is a compact n-manifold, $IH_i^{\bar{p}}(X)$ coincide with the usual homology $H_i(X)$ for every perversity and every stratification. The second is that intersection homology of n-dimensional pseudomanifolds satisfies a certain form of duality

$$IH_i^{\bar{p}}(X; \mathbb{Q}) = IH_{n-i}^{\hat{p}}(X; \mathbb{Q}),$$

where $\hat{p}$ is the dual perversity of the perversity $\bar{p}$, defined by $\hat{p}(i) + \bar{p}(i) = i - 2$ for every i.

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When X is a complex algebraic variety, X can be stratified with only even dimensional strata and it follows that for the middle perversity $\bar{p}$ defined by $\bar{p}(i) = [(i-1)/2]$,

$$IH_i^{\bar{p}}(X;\mathbb{Q}) = IH_{n-i}^{\bar{p}}(X;\mathbb{Q}).$$

While general intersection homology groups are remarkable invariants of general singular spaces and pseudomanifolds, it is fair to say that the case of middle perversity intersection homology of algebraic varieties accounts for most applications. Those range from algebraic geometry and representation theory to the combinatorial theory of convex polytopes.

In this paper we explore an extension of intersection homology groups $IH_*^J(X)$ when a single perversity is replaced by a whole set J of perversities. The purpose of this study is to obtain further interesting homology-like invariants for singular spaces. As intersection homology (in middle perversity) leads to remarkable combinatorial invariants of convex polytopes, hoping to find (eventually) additional combinatorial invariants for polytopes was another motivation.

Our analysis of these invariants follows a paper by Henry King [5], which itself builds upon the work of Goresky and MacPherson in [4]. King considered singular intersection homology theory and presented a direct proof (avoiding Deligne's sheaf) that singular intersection homology is a topological invariant of X. In the setting of singular simplices, topological invariance is equivalent to showing that the invariants do not depend on the stratification. King's proof of the topological invariance, which itself is based upon the Goresky-MacPherson proof in [4] (though King avoids using sheaves), is based upon several properties of intersection homology. It requires extensions of several basic properties of homology and, in addition, an understanding of intersection homology of cones over spaces and of products of spaces with manifolds. King's Theorem 9 gives an inductive proof of topological invariance and the steps in his argument can be regarded as a "road map" for such a proof in more general cases. King's Theorem 10 asserts that invariants of singular spaces with certain natural properties are equivalent if they have the same behavior with respect to forming a cone over a space. The behavior of intersection homology with respect to forming a cone is very simple and thus King's result can be regarded as an invitation for further invariants

For a compact filtered space X, let cX be an open cone over X (with the natural filtration whereby $(cX)^i = X^{i-1} \times \mathbb{R}$ and $(cX)^0$ is the cone point). A main result of this paper is a "cone formula", namely, a formula for $IH_*^J(cX)$. We also establish some basic properties such as Mayer-Vietoris and Künneth theorems. Our cone formula is rather complicated and we cannot use it so far to prove that our invariants do not depend on the stratification. Proving (or disproving) topological invariance remains the main open problem. Another problem would be to understand if some form of duality holds in this greater generality.

2 Definition and most basic properties

All spaces will be filtered spaces X. An n-dimensional filtered space is a Hausdorff, metrizable, and separable space $X = X^n$ filtered by closed subsets X^i, called *skeleta*:

$$X = X^n \supset X^{n-1} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset.$$

The sets $X^i - X^{i-1}$, if not empty, are called *strata*. Such a space is called an n-dimensional locally-conelike stratified space if each $X^i - X^{i-1}$ is a either empty or a topological i-dimensional manifold, and if for each $x \in X^i - X^{i-1}$, $0 \le i \le n-1$, there is a neighborhood of x filtered homeomorphic to $\mathbb{R}^i \times cY$, where Y is a compact filtered space and $\mathbb{R}^i \times cY$ is given the filtration induced by that of Y (see [5]). Here cY is the *open* cone on Y. Throughout, cX will denote the *open* cone if X is a space, and $c\xi$ will denote the *closed* cone if ξ is a chain.

Let $J = \{\bar{p}_1, \cdots, \bar{p}_m\}$ be a collection of traditional perversities, i.e. for each j, $\bar{p}_j(2) = 0$ and $\bar{p}_j(k) \le \bar{p}_j(k+1) \le \bar{p}_j(k) + 1$.

Recall ([5]) that a singular simplex $\sigma \in C_i(X; G)$, $\sigma : \Delta^i \to X$, is said to be $\bar{p}$ allowable if for each k, $\sigma^{-1}(X^{n-k})$ is a subset of the $i-k+\bar{p}(k)$ skeleton of the model simplex Δ^i. $IC^{\bar{p}}_*(X)$ is the complex of chains ξ such that each simplex of ξ and $\partial\xi$ (with non-zero coefficient) is allowable.

Let $IC^J_i(X; G)$ be the subgroup of $C_i(X; G)$ consisting of i-chains ξ such that $\xi = \sum_j \xi_{\bar{p}_j}$ and $\partial\xi = \sum_j \zeta_{\bar{p}_j}$, where $\xi_{\bar{p}_j}$ and $\zeta_{\bar{p}_j}$ are G-linear combinations of $\bar{p}_j$ allowable simplices. In other words, each simplex of ξ and $\partial\xi$ must be allowable with respect to some perversity in J. Note that $\zeta_{\bar{p}_j}$ is not necessarily equal to $\partial\xi_{\bar{p}_j}$. Let $IH^J_*(X; G)$ be the homology groups of $IC^J_*(X; G)$. We will generally omit the coefficient group G to simplify notation.

Note that if J contains only a single perversity $\bar{p}$, then $IH^J_* = IH^{\bar{p}}_*$. If $J = \emptyset$, then we explicitly take $IC^J_i = 0$.

If U is an open subset of X, then we define $IH^J_*(X, U)$ in the usual way.

Remark 2.1. Although we use the language of singular chains throughout, we observe that all constructions and theorems to follow could also be formulated in the world of PL chains on piecewise-linear stratified spaces. Instead of modifying the singular chain definition of intersection homology as given by King in [5], one would instead modifying (in the obvious analogous ways) the original Goresky-MacPherson definition of intersection homology in [3].

The following basic lemma will be useful throughout the paper.

Lemma 2.2 (Subdivision). *Suppose ξ is a cycle in $IC^J_i(X, U)$. Then ξ is homologous to any subdivision ξ' of ξ.*

Proof. The proof is essentially the same as that given in [1] for the case in which J contains a single perversity - if σ is $\bar{p}_j$ allowable for some $\bar{p}_j$, then each simplex in any subdivision of σ is also $\bar{p}_j$ allowable, so ξ' is allowable. Furthermore, an allowable homology between ξ and ξ' is constructed by suitably subdividing $\Delta \times [0, 1]$ for each simplex $\sigma : \Delta \to X$ so that one end, say $\Delta \times 0$, remains unsubdivided, and the other end, $\Delta \times 1$, is subdivided as per ξ'. These subdivisions can be chosen so that the chain determined by the composition of σ with the projection of the subdivided $\Delta \times I$ to Δ is $\bar{p}_j$ allowable. Employing this process compatibly for each σ and with the relevant coefficients provides the desired homology. More precise details can be found in [1]. The main point here is that we can maintain the decomposition of ξ into pieces allowable with respect to different perversities. $\square$

3 The cone formula

In this section we find the formula for the multiperverse intersection homology of a cone. This formula plays a fundamental role in single perversity intersection homology theory.

