Intersection homology of regular and cylindrical neighborhoods

Greg Friedman
Yale University, Dept. of Mathematics
10 Hillhouse Ave.
PO Box 208283
New Haven, CT 06520
friedman@math.yale.edu
Tel. 203-432-6473 Fax: 203-432-7316

November 15, 2002

Proposed running head: IH OF NEIGHBORHOODS

Abstract

We develop a spectral sequence for the intersection homology of regular neighborhoods in PL stratified pseudomanifolds and of certain cylindrical neighborhoods of pure subsets in homotopically stratified spaces. We also develop a theory of simplicial homology with stratified systems of local coefficients and show that the E^2 terms of our spectral sequences are computable in terms of the homology of the "base" space with a suitably defined stratified system of local coefficients.

Contents

1 Introduction

In the early 1980s, Goresky and MacPherson introduced intersection homology theory [10], [11] to generalize ordinary homology theory to a setting more adapted for the study of stratified spaces, in particular PL pseudomanifolds, which admit natural stratifications. Since then, there has been an ongoing effort to construct intersection homology invariants for such spaces, thus advancing the results of the work on manifold invariant theory of the previous decades, including the study of signatures, characteristic classes, surgery theories, and knot theory. For examples of efforts in these directions and further references, the reader is referred to [3], [1], [22], and [7]. But, of course, following the introduction and successful application of any new theory, there is also the need for useful computational techniques. For intersection homology, there already exist generalizations of the long exact sequence of the pair and the triple and the Mayer-Vietoris sequence (see, e.g., [15]). However, due to the nature of intersection homology, it is necessary for the definition of these exact sequences that all subspaces be open subspaces. So, in order to employ these exact sequences as useful tools, it is necessary to know something about the intersection homology of open subspaces and of the maps on intersection homology induced by inclusions of open subspaces. In particular, it is important to be able to compute the intersection homology of open neighborhoods of the strata of stratified pseudomanifolds, or, more specifically, the open neighborhoods of unions of components of strata.

This last goal is the subject of this paper. In particular, we are interested in the computation of the intersection homology of open regular neighborhoods and deleted open regular neighborhoods (regular neighborhoods with the base subset removed) of *pure* subsets of PL pseudomanifolds. In this case, a pure subset, as defined by Quinn [20], is a closed subset that is a union of connected components of strata. However, since intersection homology is independent of choice of stratification in a PL-pseudomanifold, any closed subpolyhedron of a PL pseudomanifold can be made to fit this bill. Now, of course, in ordinary homology theory, there is nothing to prove, since a regular neighborhood has the same homotopy type as its base (the subset it is a regular neighborhood of). But intersection homology is not a homotopy invariant [10]; it is only a stratified homotopy invariant [8], meaning that intersection homology is preserved under homotopies that preserve strata, but not necessarily otherwise. Hence, the intersection homology of a regular neighborhood truly contains information not just about the base space but also about how it lies in the space.

So how to compute the intersection homology of such neighborhoods? The solution is provided by the theories of Quinn [20] and Hughes [13], which we can employ to show that these neighborhoods are stratum-preserving homotopy equivalent to mapping cylinders of stratified fibrations. Stratified fibrations generalize ordinary fibrations by possessing a stratified homotopy lifting property. Once a fibration is in the picture, we can hope to set to work on a generalization of the Leray-Serre spectral sequence, and this is the core of our program. The difficulty is that stratified fibrations do not possess uniform fibers: the stratified homotopy type of a fiber of a stratified fibration depends on the stratum of the point it lies over. Luckily, however, there is some uniformity over any single stratum, and in the places where multiple strata come together, we find a method to meld together the various open subsets involved. The result is a spectral sequence that looks outwardly like the Leray-Serre spectral sequence but whose E^2 terms involve the homology of the base not just with a single system of local coefficients but with a *PL stratified system of local coefficients.* In other words, every stratum of the base possesses its own system of local coefficients, together with a specification of how these systems should interact under boundary chain maps. Hence, we can take homology with such stratified coefficient systems. This theory is introduced and developed in Section 2 below.

The case of regular neighborhoods only of bottom strata of PL pseudomanifolds has been studied by the author in [8] and used in the computation of intersection Alexander polynomials for non-locally-flat knots. The program of this paper provides a vast generalization, not only in the jump to neighborhoods of several strata, but also in the general context: since the stratified homotopy category of Quinn [20] is used so strongly in our techniques of proof, we set ourselves as far into this category as possible, throughout. So, in fact, we develop our spectral sequence not only for regular neighborhoods in pseudomanifolds, but also for certain types of neighborhoods of triangulable pure subsets of Quinn's manifold weakly stratified spaces. In Section 4, we shall describe these neighborhoods more fully, although let us note here that our considerations include mapping cylinder neighborhoods, which have been of some interest [2].

The structure of this paper is as follows:

In Section 2, we develop the theory of PL stratified systems of groups and show that this leads to a well-defined concept of homology with stratified systems of local coefficients. In Section 2, we explain our principal tools and concepts, including intersection homology, stratified spaces, weakly stratified spaces, homotopy links, and stratified fibrations. Section 4 brings us to our main objects of study, (cylindrical) nearly-stratum-preserving deformation retract neighborhoods, which generalize the regular neighborhoods of PL pseudomanifolds. These are defined and shown to be essentially unique.

In Section 5, we are finally able to state our main theorem and to provide some applications. We copy the statement of the theorem here, but defer explanation of the terminology to the appropriate places below.