Let $J(i,k) = \{\bar{p}_j \in J \mid 0 \le i - (k+1) + \bar{p}_j(k+1)\}$. These are the perversities $\bar{p}$ such that $IC_i^{\bar{p}}(cX)$, where X is a k-dimensional compact filtered space, includes chains whose supports intersect the cone point x.

Let $IC_*^{J(n)}(X)$ be the subcomplex of $IC_*^J(X)$ such that

$$IC_i^{J(n)}(X) = \{\xi \in IC_i^J(X) \mid \xi \in IC_i^{J(i+1,n)}(X), \partial\xi \in IC_{i-1}^{J(i,n)}(X)\}.$$

Recall that $IC_i^{\emptyset}(X) = 0$, by definition, and note that the boundary map on this complex is well-defined as the restriction of the boundary map of $IC_*^J(X)$. We define the homology groups $IH_*^{J(n)}(X)$ as the homology groups of $IC_*^{J(n)}(X)$.

Finally, let

$$IC_*^{J/n}(X) = IC_*^J(X)/IC_*^{J(n)}(X),$$

and let $IH_*^{J/n}(X)$ be the corresponding homology groups. We will prove the following Cone Formula.

Proposition 3.1 (Cone Formula). *Let X be an n-dimensional compact filtered space. Then $IH_i^J(cX) \cong IH_i^{J/n}(X \times \mathbb{R}) \cong IH_i^{J/n}(X)$ for all i.*

First, we prove the following lemma.

Lemma 3.2. *Suppose ξ is a cycle in $IC_i^J(cX)$, where X is an n-dimensional compact filtered space. Let x denote the cone point of cX. Then ξ is homologous (in $IC_*^J(cX)$) to a chain of the form $c\psi + \gamma \in IC_i^J(cX)$, where*

1. *$|\gamma| \cap x = \emptyset$,*
2. *$\psi = -\partial\gamma$, $|\psi| \cap x = \emptyset$, $\psi \in IC_{i-1}^{J(i,n)}(cX - x)$, and $c\psi$ is the (closed) cone on ψ.*

In particular, ξ is homologous to a cycle whose simplices intersect x at most at vertices.

Proof. We begin by subdividing ξ and then breaking it up into pieces, one of which will be γ and one of which we will modify to be $c\psi$. Let U be an open neighborhood of the cone point $x \in cX$ that is disjoint from all simplices of ξ whose supports do not intersect x. Let V be $cX - x \cong X \times \mathbb{R}$. Let ξ' be a subdivision of ξ that is $\{U, V\}$ small, so that ξ' is homologous to ξ and $\xi' = \eta + \gamma$, η and γ are allowable, $|\eta| \subset U$, and $|\gamma| \subset V$. Such subdivisions can be accomplished as per Proposition 2.9 of [1]. That proposition deals only with single perversity intersection homology, but the arguments work equally well (with obvious modifications) in our more general setting. Furthermore, by our choice of U, all simplices in η will be simplices in a subdivision of those simplices of ξ that intersect x. Since these simplices of ξ must all be $J(i,n)$ allowable, so too will all i-simplices of η be $J(i,n)$ allowable, as follows from the multiperversity generalization of the subdivision allowability arguments of Lemma 2.6 of [1]. Furthermore, we claim that $\partial\eta = -\partial\gamma$ is $J(i,n)$ allowable. This is a consequence of the actual chain map constructed in [1, Proposition 2.9] for breaking up chains into small pieces. The upshot is that one of the following scenarios holds for each $i-1$ simplex μ in η:

1. μ has no l face, $0 \leq l \leq i-1$, that is in the l skeleton of any simplex of ξ (in the sequel, we will refer to this condition as being "deeply embedded"). In this case, μ inherits the allowability of whichever simplex of ξ it is contained in, and in this case such a simplex must be $J(i,n)$ allowable. (Note: for statements such as these about subsets and inclusions, we are really referring to the simplicial models upon which the singular simplices are built - see [1] for more careful statements).

2. μ is a simplex in a subdivision of $\partial\xi$. But since ξ is a cycle, this is not possible.

We next claim that ξ' is homologous to $c(\partial\eta)+\gamma$. We check the details:

$c(\partial\eta)+\gamma$ is allowable: γ is already allowable and since $\partial(c(\partial\eta)+\gamma) = \partial\eta+\partial\gamma = \partial(\eta+\gamma) = \partial\xi' = 0$, to check allowability, we only need check the allowability of each simplex $c\sigma$ for $\sigma \in \partial\eta$. But each such σ is in $IC^{J(i,n)}_{i-1}(cX-x)$. So each σ is $\bar{p}$ allowable for some $\bar{p} \in J(i,n)$. But this implies that $c\sigma$ is allowable by the arguments in, e.g. King [5]. Geometrically, for $k < n+1$, $(c\sigma)^{-1}((cX)^{n+1-k})$ intersects skeleton of Δ^i of one dimension greater than the skeleton containing $\sigma^{-1}(cX-x)$. However, $c\sigma$ is one dimension greater than σ, so the changes offset in the allowability formula. $c\sigma^{-1}(x)$ is a single vertex, and this is also allowable for each $\bar{p} \in J(i,n)$. Thus $c(\partial\eta)+\gamma$ is allowable.

To obtain the homology, we next observe that $c\eta$ is itself allowable. To see this, we again consider $c\sigma$ for each simplex σ in η. Once again, for $k \neq n+1$, the dimension of the minimal skeleton of Δ^{i+1} containing $(c\sigma)^{-1}(cX)^{n+1-k}$ is at most one larger than the analogous dimension for σ, which is offset by the increase in dimension from σ to $c\sigma$ in the allowability formula. For $(c\sigma^{-1})(x)$, the same argument applies if $|\sigma| \cap x \neq \emptyset$, while if $|\sigma| \cap x = \emptyset$, $(c\sigma)^{-1}(x)$ will be a single vertex of Δ, which once again is allowable for each $\bar{p} \in J(i,n)$.

Now let $H = c\eta + \gamma \times [0,1]$, where $\gamma \times [0,1]$ is induced by a subdivision of the trivial homotopy (which is also readily verified to preserve allowability as in [2]). Then $\partial H = \eta - c(\partial\eta) + \gamma - \gamma = (\eta+\gamma) - (c(\partial\eta)+\gamma)$, demonstrating the desired homology. $\square$

Now we can prove the Cone Formula.

Proof of Cone Formula. The second isomorphism is simply stratum-preserving homotopy equivalence or the Künneth theorem for products with manifolds (see Proposition 4.1, below, which is independent of the current proposition). So we focus on the first isomorphism. Throughout, we identify $X \times \mathbb{R}$ with $cX - x$, where x is the cone point.

We define a homomorphism $f : IH_i^{J/n}(X \times \mathbb{R}) \to IH_i^J(cX)$ as follows. Let ξ be a relative cycle representing an element of $IH_i^{J/n}(X\times\mathbb{R})$. So ξ is J allowable and $\partial\xi$ is $J(i,n)$ allowable (since $\partial^2\xi = 0$, the condition that $\partial(\partial\xi)$ be $J(i-1,n)$ allowable is satisfied for free). Then let $f(\xi)$ be $\xi - c(\partial\xi)$ in $IC_i^J(cX)$. This chain is J allowable precisely because $\partial\xi$ is $J(i,n)$ allowable, and it is an absolute cycle. We must show that f is well defined as a map on homology. So let ξ_1 be another chain representing the same cycle as ξ in $IH_i^{J/n}(X\times\mathbb{R})$. So there exists a J-allowable $i+1$ chain Ξ such that $\partial\Xi = \xi - \xi_1 + \phi$, where ϕ is $J(i+1,n)$ allowable and $\partial\phi$ is $J(i,n)$ allowable. Note that $\partial\phi = \partial\xi_1 - \partial\xi$. Then $\Xi - c\phi$ is J-allowable (again because ϕ is $J(i+1,n)$ allowable and $\partial\phi$ is $J(i,n)$ allowable, so that coning on ϕ gives allowable chains). But now $\partial(\Xi - c\phi) = \partial\Xi - \partial(c\phi) = \partial\Xi - (\phi - c(\partial\phi)) = \xi - \xi_1 + \phi - (\phi - c(\partial\xi_1 - \partial\xi)) = \xi - c\partial\xi - (\xi_1 - c\partial\xi_1) = f(\xi) - f(\xi_1)$. So f is well-defined.

By Lemma 3.2, f is a surjective homomorphism. To check that it is injective, suppose that $f(\xi) = \xi - c\partial\xi$ is the image of a chain representing a cycle in $IH_i^{J/n}(X\times\mathbb{R})$ and that $f(\xi) = 0$ in $IH_i^J(cX)$. Then there is a J-allowable $i+1$ chain Ξ such that $\partial\Xi = \xi - c\partial\xi$. We will split Ξ into pieces. Let $V = cX - x$, and let U be an open neighborhood of the cone point x of cX such that any simplex σ in Ξ with $x \notin |\sigma|$ satisfies $|\sigma| \cap U = \emptyset$. In particular, $|\xi|\cap U = \emptyset$. Now, as in the proof of Lemma 3.2, we may employ a multiperversity generalization of Proposition 2.9 of [1] to break Ξ into $\Xi = \Xi_U + \Xi_V$, where $|\Xi_U| \subset U$, $|\Xi_V| \subset V$, and each of these chains is allowable. Again, the argument of [1] carries over because, by construction, each simplex of Ξ_U and Ξ_V is a simplex in a subdivision of Ξ, and each such simplex inherits the allowability of its parent simplex, and similarly each i simplex of $\partial\Xi_U$ or $\partial\Xi_V$ is either deeply enough embedded in a simplex of Ξ that it inherits its allowability or it is a subdivision simplex of a simplex already in $\partial\Xi$ and hence inherits allowability from that.

Now we consider Ξ_V and $\partial\Xi_V$. Since $\partial(\Xi_U + \Xi_V)$ is just a subdivision of $\partial\Xi = \xi - c\partial\xi$ and since $|\xi| \cap U = \emptyset$, a subdivision of ξ forms part of the boundary of Ξ_V. In fact, $\partial\Xi_V = \xi' + \omega + \phi$, where ξ' is a subdivision of ξ, ω is a subchain of a subdivision of $-c\partial\xi$ and ϕ is a new bit of boundary that results from cutting Ξ_U and Ξ_V apart. But now since each i simplex in ω lies in a subdivision of a simplex of $c\partial\xi$, which was $J(i,n)$ allowable, ω is $J(i,n)$ allowable. Similarly since $|\phi|$ must lie in $U\cap V \subset U$, each i-simplex of ϕ is in a subdivision of a simplex σ of Ξ whose support contains x. But each such simplex of Ξ must be $J(i+1,n)$ allowable and so ϕ is $J(i+1,n)$ allowable (again, ϕ must be one of these "deeply embedded" subdivision simplices or else it would be part of the subdivision of $\partial\Xi = \xi - c\partial\xi$). Thus $\xi' = \partial\Xi_V - \omega - \phi$. As we've seen, ω is $J(i,n)$ allowable (and hence also $J(i+1,n)$ allowable since $J(i,n) \subset J(i+1,n)$), and ϕ is $J(i+1,n)$ allowable, and $\partial(\omega+\phi) = -\partial\xi'$, which is $J(i,n)$ allowable as a subdivision of $\partial\xi$, which was $J(i,n)$ allowable to begin with. And so ξ' represents 0 in $IH_i^{J/n}(X\times\mathbb{R})$. But ξ' is a subdivision of ξ and so by the Lemma 2.2, ξ and ξ' are homologous. Thus f is injective, and we are done.

$\square$

Corollary 3.3. *If Y is an n-dimensional compact filtered space, then $IH_i^J(cY, Y\times\mathbb{R}) \cong IH_{i-1}^{J(n)}(Y)$ for all i.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
\to IH_i^J(Y) & \longrightarrow & IH_i^{J/n}(Y) & \xrightarrow{\partial_*} & IH_{i-1}^{J(n)}(Y) & \longrightarrow & IH_{i-1}^J(Y) & \longrightarrow & \\
\Big\downarrow {=} & & \Big\downarrow f_* & & \Big\downarrow c & & \Big\downarrow & & \\
\to IH_i^J(Y) & \xrightarrow{\text{inc}_*} & IH_i^J(cY) & \longrightarrow & IH_i^J(cY, Y \times \mathbb{R}) & \xrightarrow{\partial_*} & IH_{i-1}^J(Y) & \longrightarrow & .
\end{array}$$

The first line is the exact sequence induced by the short exact sequence

$$0 \longrightarrow IC_i^{J(n)}(Y) \longrightarrow IC_i^J(Y) \longrightarrow IV_i^{J/n}(Y) \longrightarrow 0.$$

The second line is the exact sequence induced by the short exact sequence

$$0 \longrightarrow IC_i^J(Y \times \mathbb{R}) \longrightarrow IC_i^J(cY) \longrightarrow IC_i^J(cY, Y \times \mathbb{R}) \longrightarrow 0.$$

Once we show that the diagram commutes, the corollary will follow from Proposition 3.1 and the five lemma.

The map f_* in the diagram is that of Proposition 3.1, while the map labeled c takes the cone on chains.

The commutativity of the right square is straightforward, while that of the left and central squares follows from the definition of f_* and some obvious diagram chasing.. ☐

As a sample computation, let us see that Corollary 3.3 provides the correct result when applied to the case where J contains a single perversity. So, let $J = \{\bar{p}\}$. Then the corollary claims that for an n-dimensional compact locally-conelike space Y, $IH_i^{\bar{p}}(cY, Y \times \mathbb{R}) \cong IH_i^J(cY, Y \times \mathbb{R}) = IH_{i-1}^{J(n)}(Y)$. Let us see how this reconciles with the known computations for $IH_i^{\bar{p}}(cY, Y \times \mathbb{R})$. We need to compute $IC_i^{J(n)}(Y)$, which by definition is

$$IC_i^{J(n)}(X) = \{\xi \in IC_i^J(X) \mid \xi \in IC_i^{J(i+1,n)}(X), \partial\xi \in IC_{i-1}^{J(i,n)}(X)\}.$$

So what are the $J(i,n)$? Again, by definition, $J(i,n) = \{\bar{p}_j \in J \mid 0 \leq i-(n+1)+\bar{p}_j(n+1)\}$. Since our current choice of J contains only $\bar{p}$, we see that $J(i,n) = \bar{p}$ if $i \geq n+1-\bar{p}(n+1)$, and that it is empty otherwise (this is the key point). Thus $IC_i^{J(n)}(Y)$ is equal to $IC_i^{\bar{p}}(Y)$ if $i \geq n+1-\bar{p}(n+1)$, it is equal to the cycles of $IC_i^{\bar{p}}(Y)$ if $i = n - \bar{p}(n+1)$, and it is the trivial group if $i < n - \bar{p}(n+1)$. Thus applying the corollary and computing $IH_{i-1}^{J(n)}(Y)$, we obtain

$$IH_i^J(cY \times \mathbb{R}, Y) \cong \begin{cases} IH_{i-1}^{\bar{p}}(Y), & i \geq n+1-\bar{p}(n+1), \\ 0, & i < n+1-\bar{p}(n+1). \end{cases}$$

Indeed, this is the classical formula.