Theorem 1.1 (Theorem 5.1). *Suppose X is a manifold weakly stratified space with pure subset K triangulable by a triangulation compatible with the filtration. Let $\mathcal{V}$ be a cylindrical NSDRN of K. Let $\mathcal{G}$ me a local coefficient system on the top stratum of $\mathcal{V}$. Then the intersection homology $IH^{\bar{p}}_*(\mathcal{V};\mathcal{G})$ is the abutment of a spectral sequence whose E^2 terms are the homology of K with a PL stratified system of coefficients, $H_p(K;\mathbb{IH}^{\bar{p}}_q(cL;\mathcal{G}))$, where $\mathbb{IH}^{\bar{p}}_q(cL;\mathcal{G})$ is a PL stratified system of coefficients whose fiber over a point $x\in K_j = K\cap X_j$ is the intersection homology of the cone of the fiber L over x of the holink evaluation $\text{holink}_s(X,K_j)\to K_j$ (with the coefficient system induced by $\mathcal{G}$). Similarly, there is a spectral sequence for the intersection homology $IH^{\bar{p}}_*(\mathcal{V}-K)$ whose E^2 terms are the homology of K with PL stratified coefficients in $\mathbb{IH}^{\bar{p}}_*(L';\mathcal{G})$, whose fiber over $x\in K_j$ is the intersection homology of L', the fiber over X of $\text{holink}_s((X-K)\cup K_j, K_j)\to K_j$ (with the coefficient system induced by $\mathcal{G}$). Furthermore, the map induced by inclusion $IH^{\bar{p}}_*(\mathcal{V}-K;\mathcal{G})\to IH^{\bar{p}}_*(\mathcal{V};\mathcal{G})$ gives a map of spectral sequences which on the E^2 terms is determined by the map of stratified systems of coefficients specified on each fiber by the inclusion $L'\hookrightarrow L$, which induces the intersection homology map $\mathbb{IH}^{\bar{p}}_*(L';\mathcal{G})\to\mathbb{IH}^{\bar{p}}_*(L;\mathcal{G})$.*

The remainder of the paper is concerned with the proof of the main theorem. In Section 6, we study the local behavior of our retract neighborhood, and in Section 7, we show how this local behavior determines a stratified system of local coefficients. Finally, in Section 8, we put our work together to prove the main theorem.

2 Homology with PL stratified systems of coefficients

In this section, we introduce PL stratified systems of groups (rings, modules, etc.) on filtered PL spaces. A *filtered space* is a space, X, together with a collection of closed subspaces

$$\emptyset = X^{-1}\subset X^0\subset X^1\subset\cdots\subset X^{n-1}\subset X^n = X.$$

The subsets X^i are called *skeleta*, not to be confused with simplicial skeleta. In particular, index does not necessarily reflect dimension in any way.

We begin by defining a PL stratified system on a filtered PL space with a fixed triangulation compatible with the filtered structure, i.e. so that each skeleton of the filtration is a subcomplex.

Remark 2.1 *(Notation).* When dealing with simplices in a simplicial complex, we will use lower case Greek, and sometimes Roman, letters. A letter with a bar over it, i.e. $\bar{\sigma}$, shall denote the standard closed simplex, while the absence of the bar, σ, denotes the open interior.

Definition 2.2. Let $(K, \mathfrak{K})$ be a pair consisting of a simplicial complex K and a finite filtration $\mathfrak{K}$ of K by subcomplexes $K = K^n \supset K^{n-1} \supset \cdots \supset K^1 \supset K^0 \subset K^{-1} = \emptyset$ (note that the index does not necessarily reflect the dimension of the complex). For each i, let $\mathcal{G}_i$ be a bundle of abelian groups, i.e. a locally-constant sheaf, over $K_i = K^i - K^{i-1}$. A *PL stratified system of groups* $\mathfrak{G}$ on $(K, \mathfrak{K})$ consists of the collection $\{\mathcal{G}_i\}$ and, for each pair of closed simplices $\bar{\tau} < \bar{\sigma}$ in K, a homomorphism $\phi_{\sigma\tau} : \mathcal{G}_{\hat{\sigma}} \to \mathcal{G}_{\hat{\tau}}$, where $\mathcal{G}_{\hat{\sigma}}$ and $\mathcal{G}_{\hat{\tau}}$ are the stalks of the appropriate $\mathcal{G}_i$ over the respective barycenters of $\bar{\sigma}$ and $\bar{\tau}$. The maps ϕ are also required to satisfy the following conditions:

1. If σ and τ, the respective interiors of $\bar{\sigma}$ and $\bar{\tau}$, lie in the same stratum $K_i = K^i - K^{i-1}$, then $\phi_{\sigma\tau}$ is the standard isomorphism induced by the bundle $\mathcal{G}_i$, i.e. that determined by the bundle over any path from $\hat{\sigma}$ to $\hat{\tau}$ lying within $\sigma \cup \tau$ (see [23, Chptr. VI]).

2. If $\bar{\xi} < \bar{\tau} < \bar{\sigma}$ and $\bar{\xi} < \bar{\tau}' < \bar{\sigma}$, then $\phi_{\tau\xi}\phi_{\sigma\tau} = \phi_{\tau'\xi}\phi_{\sigma\tau'}$.

Similarly, we can define PL stratified systems of rings, fields, R-modules, etc.

Next, we define a PL stratified system on a filtered PL space. As opposed to the previous definition, this one does not depend on assuming in advance a fixed triangulation. Rather, we define systems as above on all compatible triangulations, while imposing certain compatibility conditions on these systems.

Definition 2.3. Let $(X, \mathfrak{X})$ be a PL space X endowed with a finite PL filtration $\mathfrak{X} = \{X^i\}$, $X = X^n \supset X^{n-1} \supset \cdots \supset X^1 \supset X^0 \supset X^{-1} = \emptyset$, i.e. X is endowed with a PL structure such that each X^i is a PL subspace. A *PL stratified system of groups (rings, modules, ...)* $\mathbb{G}$ on the space $(X, \mathfrak{X})$ is given by a PL stratified system of groups $\mathfrak{G}_K$ on each triangulation K of X for which each X^i is a subcomplex. These must satisfy the following consistency axioms:

1. The bundles of groups $\mathcal{G}_i$ are defined on the strata $X^i - X^{i-1}$ and are independent of the choice of triangulation.

2. Suppose K and L are triangulations of X such that each X^i is a subcomplex and that L is a subdivision of K. Suppose that $\bar{\sigma}$ is a p-simplex of L and that $\bar{\sigma}$ is contained in a p-simplex $\bar{s}$ of K. Suppose further that $\bar{\tau}$ is a $p-1$ face of $\bar{\sigma}$ that lies in a $p-1$ face $\bar{t}$ of $\bar{s}$ and that the interiors of $\bar{\sigma}$ and $\bar{\tau}$ (and hence also of $\bar{s}$ and $\bar{t}$) lie in different strata of X. Let $\psi_{s\sigma} : \mathcal{G}_{\hat{s}} \to \mathcal{G}_{\hat{\sigma}}$ be the coefficient isomorphism uniquely determined by any path between the barycenters $\hat{\sigma}$ and $\hat{s}$ that lies in the interior of $\bar{s}$, and define $\psi_{t\tau}$ similarly. Then $\phi_{\sigma\tau} = \psi_{t\tau}\phi_{st}\psi_{s\sigma}^{-1}$.

Given a PL stratified system of groups $\mathbb{G}$ on a filtered PL space, we now see how we can consider these as stratified systems of coefficients to define the homology $H_*(X; \mathbb{G})$. We first define these homology modules with respect to a given triangulation, but we then show independence of this choice.

Definition 2.4. Let $\mathfrak{G}$ be a PL stratified system of abelian groups (or modules) on the filtered simplicial complex $(K, \mathfrak{K})$. We define the homology of K with the PL stratified system of coefficients $\mathfrak{G}$ to be the homology groups of the chain complex $C_*(K; \mathfrak{G})$ whose terms are given by $C_i(K; \mathfrak{G}) = \oplus_{\sigma^i \in K} \mathcal{G}_{\hat{\sigma}}$, the sum taken over i simplices of K, and whose boundary maps are given by $\partial(g[v_0, \dots, v_i]) = \sum_j (-1)^j \phi_{[v_0,\dots,v_i],[v_0,\dots,\hat{v}_j,\dots,v_i]}(g)[v_0, \dots, \hat{v}_j, \dots, v_i]$. Here $g[v_0, \dots, v_i]$ represents the element $g \in \mathcal{G}_{\hat{\sigma}}$ corresponding to the summand of $C_i(K; \mathfrak{G})$ over the oriented simplex $\sigma = [v_0, \dots, v_i]$, the $[v_0, \dots, \hat{v}_j, \dots, v_i]$ represent the usual oriented boundary facets of σ, and the coefficients of the boundary faces under the boundary map are given by $(-1)^j \phi_{[v_0,\dots,v_i],[v_0,\dots,\hat{v}_j,\dots,v_i]}(g)$. In other words, we weight the geometrical boundary terms by coefficients determined by the maps ϕ. (Of course we have tacitly oriented all geometric simplices, and reversal of geometric orientation algebraically reverses the sign of the coefficient of the simplex.) That $\partial^2 = 0$ follows from condition (2) of Definition 2.2 and the standard simplicial proof of cancellation of terms. We will denote these homology groups by $H_*(K; \mathfrak{G})$.

Definition 2.5. Given a PL stratified system of groups $\mathbb{G}$ on the filtered PL space X, we define the homology groups with PL stratified system of coefficients $H_*(X; \mathbb{G})$ to be the homology groups $H_*(K; \mathfrak{G}_K)$ as determined in the last definition for any given triangulation K in the PL structure of X compatible with the filtration.

Of course in order to show that these homology groups are well-defined, we must show that they are independent of the choice of triangulation.

Proposition 2.6. *Given a finitely filtered PL space $(X, \mathfrak{X})$ and a PL stratified system of groups $\mathbb{G}$ on X, the homology groups $H_*(X; \mathbb{G})$ are independent of the choice within the PL structure of X of triangulation consistent with the filtration.*

Proof. Let K and L be triangulations within the PL structure of X that are consistent with the filtration. We wish to show that $H_*(K; \mathfrak{G}_K) \cong H_*(L; \mathfrak{G}_L)$. Since any two triangulations within the PL structure have a common subdivision, it will suffice to assume that L is a subdivision of K. The proof will proceed by induction on the number of strata in the filtration and then reduction to a special case.

First, consider the bottom skeleton X^0 of X. From the definitions, $H_*(X^0; \mathbb{G}) \cong H_*(X^0; \mathcal{G}_0)$, the standard homology groups with local coefficient system $\mathcal{G}_0$ on $X_0 = X^0$. These are well known topological invariants and are therefore, in particular, invariant under subdivisions of X^0.

Now, suppose that we have shown that $H_*(X^{n-1}; \mathbb{G})$ is independent of the choice of triangulation induced on X^{n-1} by a triangulation of X. These homology groups, of course, are those obtained by restricting attention to the simplices in X^{n-1} and hence inducing a chain complex $C_*(K^{n-1}; \mathfrak{G}_K) \subset C_*(K; \mathfrak{G}_K)$ for every triangulation K of X, where K^{n-1} is the triangulation induced on X^{n-1} by the triangulation K of X. Note that $C_*(K^{n-1}; \mathfrak{G}_K)$ will clearly be a chain complex, from the definitions. We can also consider the chain complex $C_*(K^n; \mathfrak{G}_K)$ over the triangulation K^n of X^n induced by K and the induced quotient complex $C_*(K^n; \mathfrak{G}_K)/C_*(K^{n-1}; \mathfrak{G}_K)$, which we will denote by $C_*(K^n, K^{n-1}; \mathfrak{G}_K)$. We then have a short exact sequence $0 \to C_*(K^{n-1}; \mathfrak{G}_K) \to C_*(K^n; \mathfrak{G}_K) \to C_*(K^n, K^{n-1}; \mathfrak{G}_K) \to 0$, which induces a long exact sequence that we denote

$$\longrightarrow H_i(K^{n-1}; \mathfrak{G}_K) \longrightarrow H_i(K^n; \mathfrak{G}_K) \longrightarrow H_i(K^n, K^{n-1}; \mathfrak{G}_K) \longrightarrow H_{i-1}(K^{n-1}; \mathfrak{G}_K) \longrightarrow .$$

Suppose now that we have constructed a chain map $C_*(K;\mathfrak{G}_K) \to C_*(L;\mathfrak{G}_L)$ that restricts to chain maps $C_*(K^n;\mathfrak{G}_K) \to C_*(L^n;\mathfrak{G}_L)$ and $C_*(K^{n-1};\mathfrak{G}_K) \to C_*(L^{n-1};\mathfrak{G}_L)$. By homological algebra, we will then obtain a map of short exact sequences of chain complexes and of long exact sequences of homology groups. Suppose that L is a subdivision of K and recall that we are assuming by induction that $H_*(X^{n-1};\mathbb{G})$ is independent of triangulation. In fact, we will see below that we can assume the necessary isomorphism is induced by our chain map. Then to show that $H_*(X^n;\mathbb{G})$ is independent of triangulation, it will suffice, by the five lemma, to show that the induced map of "relative" homology groups is an isomorphism. We will construct the chain maps and demonstrate this last isomorphism. The proposition will then follow by induction since $X^n = X$ for sufficiently large n, as we have assumed the filtration to be finite.