4 Some more basic properties

Next we show that IH_*^J satisfies some of the other familiar properties of ordinary intersection homology.

Proposition 4.1 (Stratified homotopy invariance). *IH_*^J, $IH_*^{J(n)}$, and $IH_*^{J/n}$ are invariant under stratum-preserving homotopy equivalences. In addition, any two chains that differ only by a stratum-preserving homotopy are homologous in any of these theories.*

Proof. The proofs follow just as for the single perversity case; see [2]. The main point is that given a $\bar{p}$-allowable simplex in a filtered space, $\sigma : \Delta^i \to X$, each simplex in the chain induced by a subdivision of $\Delta^i \times [0,1]$ and a stratum-preserving homotopy $H : \Delta^i \times [0,1] \to X$ from σ is also $\bar{p}$-allowable. □

Corollary 4.2 (Künneth theorem for products with $\mathbb{R}^k$). *Let X be a filtered space, and suppose that $\mathbb{R}^k$ is unfiltered and $X \times \mathbb{R}^k$ is given the obvious product filtration. Then $IH_*^J(X \times \mathbb{R}^k) \cong IH_*^J(X)$, $IH_*^{J(n)}(X \times \mathbb{R}^k) \cong IH_*^{J(n)}(X)$, and $IH_*^{J/n}(X \times \mathbb{R}^k) \cong IH_*^{J/n}(X)$.*

Proposition 4.3. *Let X be a filtered space. Then IH_*^J admits Mayer-Vietoris sequences relating open sets U, V, $U \cap V$, and $U \cup V$. IH_*^J also admits excision $IH_*^J(X, U) \cong IH_*^J(X - K, U - K)$, for K closed in the open set U.*

Proof. Once again, the proof is a straightforward generalization of that for single perversities; see [1]. The idea is to follow the proofs for ordinary singular homology as given in, e.g., [6]. The main technical point is the ability to break a chain into small pieces subordinate to some cover using subdivision. The main difficulty in intersection homology comes in ensuring that these pieces are allowable. There is no problem with the simplices in a chain, themselves, as subdivision preserves allowability. The difficulty is in making sure that the new boundaries created when pieces of the chain are broken apart are allowable. It is shown in [1] how to do this in such a way that the newly created boundary $i-1$ simplices inherit the same allowability as the i-simplices out of which they are carved. In the current case, the J-allowability of the original simplices thus ensures the J-allowability of the new pieces and their new boundaries after performing the construction of [1]. □

Note that $IH_*^{J(n)}$ and $IH_*^{J/n}$ generally will not admit Mayer-Vietoris properties and excision. This can be seen by considering the case where $J = \{\bar{p}\}$. In this case, $IC_*^{J(n)}$ is a truncation of $IC_*^{\bar{p}}$, so that $IH_*^{J(n)}$ will equal $IH_*^{\bar{p}}$ for large $*$ and 0 below a certain cutoff dimension. Since $IH_*^{\bar{p}}$ itself admits Mayer-Vietoris sequences, $IH^{J(n)}$ will simply truncate that Mayer-Vietoris sequence, not preserving exactness in general. This failure can also be seen as a consequence of the fact that the allowability conditions on boundary simplices of chains in $IC_*^{J(n)}$ is more stringent than that on the simplices themselves. Thus the above arguments do not guarantee that the process of [1] yields allowable chains once we break a chain into pieces.

Proposition 4.4 (Künneth theorem for products with manifolds). *If X is a filtered space, M is an unfiltered k-manifold, and $X \times M$ is given the product filtration, then $IH_*^J(X \times M) = (\bigoplus_{p+q=*} IH_p^J(X) \otimes IH_q^J(M)) \oplus (\bigoplus_{p+q=*-1} IH_p^J(X) * IH_q^J(M))$.*

Proof. This property is proven in [5] for single perversity intersection homology by noting that it is a consequence of the existence of Mayer-Vietoris sequences and the Künneth theorem for products with $\mathbb{R}^k$. Since IH_*^J also has these properties, the same proof applies. □

5 Dimension relations

In this section, we demonstrate certain formulas that hold among the groups IH_*^J, $IH_*^{J(n)}$, and $IH_*^{J/n}$ when the spaces involved no longer have dimension n, themselves, but rather a smaller dimension t. In particular, we show that in this case some otherwise complicated homomorphisms become either trivial or isomorphisms.

Lemma 5.1. *Let Y be a t-dimensional compact filtered space, and suppose $n \geq t$. Then $IH_i^{J(n)}(cY) = 0$ and $IH_i^J(cY) \cong IH_i^{J/n}(cY)$ for all i.*

Proof. The second statement follows from the first using the long exact sequence

$$IH_i^{J(n)}(cY) \longrightarrow IH_i^J(cY) \longrightarrow IH_i^{J/n}(cY) \longrightarrow IH_{i-1}^{J(n)}(cY).$$

To begin, we assume $t = n$.

Suppose ξ represents a cycle in $IH_i^{J(n)}(cY)$. Then each simplex in ξ is allowable with respect to a perversity $\bar{p}$ such that $0 \leq i+1-(n+1)+\bar{p}(n+1)$. We claim that each simplex in $\bar{c}\xi$ is also allowable with respect to such a perversity, which would demonstrate that each cycle bounds. The argument is essentially that which occurs for the intersection homology of a cone with a single perversity once we have passed the critical dimension at which all intersection homology dies. A thorough treatment of that case can be found in [5] and [1]. We indicate the ideas here:

Suppose σ is a simplex in ξ. If $\sigma^{-1}((cY)^{n+1-k})$, $k \leq n$, lies in the ℓ skeleton of Δ^i, then $(c\sigma)^{-1}((cY)^{n+1-k})$ lies in the $\ell+1$ skeleton of Δ^{i+1}. Allowability of σ implies that $\ell \leq i-k+\bar{p}(k)$ for some $\bar{p}$ in $J(i+1,n)$. This implies that $\ell+1 \leq i+1-k+\bar{p}(k)$, which is necessary for allowability for $c\sigma$ with respect to $\bar{p}$. This argument also works if $k = n+1$ and $|\sigma| \cap y \neq \emptyset$. The only other case to check is when $k = n+1$ and $|\sigma| \cap y = \emptyset$. In this case $(c\sigma)^{-1}(y)$ is in the 0 skeleton of $c\sigma$, but we must have $0 \leq i+1-(n+1)+\bar{p}(n+1)$ since $\bar{p} \in J(i+1,n)$. Thus $c\sigma$ is allowable. Since this works for each σ in ξ, we see that $c\xi$ is $J(i+1,n)$ allowable (and hence $J(i+2,n)$ allowable), and $\partial(c\xi) = \xi$, which was given as $J(i+1,n)$ allowable. Thus $c\xi \in IC_{i+1}^{J(n)}(cY)$, and $IH_*^{J(n)}(cY) = 0$.