To show independence of triangulation of the groups $H_*(K^n, K^{n-1};\mathbb{G})$, let us consider the chain complex $C_*(K^n, K^{n-1};\mathfrak{G}_K)$. Notice that we are quotienting out the direct sum of all of the terms of the chain groups of $C_*(K^n;\mathfrak{G}_K)$ generated by simplices in K^{n-1}. So this chain group is isomorphic to the unique chain group given by the PL stratified system of groups determined by $\mathcal{G}_n$ on $K^n - K^{n-1}$ and by 0 on all lower strata. Let us call this coefficient system $\mathbb{G}_n$. To show that $H_*(X^n, X^{n-1};\mathbb{G})$ is independent of triangulation, it therefore suffices to prove this for $H_*(X^n;\mathbb{G}_n)$. Our method for proving this will be based on the usual constructions for ordinary simplicial homology (see, e.g., [18]). We will use these standard methods to create geometric chain maps and homotopies and then fill in the proper coefficients to make the theory work algebraically. In some sense this generalizes the method for showing that ordinary relative homology is independent of triangulation, and, in fact, if $\mathcal{G}_n$ is a constant coefficient system with stalk G, then $H_*(X^n;\mathbb{G}_n)$ is exactly the relative homology $H_*(X^n, X^{n-1};G)$.

So let us begin by constructing a chain map $f : C_*(K^n;\mathfrak{G}_K) \to C_*(L^n;\mathfrak{G}_L)$, where L is a subdivision of K. In fact, we can define $f : C_*(K;\mathfrak{G}_K) \to C_*(L;\mathfrak{G}_L)$ on the entire complex and then restrict. We are free to choose compatible orientations of the simplices of K and L. For each p-simplex $\bar{s} \in K$ with coefficient g in the group over its barycenter, define $f(g\bar{s}) = \sum_\sigma \psi_{s\sigma}(g)\bar{\sigma}$, where the sum is taken over all p-simplices σ of L such that $\bar{\sigma} \subset \bar{s}$. Clearly then the barycenters of $\bar{s}$ and all such $\bar{\sigma}$ lie within s and each $\psi_{s\sigma}$ is well-defined as the isomorphism induced in the appropriate group stratum $\mathcal{G}_i$ by any path from $\hat{s}$ to $\hat{\sigma}$ that is contained entirely in the interior of $\bar{s}$. That f is a chain map follows immediately from the consistency axioms in the definition of a PL stratified system of groups $\mathbb{G}$ on X. The chain map f clearly induces chain maps on $C_*(K^n;\mathfrak{G}_K) \to C_*(L^n;\mathfrak{G}_L)$ by restriction and $C_*(K^n, K^{n-1};\mathfrak{G}_K) \to C_*(L^n, L^{n-1};\mathfrak{G}_L)$ by then taking quotients. We denote all of these maps by the same symbol f. It is this set of chain maps that induce a map of the homology exact sequences with coefficients of (K^n, K^{n-1}) and (L^n, L^{n-1}). This chain map induces the standard isomorphism of homology with local coefficients over X^0, so we can assume inductively that it generates the isomorphism for $H_*(X^j;\mathbb{G})$ for $j \leq n-1$. Therefore, by these induction hypotheses, it suffices to show that f induces an isomorphism of the appropriate relative homology groups.

We now construct a chain inverse $h : C_*(L^n, L^{n-1};\mathfrak{G}_L) \to C_*(K^n, K^{n-1};\mathfrak{G}_K)$ to $f : C_*(K^n, K^{n-1};\mathfrak{G}_K) \to C_*(L^n, L^{n-1};\mathfrak{G}_L)$. It is well known (see [18]) that for $\mathbb{Z}$ coefficients there exists such a chain homotopy inverse $C_*(L^n) \to C_*(K^n)$ to the $\mathbb{Z}$ coefficient version of f and that this inverse induces a well-defined chain map $C_*(L^n, L^{n-1}) \to C_*(K^n, K^{n-1})$

such that each $\bar{\sigma} \in C_*(L)$ gets mapped geometrically to a face of $\bar{s}$, where $\bar{s}$ is the simplex of K^n of lowest dimension such that $\bar{\sigma} \subset \bar{s}$. Of course if $\bar{\sigma}$ gets taken geometrically to a face of lower dimension than itself, the corresponding algebraic map in the chain complex is 0. We use this chain map as the geometric underpinning for our h and show how to determine the coefficient map. So let $g\bar{\sigma}$ be a simplex in L^n with coefficient over its barycenter. Of course if $\bar{\sigma} \in L^{n-1}$, then its coefficient in $C_*(L^n, L^{n-1}; \mathfrak{G}_L)$ must be 0, and so $h(g\bar{\sigma}) = h(0\bar{\sigma}) = 0$. If $\bar{\sigma}$ does not lie in L^{n-1}, then our geometric chain map takes it to a face of $\bar{s}$ that either lies in K^{n-1} or does not. If the geometric image of $\bar{\sigma}$ lies in K^{n-1}, then set $h(g\bar{\sigma}) = 0$. Otherwise, the barycenters of $\bar{\sigma}$ and its image simplex $\bar{\xi}$ both lie in $K^n - K^{n-1}$, and we can define a coefficient map from the group over $\hat{\sigma}$ to the group over $\hat{\xi}$ by that determined by any path between the barycenters lying in $\bar{s} - \bar{s} \cap K_{n-1}$. To see that this is a well-defined chain map, note that if we restrict the coefficient systems to $\bar{s}$, which is contractible, then there are isomorphisms from $C_*(K(s), K(s) \cap K^{n-1}; \mathfrak{G}_K)$ to $C_*(K(s), K(s) \cap K^{n-1}; G_n)$ and similarly for L, where $K(s)$ is K restricted to $\bar{s}$ and G_n is the group over $\hat{s}$. So the chain complexes $C_*(K(s), K(s) \cap K^{n-1}; G_n)$ and $C_*(L(s), L(s) \cap L^{n-1}; G_n)$ are the relative chain complex with coefficients in the constant group G_n. Then, there is a commutative diagram

$$\begin{array}{ccc} C_*(L(s), L(s) \cap L^{n-1}; \mathfrak{G}_L) & \xrightarrow{\cong} & C_*(L(s), L(s) \cap L^{n-1}; G_n) \\ {\scriptstyle h}\downarrow & & \downarrow \\ C_*(K(s), K(s) \cap K^{n-1}; \mathfrak{G}_K) & \xrightarrow{\cong} & C_*(K(s), K(s) \cap K^{n-1}; G_n), \end{array} \tag{1}$$

where the right-hand vertical map is that induced on a constant coefficient system by our geometric chain map. In fact, this commutativity is tautological, as we define the coefficient maps of the horizontal chain isomorphism also by using paths contained in s between barycenters of simplices in s in order to determine canonical isomorphisms of stalks over the contractible space s. It now follows from commutativity and the standard theory of relative chain complexes with constant coefficient that h is a well-defined chain map over $\bar{s}$ (see, e.g. [18]). So our chain map h is well-defined locally, but that is all that is required for a chain map to be well-defined globally.

Finally, we need to show that f and h are chain homotopy inverses, i.e. that there exist homomorphisms $D : C_p(K^n, K^{n-1}; \mathfrak{G}_K) \to C_{p+1}(K^n, K^{n-1}; \mathfrak{G}_K)$ and $E : C_p(L^n, L^{n-1}; \mathfrak{G}_L) \to C_{p+1}(L^n, L^{n-1}; \mathfrak{G}_L)$ such that $D\partial + \partial D = hf - \mathrm{id}$ and $E\partial + \partial E = fh - \mathrm{id}$. Again, we will consider the chain maps locally and resort to the well known fact that D and E exist for relative homology with constant coefficients. So again consider diagram (1) and the chain maps f' and h' between $C_*(L(s), L(s) \cap L^{n-1}; G_n)$ and $C_*(K(s), K(s) \cap K^{n-1}; G_n)$ induced by f and h. By relative constant coefficient theory (see [18]), there exist the desired chain homotopies D' and E' carried by $\bar{s}$ and such that $(D'\partial + \partial D')(\bar{s}) = (h'f' - \mathrm{id})(\bar{s})$ and $(E'\partial + \partial E')(\bar{\sigma}) = (f'h' - \mathrm{id})(\bar{\sigma})$, for each $\bar{\sigma}$ such that $\bar{s}$ is the minimal dimension simplex with $\bar{\sigma} \subset \bar{s}$. Using the isomorphisms of diagram (1), we can thus define D and E locally on $C_p(K^n, K^{n-1}; \mathfrak{G}_K)$ and $C_p(L^n, L^{n-1}; \mathfrak{G}_L)$, but again, local definition on generators is sufficient to define D and E consistently and with the desired properties on the entire chain complexes.

This completes the proof.

□

It follows from these methods that homology groups with PL systems of local coefficients are invariants in the proper category, i.e. under filtered PL homeomorphisms that are covered by bundle maps on the strata that are consistent with the given morphisms between between coefficients of different strata. We leave it to the reader to formulate a precise definition, as in the sequel we will limit ourselves to specific spaces.

2.1 Examples

Let us present some simple examples of homology with PL stratified systems of groups.

1. Let X be a filtered PL space. Of course, X equipped with any constant coefficient system satisfies our definitions and yields standard homology.

2. Let X be a PL space filtered by two skeletons $X^0 \subset X^1$. Let $\mathcal{G}_1$ be the constant bundle of the abelian group G on $X_1 = X^1 - X^0$, and let $\mathcal{G}_0$ be the constant bundle of the trivial group on X_0. This defines a unique PL-stratified system of groups on X whose homology is the relative homology $H_*(X, X^0; G)$.

3. Let $X = X^n \supset X^{n-1} \supset \cdots X^0$ be a filtered PL space. Let each bundle of groups $\mathcal{G}_i$ on X_i be a constant bundle with fiber G_i. Suppose for each $i > j$ we have a homomorphism $\phi_{ij} : G_i \to G_j$ such that for any $i > l > j$, $\phi_{lj}\phi_{il} = \phi_{ij}$. This determines a unique PL stratified system of groups on the space; compatibility amongst the triangulations is clear. As a more refined example, note that we only need have ϕ_{ij} defined if $\bar{X}_i \cap X_j \neq \emptyset$.

4. Let us provide a concrete realization of the previous example. Consider the disk $X = D^2$ realized as the standard 2-simplex σ^2. Let us filter X so that $X^2 = X$, X^0 is a single vertex v_0, and X^1 is a single edge $[v_0, v_1]$. Let each $\mathcal{G}_i$ be a constant $\mathbb{Z}$ bundle, but let $\phi_{2,1}$ and $\phi_{1,0}$ each be multiplication by 2. Now, the vertices are all homologous (using the edges $[v_1, v_2]$ and $[v_2, v_0]$), and clearly the lone two simplex is not a cycle. However, the cycles in $C^1(X; \mathbb{G})$ are generated by $[v_0, v_1] + 2[v_1, v_2] + 2[v_2, v_0]$, while $\partial[v_0, v_1, v_2] = 2[v_0, v_1] + [v_1, v_2] + [v_2, v_0]$. Hence $H_2(X; G) = 0$ and $H_0(X; \mathbb{G}) = \mathbb{Z}$, but $H_1(X; \mathbb{G}) = \mathbb{Z}$.

5. As a simple example with a non-constant coefficient system, consider $X = D^2$ filtered by $X^1 = X$ and $X^0 = x_0$, the center of the disk. Let $\mathcal{G}_1$ be the non-trivial $\mathbb{Z}$ bundle on $X_1 = X - X^0$, and let $\mathcal{G}_0 = \mathbb{Z}_2$. For each simplex σ that has x_0 as a vertex, let $\phi_{\sigma x_0}$ be reduction mod 2, $\mathbb{Z} \to \mathbb{Z}_2$. Then $H_2(X; \mathbb{G}) = H_1(X; G) = 0$ and $H_0(X; \mathbb{G}) = \mathbb{Z}_2$, since for any vertex v, $2v$ is the boundary of a 1-chain connecting it to x_0 (alternatively, every vertex is homologous to x_0).