Now, suppose that $t < n$. In each of the allowability arguments above, we must replace n with t. The first part of the argument goes through except for this cosmetic change. The only case that needs new verification is the case where $|\sigma| \cap y = \emptyset$. In this case we need $0 \leq i+1-(t+1)+\bar{p}(t+1)$. But we do know that $0 \leq i+1-(n+1)+\bar{p}(n+1)$, since $\bar{p} \in J(i+1,n)$, and so it suffices to have $-(t+1)+\bar{p}(t+1) \geq -(n+1)+\bar{p}(n+1)$, or $n+1-(t+1) \geq \bar{p}(n+1)-\bar{p}(t+1)$. But this is certainly true since $\bar{p}(k+1) \leq \bar{p}(k)+1$, by assumption for all k. □

Before moving on to the next lemma, we observe that for A an open subset of X, $IC_*^{J/n}(X,A)$ can be defined as the quotient of either inclusion $IC_*^{J(n)}(X,A) \hookrightarrow IC_*^J(X,A)$ or $IC_*^{J/n}(A) \hookrightarrow IC_*^{J/n}(X)$. In fact, we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & IC_*^{J(n)}(A) & \longrightarrow & IC_*^{J(n)}(X) & \longrightarrow & IC_*^{J(n)}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & IC_*^{J}(A) & \longrightarrow & IC_*^{J}(X) & \longrightarrow & IC_*^{J}(X,A) & \longrightarrow & 0
\end{array}$$

The vertical arrows readily can be checked to be injections, and so the snake lemma gives us the cokernel short exact sequence which is precisely

$$0 \longrightarrow IC_*^{J/n}(A) \longrightarrow IC_*^{J/n}(X) \longrightarrow IC_*^{J/n}(X,A) \longrightarrow 0.$$

Lemma 5.2. *Let Y be a t-dimensional compact filtered space, and suppose $n \geq t$. Then $IH_i^{J/n}(cY, Y) = 0$, and $IH_i^{J/n}(cY) \cong IH_i^{J/n}(Y)$ for all i.*

Proof. Consider the exact sequence

$$\longrightarrow IH_i^{J/n}(Y) \longrightarrow IH_i^{J/n}(cY) \longrightarrow IH_i^{J/n}(cY,Y) \longrightarrow$$

By Lemma 5.1, $IH_i^J(cY) \to IH_i^{J/n}(cY)$ is an isomorphism, and $IH_i^{J/n}(Y) \cong IH_i^J(cY)$, by Proposition 3.1. These isomorphisms can be seen to commute (recalling that $J(i,n) \subset J(i+1,n)$), and so $IH^{J/n}(Y) \to IH^{J/n}(cY)$ is an isomorphism, proving the claim. ☐

6 Towards topological invariance?

In this section, we outline a proposed proof of the topological invariance of IH_*^J for locally-conelike stratified spaces. This proposed proof follows the outline of King's proof in [5] for the single perversity case, which itself follows the main theme of the Goresky-MacPherson proof in [4], though avoiding the use of sheaves. We are currently unable to complete the proof at one step, but we will indicate how proving that step completes the whole proof. We will also show that topological invariance holds in a certain special case.

We begin by recalling the main ideas of the proof as presented in [5] of topological invariance of $IH_*^{\bar{p}}$. The idea is to start with the functor $X \to X^*$ taking a locally conelike stratified space (called a CS set in [5]) to the CS set that is topologically equivalent to X but that is stratified with the coarsest intrinsic stratification. This functor yields maps $IH_*^{\bar{p}}(X) \to IH_*^{\bar{p}}(X^*)$, since the property $\bar{p}(k) \leq \bar{p}(k+1)$ implies that if a simplex is allowable in X then it is also allowable in X^*. This will also be true for $IH_*^J(X)$. A CS set X has property $\mathcal{H}$ if $IH_*^{\bar{p}}(X) \to IH_*^{\bar{p}}(X^*)$ is an isomorphism for all perversities $\bar{p}$.

The proof then inducts on *depth*, the difference in dimension between the highest and lowest dimensional non-empty strata of X. Let $P(j)$, $Q(j)$, and $R(j)$ be the following statements:

P(j) All CS sets of depth $\leq j$ have property $\mathcal{H}$.

Q(j) All CS sets of the form $M \times cW$, M a manifold, W a compact filtered space, depth$(W) \leq j$ have property $\mathcal{H}$.

R(j) All CS sets of the form $\mathbb{R}^k \times cW$, W a compact filtered space, depth$(W) \leq j$ have property $\mathcal{H}$.

King goes on to show that $P(j)$ implies $R(j)$, $R(j)$ implies $Q(j)$, and $P(j)$ and $Q(j)$ together imply $P(j+1)$. This implies the topological invariance of $IH_*^{\bar{p}}$ by induction, since $P(1)$ is clearly true.

It is also clear that this framework would imply topological invariance of IH_*^J if one could prove this sequence of implications with property $\mathcal{H}$ replaced by property $\mathcal{H}^J$ stating that $IH_*^J(X) \to IH_*^J(X^*)$ is an isomorphism for any J.

In fact, the implications $R(j)$ implies $Q(j)$, and $P(j)$ and $Q(j)$ together imply $P(j+1)$, as proven in [5], depend only on geometric properties of CS sets, the Mayer-Vietoris theorem, and the Künneth theorem of a CS set with a manifold. Since these properties all hold for IH_*^J, so do these implications. The problematic one is $P(j) \Rightarrow R(j)$, which relies on the actual cone formula properties of $IH^{\bar{p}}$.

The setting for attempting to prove $P(j)$ implies $R(j)$ is geometrically the following: We have a CS set $\mathbb{R}^k \times cW$, W a compact filtered space such that depth$(W) \leq j$, and by the arguments of Proposition 6 of [5], $(\mathbb{R}^k \times cW)^* = \mathbb{R}^m \times cY$, where Y is a compact filtered space. We take $\dim(W) = s$ and $\dim(Y) = t$ so that $t = k - m + s$. Letting $h : \mathbb{R}^k \times cW \to \mathbb{R}^m \times cY$ represent the (non-stratum-preserving) homeomorphism, it is also shown that $h^{-1}(\mathbb{R}^m \times *) = \mathbb{R}^k \times cA$, where A is an $m-k-1$ homology sphere and a union of components of strata of W. One also notes that $\mathbb{R}^m \times cY - h(\mathbb{R}^k \times *)$, where $*$ is the cone point of cW, carries the intrinsic stratification of $\mathbb{R}^{k+1} \times W = \mathbb{R}^k \times (cW - *)$. Since each of these spaces has depth less than $\mathbb{R}^k \times cW$, they can be assumed to have property $\mathcal{H}$ by induction. This is the basis for the remainder of the argument showing that $IH_*^{\bar{p}}(\mathbb{R}^k \times cW) \cong IH_*^{\bar{p}}(\mathbb{R}^m \times cY)$, which is then based upon both some diagram chasing and some calculations involving the explicit cone formula for intersection homology with a single perversity. Unfortunately, points of this proof rely heavily on the precise formula - its vanishing in certain ranges and its equivalence to the intersection homology of the link in the other range. Thus any attempt to apply the overall argument to IH_*^J must be modified.