6. Consider S^2 with twisted $\mathbb{Z}$ coefficients on $X_1 = S^2 - \{n, s\}$, where n and s are the north and south poles, but again with $\mathbb{Z}_2$ coefficients on $X^0 = \{n, s\}$ and the reduction mod 2 homomorphism for ϕ. The homology is the same as for the D^2 of the preceding example: there is a new 1-cycle generated by twice any longitude line, but it bounds. However, suppose that S^2 has the same filtration and that the coefficients on the north and south pole remain $\mathbb{Z}_2$, but let the fiber over $X_1 = X - X^0$ be $\mathbb{Z}_{2^n}$ with the action of a generator of the fundamental group of X_1 given by multiplication by $2^{n-1}+1$. Again, let

ϕ be reduction mod 2. Then $H_0(S^2;\mathbb{G})$ is still $\mathbb{Z}_2$, but $H_2(S^2;\mathbb{G}) = \mathbb{Z}_{2^{n-1}}$, generated by the chain that takes coefficient 2 on each 2-simplex, and $H_1(S^2;\mathbb{G}) = \mathbb{Z}_{2^{n-2}}$, generated by twice any longitude, but 2^{n-2} of which is bounded by the 2-chain with coefficient 1 on each 2-simplex of S^2 (assuming each simplex oriented so that the sum of all oriented simplices would be a cycle with constant $\mathbb{Z}$ coefficients).

Notice that in the last example, there is no way of extending the coefficient system defined on the stratum X_1 to the entire space X.

3 Background definitions, notations, and conventions

In this section, we present the staple definitions that we will be use throughout. The initial definitions of intersection homology are due to Goresky and MacPherson [10], [11], but we will mostly use the singular chain variant described by King [15], [14]. The notions of weakly stratified space are due to Quinn [20], who also developed the theory of holinks which can also be found in Chapman [4]. Stratified fibrations are due to Hughes [13].

3.1 Intersection homology

As defined in [10], a (traditional) *perversity* $\bar{p}$ is a sequence of integers $\{\bar{p}(0), \bar{p}(1), \bar{p}(2), \ldots\}$ such that $\bar{p}(i) \leq \bar{p}(i+1) \leq \bar{p}(i)+1$ and such that $\bar{p}(0) = \bar{p}(1) = \bar{p}(2) = 0$ (much of the following would hold using more general perversities (see [15], [3], [9]), but for the other common case, in which $\bar{p}(2) = 1$ (superperversities), it is not clear that intersection homology with local coefficients can be well defined in a geometric manner, i.e. without sheaves; hence we omit discussion).

A *filtered space* is a space, X, together with a collection of closed subspaces

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n = X.$$

If we want to emphasize both the space and the filtration, we will refer to the filtered space $(X, \{X^i\})$. Note that $X^i = X^{i+1}$ is possible. We will refer to n as the (stratified) *dimension* of X and X^{n-k} as the $n-k$ *skeleton* or the *codimension* k *skeleton*. The sets $X_i = X^i - X^{i-1}$ are the *strata* of X. We call a space either *unfiltered* or *unstratified* if we do not wish to consider any filtration on it.

We can now define the singular intersection homology of X for a perversity $\bar{p}$, $IH^{\bar{p}}_*(X)$, in the usual manner: as the homology of the chain complex $IC^{\bar{p}}_*(X)$, which is the submodule generated by the $\bar{p}$-allowable chains of the singular chain complex, $C_*(X)$. A singular i-simplex $\sigma : \Delta^i \to X$ is $\bar{p}$-*allowable* if $\sigma^{-1}(X_{n-k} - X_{n-k-1})$ is contained in the $i-k+\bar{p}(k)$ skeleton of the (polyhedral) simplex Δ^i (with the usual filtration by simplicial skeletons). An i-chain is $\bar{p}$-*allowable* if it and its boundary (in the chain complex sense) are linear combinations of $\bar{p}$-allowable i-simplices and $i-1$ simplices, respectively. Similarly, we can define the intersection homology with coefficients in a group or with local coefficients, though for intersection homology it is only necessary to define the local coefficient system on the top stratum $X - X^{n-2}$ (see [8] for more details).

If X is a stratified PL-pseudomanifold (definition below), then the above definition of intersection homology is equivalent to that of Goresky and MacPherson (with compact supports, if using the sheaf language) and is a topological invariant, independent of the choice of stratification [15]. Note, however, that for a general filtered space, X, the intersection homology may not be a topological invariant; it may depend upon our choice of filtration.

In a filtered space, since the intersection homology definitions depend only on the codimensions of the strata, we see that we can reindex a filtration by addition of a fixed integer, accompanied by the same change to the stratified dimension, without affecting the intersection homology of the space. In what follows, we shall often defer from the norm by *not* reindexing in certain situations where it is standard to do so when working with stratified pseudomanifolds. In particular, for simplicity, we will usually give filtered subspaces, $Y \subset X$, the same dimension as X and filtration indexing $Y^i = Y \cap X^i$, even though the common practice for stratified pseudomanifolds is to reindex to the simplicial dimensions of Y and Y^i by subtracting the codimension of Y in X from each index.

If X and Y are two filtered spaces, we call a map $f : X \to Y$ *filtered* if the image of each component of a stratum of X lies in a stratum of Y. *N.B. This property is often referred to as "stratum-preserving", e.g. in [20] and [8]. However, we must reserve the term "stratum-preserving" for other common uses.* In general, it is not required that a filtered map take strata of X to strata of Y of the same (co)dimension. However, if f preserves codimension, or if X and Y have the same stratified dimension and $f(X_i) \subset Y_i$, then f will induce a well-defined map on intersection homology (see [8, Prop. 2.1] for a proof). In this case, we will call f *well-filtered*. We call a well-filtered map f a *stratum-preserving homotopy equivalence* if there is a well-filtered map $g : Y \to X$ such that fg and gf are homotopic to the appropriate identity maps by homotopies such that each $x \times I$ lies in a single stratum. Stratum-preserving homotopy equivalences induce intersection homology isomorphisms [8]. If stratum-preserving homotopy equivalences between X and Y exist, we say that X and Y are *stratum-preserving homotopy equivalent*, $X \sim_{sphe} Y$.

In the sequel, all maps inducing inducing intersection homology homomorphisms will clearly be well-filtered. Hence, we will usually dispense with discussion of this point.

3.2 Stratified PL-pseudomanifolds

In this section, we provide the definitions of stratified PL pseudomanifolds and their regular neighborhoods.