We do not at present know how to show that $IH_*^J(\mathbb{R}^k \times cW) \cong IH_*^J(\mathbb{R}^m \times cY)$, but we provide the partial result that $IH_*^J(cY)$ is a direct summand of $IH_*^J(cW)$, under the induction hypothesis $P(j)$.

So, consider the locally-conelike stratified space $(\mathbb{R}^k \times cW)^* = cY \times \mathbb{R}^m$, where W is a compact filtered space of depth $\leq j$ and Y is a t-dimensional compact filtered space. Let $t + m = n$, and let y be the cone point of y. Let $V = cY \times R^m - \mathbb{R}^k \times a$, where a is the cone point of cA so that $R^k \times a \subset y \times \mathbb{R}^m$. (Of course $\mathbb{R}^k \times a = \mathbb{R}^k \times w$, where w is the cone point of cW, under the homeomorphism h). So V carries the intrinsic stratification of $\mathbb{R}^{k+1} \times W = \mathbb{R}^k \times (cW - *)$, which is stratum-preserving homotopy equivalent to W. Thus if $P(j)$ holds for IH_*^J, then $IH_*^J(V) \cong IH_*^J(W)$.

Lemma 6.1. *With Y and V as just described, $IH_i^J(cY)$ is a direct summand in $IH_i^{J/n}(V)$ for all i.*

Note: no induction assumption is needed here.

Proof. Let $*$ be any point in $\mathbb{R}^m - \mathbb{R}^k \times a$ and let $\mathfrak{i} : cY \times * \hookrightarrow V$ be the inclusion. Since this inclusion preserves codimensions of strata, it induces a map $\mathfrak{i}_* : IH_*^J(cY) \to IH_*^J(V)$. This can be extended to a map $j_* : IH_i^J(cY) \to IH_i^{J/n}(V)$ by composing i_* with the natural map $p_* : IH_i^J(V) \to IH_i^{J/n}(V)$.

We find a map that splits j_*. Note that we have a codimension preserving retraction $r : V \to cY$ induced by projecting $cY \times \mathbb{R}^m$ to $cY \times *$. The retraction r induces a map r_* of all intersection homology groups of V to those of cY. In particular, we have $r_* : IH_i^{J/n}(V) \to IH_i^{J/n}(cY)$. We compose r_* with $\rho_* : IH_i^{J/n}(cY) \to IH_i^{J/t}(cY)$, which exists since $J(i,t) \supset J(i,n)$ since $t \leq n$. But $IH_i^{J/t}(cY)$ is isomorphic to $IH_i^J(cY)$ by Lemma 5.1. Let's call this isomorphism $k_* : IH_i^{J/t}(cY) \to IH_i^J(cY)$. We claim that $\rho_* r_* j_* k_*$ is the identity on $IH_i^{J/t}(cY)$, which implies that $k_* \rho_* r_*$ splits j_*.

To verify the claim, we need only start with a cycle ξ in $IC_i^{J/t}(cY)$ and chase it through the maps. By the chain of isomorphisms $IH_i^{J/t}(Y) \cong IH_i^J(cY) \cong IH^{J/t}(cY)$ and their constructions in Proposition 3.1 and Lemma 5.1, it suffices to assume that $\xi = c\psi + \gamma$, where γ is $J(i+1,t)$ allowable and supported in $Y \times \mathbb{R} \subset cY$ and that ψ is a $J(i,t)$ allowable $i-1$ chain, so that $c\psi$ is a $J(i,t)$ allowable i chain. This chain suffices to represent both our given cycle in $IH_i^{J/t}(cY)$ and its image under k_* in $IH_i^J(cY)$. Now, let's apply our other maps:

Under $\mathfrak{i}$, ξ is simply included into $cY \times * \subset V$. The map p_* is induced from the quotient $IC_i^J(V) \twoheadrightarrow IC_i^{J/n}(V)$, so we can keep ξ as a representative chain. The retract r collapses V back to $cY \times *$, still not effecting our cycle, and finally ρ_* is again induced by a quotient. Thus our representative cycle represents the same cycle after applying $\rho_* r_* j_* k_*$. ☐

Corollary 6.2. *If V, Y, and W are as above and we assume by $P(j)$ for IH_*^J that $IH_*^J(W) \cong IH_*^J(V)$, then $IH_*^J(cY)$ is a direct summand in $IH_*^{J/n}(W) \cong IH_*^J(cW)$.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} IH_*^J(W) & \longrightarrow & IH_*^{J/n}(W) \\ \downarrow \cong & & \downarrow \\ IH_*^J(V) & \longrightarrow & IH_*^{J/n}(V) \\ & \nwarrow & \uparrow \\ & & IH_*^J(cY), \end{array}$$

where the map $IH_*^J(cY) \to IH_*^{J/n}(V)$ is a split injection by the preceding lemma and the diagonal map is is induced by inclusion. The corollary now follows by diagram chasing. ☐

Remark 6.3. There are essentially two sticking points in finishing the proof of $P(j) \Rightarrow R(j)$ in the J case. The first is to show that $IH_*^{J/n}(cY) \cong IH^{J/n}(V)$, not just that it is a direct summand. The second is the show that if $IH_*^J(V) \cong IH_*^J(W)$, then $IH_*^{J/n}(V) \cong IH^{J/n}(W)$, or, equivalently, that if two spaces share the same intersection homology IH_*^J, then so do their cones.

Using Lemma 6.1, we can also prove the following basic case of invariance of IH_*^J under restratification:

Proposition 6.4. *Let Y be an $n-1$-dimensional compact locally-conelike stratified space. Let SY be the suspension of Y. Then $IH_*^J(cSY) \cong IH_*^J(cY)$.*

Proof. As noted in [5, Section 2], as topological spaces (ignoring the stratifications), $cSY \cong cY \times \mathbb{R}$ (recall that cX denotes the open cone on the space X). To see this, note that each point $y \in Y$ generates an open half-plane in $cY \times \mathbb{R}$ and an open cone on a closed interval in cSY. It is not hard to see that a standard homeomorphism between the open half-plane and the open cone on the closed interval can be used to construct a homeomorphism $cSY \cong cY \times \mathbb{R}$. Furthermore, if Y is filtered and we give suspensions and cones the naturally induced filtrations, then the only difference in the filtrations of the two spaces will be the existence of the extra cone point v in the stratification of cSY.