Let $c(Z)$ denote the open cone on the space Z, and let $c(\emptyset)$ be a point. Generally, a *stratified paracompact Hausdorff space* Y (see [11] or [3]) is defined by a filtration

$$Y = Y^n \supset Y^{n-1} \supset Y^{n-2} \supset \cdots \supset Y^0 \supset Y^{-1} = \emptyset$$

such that for each point $y \in Y_i = Y^i - Y^{-1}$, there exists a *distinguished neighborhood* U of y such there is a compact Hausdorff space L (called the *link* of the component of the stratum), a filtration

$$L = L^{n-i-1} \supset \cdots \supset L^0 \supset L^{-1} = \emptyset,$$

and a homeomorphism

$$\phi : \mathbb{R}^i \times c(L) \to U$$

that takes $\mathbb{R}^i \times c(L^{j-1})$ onto Y^{i+j}.

N.B. For these spaces, the dimension of the skeleta *does* coincide with geometric dimension.

A *PL-pseudomanifold* of dimension n is a PL space X (equipped with a class of locally finite triangulations) containing a closed PL subspace Σ of codimension at least 2 such that $X - \Sigma$ is a PL manifold of dimension n dense in X. A *stratified PL-pseudomanifold* of dimension n is a PL pseudomanifold equipped with a filtration such that $\Sigma = X^{n-2}$ and the local normal triviality conditions of a stratified space hold with the trivializing homeomorphisms ϕ being PL homeomorphisms. In fact, for any PL-pseudomanifold X, such a stratification always exists such that the filtration refines the standard filtration of X by k-skeletons with respect to some triangulation [5, Chptr. I]. Furthermore, intersection homology is known to be a topological invariant of such spaces; in particular, it is invariant under choice of triangulation or stratification (see [11], [5], [15]).

Now, suppose that the filtered space $(X, \{X^i\})$ is a stratified PL-pseudomanifold with each X^i a subpolyhedron, so that, in particular, there exists a triangulation of X in which each X^i is triangulated as a subcomplex. Note that X is also a filtered polyhedron in the sense considered by Stone in [21]. Given such a filtered polyhedron and another subpolyhedron Y, then the filtered subpolyhedron $(V, \{V^i\})$ is defined (as in Stone [21, p. 4]) to be a *regular neighborhood* with *frontier* $\text{fr}(V)$ if there are a triangulation B of X in which Y and the X^i are triangulated by full subcomplexes A and B^i and a first derived subdivision B' in which

1. $V = |\bar{N}(Y, B')|$,
2. $V^i = |\bar{N}(Y, (B^i)')|$,
3. $\text{fr}(V) = |\text{lk}(Y, B')|$,

where, as usual, $|Z|$ represents the topological space underlying the simplicial complex Z and $\bar{N}(Y, B')$ is the (closed) first derived neighborhood of Y. (Stone in fact defines a more general relative regular neighborhood, but we will not need the added generality.) It is further shown in [21] that such neighborhoods satisfy the standard properties for regular neighborhoods, such as uniqueness via PL-isotopies and the generalized annulus property. We can also define an open regular neighborhood by taking the interior of V (meaning here $V - \text{fr}(V)$) and its intersection with each of the V^i. In what follows, "regular neighborhood" will always mean an open regular neighborhood in this sense.

3.3 Stratified fibrations

Definition 3.1. If Y is a filtered space, a map $f : Z \times A \to Y$ is *stratum-preserving along* A if, for each $z \in Z$, $f(z \times A)$ lies in a single stratum of Y. If $A = I$, f is called a *stratum-preserving homotopy*.

If Z is a filtered space and A is unfiltered, then unless otherwise noted, we will assume that $Z \times A$ has the product filtration: $(Z \times A)^i = Z^i \times A$. In particular, this will often be the case for $Z \times I$.

Definition 3.2. Suppose Y and B are filtered spaces. A map $p : Y \to B$ is a *stratified fibration* if, given any filtered space Z and the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p} \\ Z \times I & \xrightarrow{F} & B, \end{array} \tag{2}$$

with F a stratum-preserving homotopy, there exists a *stratified solution* that is stratum-preserving along I, i.e. a map $\tilde{F} : Z \times I \to Y$ such that $p\tilde{F} = F$ and, for each $z \in Z$, $\tilde{F}(z, 0) = f(z)$ and $\tilde{F}(z \times I)$ lies in a single stratum of Y.

It is shown in [8] that stratified fibrations over unfiltered base spaces satisfy stratified analogues of several of the most important properties of ordinary fibrations. In particular, any stratified fibration over an unfiltered base spaces induces a local coefficient system with fibers $IH_*^{\bar{p}}(L)$, where L is the fiber of the stratified fibration, and any stratified fibration over an unfiltered contractible base space possesses a trivialization, a stratum-preserving fiber homotopy equivalence to a product space (recall that a fiber homotopy from a fibration $p : X \to B$ to a fibration $p' : X' \to B'$ is a homotopy $H : X \times I \to X'$ of fibrations such that $p'(H(x \times I)) = p'(H(x, 0))$ for all $x \in X$).

We will also need some results concerning the mapping cylinders of stratified fibrations.

We will always use mapping cylinders (including cones) with the *teardrop topology* (see [12, §2] or [24, §3]). The *teardrop topology* on the mapping cylinder M of the map $p : Y \to B$ is the topology generated by the basis consisting of open subsets of $Y \times (0, 1]$ and sets of the form $[p^{-1}(U) \times (0, \epsilon)] \cup U$, where U is an open set in B.

If $p : Y \to B$ is a stratified fibration, Y filtered by the sets $\{Y^i\}$, we must face the question of how to filter the mapping cylinder M. Naturally, we want all components of strata to be components of strata of B or of the form $Y_i \times (0, 1]$, but this does not leave it clear how to index these strata, particularly the indexing relation between strata of B and those imposed from Y. (Although, of course, we must remain under the restriction that X^j is a close subset of X^i for $j < i$.) Fortunately, in the applications below, the mapping cylinders will come equipped with a natural filtration of this form.

Cones are to be considered as special cases of mapping cylinders, those with base space a point. So all cones will also be considered to have the teardrop topology.

In the special case where the base space is unfiltered, we have the following nice theorem.