Interpreting Lemma 6.1 in this situation, we see that $IH_*^J(cY)$ (which is isomorphic to $IH^{J/n}(Y)$ by Lemma 5.1) is a summand in $IH_*^{J/n}(V)$, where $V = (cY \times \mathbb{R}) - v$. But $cY \times \mathbb{R} - v$ has a stratum-preserving deformation retract to $(cY \times -1) \cup (Y \times [-1,1]) \cup (cY \times 1) \cong SY$. So $IH_*^{J/n}(V) \cong IH_*^J(cSY)$. So we just need to show that the summand injection is in fact an isomorphism, which will be the case if each homology class in $IH_*^{J/n}(V)$ has a representative with support in $cY \times 1$

So suppose that ξ is a cycle representing an element of $IH_i^{J/n}(V)$. If $|\xi| \cap \mathbb{R}^- \times y = \emptyset$, where y is the cone point of cY and $\mathbb{R}^- = (0, -\infty)$, then we can use stratified homotopy invariance to push ξ into $cY \times 1$, whence ξ represents an element of $IH_i^{J/n}(cY)$. But if $|\xi| \cap R \times y \neq \emptyset$, then for each simplex σ of ξ with $\sigma \cap \mathbb{R}^- \times y \neq \emptyset$, if σ is $\bar{p}$ allowable, then we must have $0 \leq i - n + \bar{p}(n)$, since $R^- \times y$ has codimension n in V. So $\bar{p} \in J(i, n-1)$. Now we note that $J(i, n-1) \subset J(i+1, n)$ because $\bar{p}(n+1) \geq \bar{p}(n)$ for each perversity. So after subdividing and regrouping, we can write

$$\xi = \zeta + \gamma,$$

where $|\zeta| \cap \mathbb{R}^- \times y = \emptyset$ and $\gamma \subset IC_i^{J(i+1,n)}(V)$. We cannot, however, say that $\gamma = 0$ in $IC_i^{J/n}(V)$ because we do not know that $\partial\gamma$ is $J(i,n)$ allowable. We can, however, write

$$\partial\gamma = \eta + \omega,$$

where η consists of the simplices shared by $\partial\gamma$ and (negatively) $\partial\zeta$ and ω consists of simplices subdivided from $\partial\xi$. Note that this makes ω $J(i,n)$ allowable since ξ is a cycle in $IC_i^{J/n}(V)$. Furthermore, by applying the usual arguments (see [1] and the discussion above in the proof of Lemma 3.2), η will be $J(i+1,n)$ allowable, since γ is. Also, $|\eta| \cap \mathbb{R}^- \times y = \emptyset$.

Now, we do the following. By initial homotopies, we may assume that $|\xi| \subset (cY \times 1) \cup (Y \times [-1,1]) \cup (cY \times -1)$. We can also assume to have subdivided sufficiently that

$|\gamma| \subset cY \times -1$ (if this is not possible, then $|\xi| \cap cY \times -1 = \emptyset$, and we are done already). Consider now

$$\Gamma = (\gamma \times 1) - (\eta \times [-1,1]) - (\gamma \times -1),$$

where the second factor in each term refers to a chain in $\mathbb{R}$. In particular, $\eta \times [-1,1]$ is the product chain, triangulated compatibly with $\eta \times -1$ and $\eta \times 1$. $\eta \times [-1,1]$ is also $J(i+1,n)$ allowable since η is (see [2]). Thus Γ is $J(i+1,n)$ allowable. Furthermore,

$$\begin{aligned}\partial\Gamma &= (\eta \times 1) + (\omega \times 1) - ((\eta \times 1) - (\eta \times -1) + (\partial\eta \times [-1,1])) - ((\eta \times -1) + (\omega \times -1)) \\ &= (\omega \times 1) - (\partial\eta \times [-1,1]) - (\omega \times -1).\end{aligned}$$

We observe that $\partial\eta = -\partial\omega$, and again we can assume from our chain division process that $\partial\omega$ inherits the allowability of ω. Thus $\partial\omega$, and hence $\partial\eta \times [-1,1]$ are $J(i,n)$ allowable. Thus, overall, Γ is $J(i+1,n)$ allowable and $\partial\Gamma$ is $J(i,n)$ allowable, i.e. $\Gamma = 0 \in IC_i^{J/n}(V)$.

But now we can form $\xi' + \Gamma$, where ξ' is the subdivision of ξ that we have been using, and observe that the $\gamma \times -1$ terms cancel, leaving no simplices in $\xi' + \Gamma$ that intersect $\mathbb{R}^- \times y$. So $\xi' + \Gamma$ can be homotoped into $cY \times 1$, and since $\xi' + \Gamma$ and ξ represent the same element of $IH_i^{J/n}(V)$, we see that $IH_*^{J/n}(cY) \to IH_*^{J/n}(V)$ is indeed a surjection. □

7 A sample computation

In this section, we provide a sample calculation of IH_*^J in a simple case. This is provided mainly to prove that $IH_*^J \not\cong IH_*^{\bar{s}}$, where $\bar{s}$ is the perversity given by $\bar{s}(k) = \sup\{\bar{p}(k) \mid \bar{p} \in J\}$.

Let X^3 be a CS set of dimension 3 such that the singular set Σ consists of a single point x, i.e., $X = X^3 \supset X^2 = X^1 = X^0 = x$. Let $\bar{p}$ be the perversity $(0,1,1,1)$ (beginning with $\bar{p}(2) = 0$), and let $\bar{q}$ be the perversity $(0,0,1,2)$. Let $J = \{\bar{p}, \bar{q}\}$. We will calculate $IH_*^J(c(X \times S^1))$.

We begin by noting that the singular set of $X \times S^1$ is the single stratum $x \times S^1$, and it has codimension 3. Since $\bar{p}(3) > \bar{q}(3)$, $IH_*^J(X \times S^1) = IH^{\bar{p}}(X \times S^1)$. This will be typical of a space Z with only one singular stratum of codimension k: $IH^J(Z)$ will equal $IH_*^{\bar{r}}(Z)$, where $\bar{r}$ is any perversity in J such that $\bar{r}(k) \geq \bar{u}(k)$ for all $\bar{u} \in J$.

Next we consider $IH_*^{J(4)}(X \times S^1)$. To compute these groups, we need to know what $J(i,4)$ is for each i. Recall that $J(i,k) = \{\bar{p}_j \in J \mid 0 \leq i - (k+1) + \bar{p}_j(k+1)\}$. In this case, $J(i,4) = \{\bar{p}_j \in J \mid 0 \leq i - 5 + \bar{p}_j(5)\}$. Thus $J(i,4)$ contains $\bar{p}$ when $0 \leq i-4$, and it contains $\bar{q}$ when $0 \leq i-3$. In other words,

$$\begin{aligned} J(0,4) &= \emptyset \\ J(1,4) &= \emptyset \\ J(2,4) &= \emptyset \\ J(3,4) &= \bar{q} \\ J(4,4) &= J. \end{aligned}$$

Now, $IC_i^{J(4)}(X\times S^1) = \{\xi \in IC_i^J(X\times S^1) \mid \xi \in IC_i^{J(i+1,4)}(X\times S^1), \partial\xi \in IC_{i-1}^{J(i,4)}(X\times S^1)\}$. So

$$\begin{aligned}
&IC_0^{J(4)}(X\times S^1) = \emptyset\\
&IC_1^{J(4)}(X\times S^1) = \emptyset\\
&IC_2^{J(4)}(X\times S^1) = \{\xi \in IC_2^{\bar q}(X\times S^1) \mid \partial\xi = 0\}\\
&IC_3^{J(4)}(X\times S^1) = \{\xi \in IC_3^{J}(X\times S^1) = IC_3^{\bar p}(X\times S^1) \mid \partial\xi \in IC^{\bar q}(X\times S^1)\}\\
&IC_4^{J(4)}(X\times S^1) = \{\xi \in IC_4^{J}(X\times S^1) = IC_4^{\bar p}(X\times S^1)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&IH_0^{J(4)}(X\times S^1) = 0\\
&IH_1^{J(4)}(X\times S^1) = 0\\
&IH_2^{J(4)}(X\times S^1) = \operatorname{im}(IH_2^{\bar q}(X\times S^1) \to IH_2^{\bar p}(X\times S^1))\\
&IH_3^{J(4)}(X\times S^1) = IH_3^{\bar p}(X\times S^1)\\
&IH_4^{J(4)}(X\times S^1) = IH_4^{\bar p}(X\times S^1).
\end{aligned}$$

From the long exact sequence for $IH_*^{J(n)}$, IH_*^J, and $IH_*^{J/n}$, and recalling $IH_*^J(X\times S^1) = IH^{\bar p}(X\times S^1)$, we get

$$\begin{aligned}
&IH_0^{J/4}(X\times S^1) = IH_0^{\bar p}(X\times S^1)\\
&IH_1^{J/4}(X\times S^1) = IH_1^{\bar p}(X\times S^1)\\
&IH_2^{J/4}(X\times S^1) = \operatorname{cok}(IH_2^{\bar q}(X\times S^1) \to IH_2^{\bar p}(X\times S^1))\\
&IH_3^{J/4}(X\times S^1) = 0\\
&IH_4^{J/4}(X\times S^1) = 0,
\end{aligned}$$

which are the groups of $IH_*^J(c(X\times S^1))$ by the cone formula.

We note by contrast that the supremum perversity for J is $\bar s = (0,1,1,2)$, and by the standard cone formula,

$$IH_i^{\bar s}(c(X\times S^1)) = \begin{cases} 0, & i \geq 2,\\ IH_i^{\bar s}(X\times S^1) = IH_i^{\bar p}(X\times S^1), & i < 2. \end{cases}$$

These groups can well be different, as it is not difficult to find examples for which $\operatorname{cok}(IH_2^{\bar q}(X\times S^1) \to IH_2^{\bar p}(X\times S^1)) \neq 0$. For example, suppose that X is the suspension of the torus T^2. Then by the Künneth theorem for products with a manifold and some elementary computation of $IH_*(X)$, one has

$$IH_2^{\bar q}(X\times S^1) \cong (IH_2^{\bar q}(X)\otimes H_0(S_1)) \oplus (IH_1^{\bar q}(X)\otimes H_1(S_1)) \cong (0\otimes\mathbb{Z}) \oplus (\mathbb{Z}^2\otimes\mathbb{Z}) \cong \mathbb{Z}^2$$

$$IH_2^{\bar{p}}(X \times S^1) \cong (IH_2^{\bar{p}}(X) \otimes H_0(S_1)) \oplus (IH_1^{\bar{p}}(X) \otimes H_1(S_1)) \cong (\mathbb{Z}^2 \otimes \mathbb{Z}) \oplus (0 \otimes \mathbb{Z}) \cong \mathbb{Z}^2.$$

These groups are isomorphic (in fact they must be, at least with rational coefficients, by the Goresky-MacPherson Poincaré Duality [3]). However, we observe that the generators of $IH_2^{\bar{q}}(X \times S^1)$ are simply the products of generators of $H_1(T^2)$ with S^1, e.g. each has the form $\xi \times S^1$. These terms goes to 0 in $IH_2^{\bar{p}}(X \times S^1)$, where they are bounded by chains of the form $\bar{c}\xi \times S^1$, where $\bar{c}\xi$ is the cone on ξ to one "pole" of the suspension. Thus the homomorphism $IH_2^{\bar{q}}(X \times S^1) \to IH_2^{\bar{p}}(X \times S^1)$ is trivial, and the cokernel is $\mathbb{Z}^2$.

8 Duality?

A natural question to ask is whether IH_*^J satisfies any form of Poincaré duality on locally-conelike stratified spaces. We note that the most obvious thing to try does not work.

For a perversity, $\bar{p}$, let $\hat{p}$ be the dual perversity: $\bar{p}(k) + \hat{p}(k) = k - 2$. Then it is natural to try to set $\hat{J} = \{\hat{p} \mid \bar{p} \in J\}$.

However, suppose that X has only a point singularity of codimension n, that $J = \{\bar{p}, \bar{q}\}$ and that $\bar{p}(k) > \bar{q}(k)$. Then we note that $IH_*^J(X;\mathbb{Q}) = IH_*^{\bar{p}}(X;\mathbb{Q}) \cong IH_{n-*}^{\hat{p}}(X;\mathbb{Q})$. But since $\bar{p}(k) > \bar{q}(k)$, $\hat{q}(k) > \hat{p}(k)$ and $IH_{n-*}^{\hat{J}}(X;\mathbb{Q}) = IH_{n-*}^{\hat{q}}(X;\mathbb{Q})$. So this is not the correct way to get duality.

9 Questions

1. Is IH_*^J a topological invariant of locally-conelike stratified spaces?
2. If $\dim X = \dim Y$ and $IH_*^J(X) \cong IH_*^J(Y)$, does $IH_*^J(cX) \cong IH_*^J(cY)$?
3. Can IH^J always be calculated from knowledge of all $IH^{\bar{p}}$?
4. Does IH_*^J possess a generalization of Poincaré duality?
5. Is there a "nice" formula for $IH_*^J(SX)$ in terms of $IH_*^J(X)$?
6. Is there a Deligne-type sheaf theoretic description of IH_*^J?
   - If so, can it be axiomatized?
   - Is there a duality theory via Verdier duality?

   Note: The complexes of intersection chains discussed here can indeed be "sheafified" to yield a complex of sheaves whose hypercohomology coincides with $IH_*^J(X)$. In the PL category, this can be done using Borel-Moore chains as Goresky and MacPherson did in [4]; for singular chains, this can be done as one of the authors did for ordinary singular intersection homology in [1]. The question here is whether this complex of sheaves is quasi-isomorphic to one that can be described, as the Deligne sheaf for classical intersection homology can, either by a simple set of axioms or by some sequence of simple sheaf operations in the derived category.

7. Is there a simple collection of spaces $\{X_i\}$ sufficient to determine J given $IH_*^J(X_i)$ for all i?

8. What spaces are distinguished by IH_*^J but not by any $IH_*^{\bar{p}}$?

9. Is there a simple collection of spaces $\{X_i\}$ which represents all possible functions from multiperversities J to sequences of nonnegative integers $(\dim IH_i^J(X, \mathbb{Q}) := 0, 1, \ldots, n)$ for n-pseudomanifolds. (This question is of interest already for the usual intersection homology groups.)

10. Is it possible to prove PL-invariance of $IH_*^J(X)$ for simplicial pseudomanifolds X based on the Goresky and MacPherson treatment of the behavior under subdivisions. (As far as we know, showing independence from the stratification along this line is not known even for the original intersection homology groups.)

Concerning Problem 7 note that for ordinary intersection homology groups, if we consider the set of open cones over products of spheres (or suspensions over products of spheres), then for an unknown perversity $\bar{p}$, given $IH_i^{\bar{p}}(X)$ for these spaces we can determine $\bar{p}$ uniquely. For IH^J a larger family may be needed. The class of spaces containing spheres and closed under forming cones and taking Cartesian products may be relevant to problems 7 and perhaps also to Problems 8 and 9.

Finally we note that multiperversities may not be the end of the road. We can define more and more refined conditions on allowable singular chains by considering next chains in $IC^{J_1} \cap IC^{J_2} \cap \ldots$ for a set $\{J_1, J_2, \ldots\}$ of multiperversities. And continue by alternating between sums and intersections. We will not say more about this further generalization but rather introduce a notation: *Tolerance* will be a collective name for perversities, sets of perversities, sets of sets of perversities etc. (Thus, 0-tolerance is a single perversity.) Since duality appears to replace sums and intersections, duality which exists for 0-tolerance may be regained by taking tolerance to the limit.

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