Proposition 3.3. *Let $p : X \to B$ be a stratified fibration with B unfiltered, let M be the mapping cylinder of p filtered so that B is the bottom skeleton and all other skeleta are the mapping cylinders of the restrictions of p to the corresponding skeleta of X. The actual indexing of the skeleta of M can be arbitrary so long as it preserves the ordering $X^i \subset X^j$ for $i < j$. Let $P : M \to B$ be the projection that takes $b \in B$ to itself and $(x, t) \in X \times (0, 1] \subset M$ to $p(x)$. Then P is a stratified fibration.*

Proof. Let Z be a space and consider the commutative diagram

$$\begin{array}{ccc} Z \times 0 & \xrightarrow{f} & M \\ \text{inc.} \Big\downarrow & & \Big\downarrow P \\ Z \times I & \xrightarrow{F} & B, \end{array}$$

where F is a stratum-preserving homotopy. We must show that this lifting problem has a stratified solution.

Let $A = f^{-1}(B)$. Note that, since F is stratum-preserving and the diagram commutes, $F(A \times I) \subset B$ and $F((Z - A) \times I) \subset M - B$. Also, f restricts to a map $f : (Z - A) \times 0 \to X \times (0, 1] \subset M$. Let $\pi_1 : X \times (0, 1] \to X$ and $\pi_2 : X \times (0, 1] \to (0, 1]$ be the projections, and consider the diagram

$$\begin{array}{ccc} (Z - A) \times 0 & \xrightarrow{\pi_1 f} & X \\ \text{inc.} \Big\downarrow & & \Big\downarrow p \\ (Z - A) \times I & \xrightarrow{F} & B. \end{array} \tag{3}$$

This diagram commutes by the commutativity of the first diagram since $p\pi_1 = P$ for points in $X \times (0, 1]$. Since $p : X \to B$ is a stratified fibration, this map has a stratified solution $G : (Z - A) \times I \to X$.

Define $H : Z \times I \to X$ as follows:

$$H(z, t) = \begin{cases} F(z, t), & z \in A, \\ (G(z, t), \pi_2 f(z)), & z \in Z - A. \end{cases}$$

Clearly, H solves the stratified lifting problem (3.3) setwise, it remains to show that H is continuous. Continuity is clear at points in the open set $(Z - A) \times I$, so we check continuity at points $(z_0, t_0) \in A \times I$. Let V be a neighborhood of $H(z_0, t_0)$ in M. Using the basis of the teardrop topology on the mapping cylinder and that $H(z_0, t_0) \in B$, we can choose V of the form $[p^{-1}(U) \times (0, \epsilon)] \cup U$, where U is open in B and contains $H(z_0, t_0)$. We will find a neighborhood W of (z_0, t_0) in $Z \times I$ that maps into V under H.

Let $\Upsilon = f^{-1}([p^{-1}(B) \times (0, \epsilon)] \cup B)$, and let $W = F^{-1}(U) \cap (\Upsilon \times I)$. Since f and F are continuous and U and $[p^{-1}(B) \times (0, \epsilon)] \cup B$ are open in B and M, respectively, W is open. We first show that $(z_0, t_0) \in W$. The point $(z_0, t_0) \subset F^{-1}(U)$, since $F(z_0, t_0) = H(z_0, t_0)$ and, by assumption, U is a neighborhood of $H(z_0, t_0)$ in B. To check that $(z_0, t_0) \in \Upsilon \times I$, it suffices to see that $z_0 \in \Upsilon$. But $z_0 \in A = f^{-1}(B)$, so clearly $z_0 \in \Upsilon = f^{-1}([p^{-1}(B) \times (0, \epsilon)] \cup B)$. Next, we check that $H(W) \subset V$. We check this separately for points $(z, t) \in W \cap (A \times I)$ and $(z, t) \in W \cap ((Z - A) \times I)$. If $(z, t) \in W \cap (A \times I)$, then $H(z, t) = F(z, t)$ and by choice of W, $F(z, t) \in U \subset V$. If $(z, t) \in W \cap ((Z - A) \times I)$, then $H(z, t) = (G(z, t), \pi_2 f(z))$. Since $(z, t) \in \Upsilon \times I$, $\pi_2 H(z, t) = \pi_2 f(z) \in (0, \epsilon)$, and since $pG(z, t) = F(z, t) \in U$ by the construction of G and W, $\pi_1 H(z, t) = G(z, t) \in p^{-1}(U)$. Hence $H(W) \subset V$, and H is continuous. $\square$

3.4 Holinks and manifold weakly stratified spaces

A map of pairs of spaces $f : (A, B) \to (X, Y)$ is said to be *strict* if $f(B) \subset Y$ and $f : (A - B) \subset X - Y$. The space of such strict maps with the compact-open topology is denoted $\text{map}_s(A, B; X, Y)$. We can then define the *homotopy link* $\text{holink}(X, Y) = \text{map}_s(I, \{0\}; X, Y)$, i.e. the space of maps of the unit interval into X which take the point 0 into Y and all other points into the complement of Y. If X is a filtered space, we will also be interested in the *stratified homotopy link* $\text{holink}_s(X, Y)$, which is the subspace of $\text{holink}(X, Y)$ consisting of maps $I \to X$ which take 0 into Y and $(0, 1]$ into a single stratum of X. $\text{holink}_s(X, Y)$ is itself filtered by the subsets which take $(0, 1]$ into a particular skeleton of X. Note that these are subspaces of X^I and so are metrizable if X is. If X is metrizable and we are given a function $\delta : Y \to (0, \infty)$, the *controlled homotopy links* $\text{holink}^\delta(X, Y)$ and $\text{holink}_s^\delta(X, Y)$ are defined to be the respective subsets of $\text{holink}(X, Y)$ and $\text{holink}_s(X, Y)$ consisting of paths w that lie entirely within distance $\delta(w(0))$ of $w(0)$. There are natural projections $\pi : \text{holink}(X, Y) \to Y$ defined by evaluation at 0. We will also consider the restriction of these holink evaluations to the various subsets we have described.

A subspace $Y \subset X$ is *tame* if Y has a neighborhood $N \subset X$ such that there exists a strict map $(N \times I, Y \times I \cup N \times 0) \to (X, Y)$ which is the identity on Y and the inclusion on $N \times 1$, i.e. if there is a deformation retraction $N \to Y$ such that all points in $N - Y$ remain in $N - Y$ until time 0 (such a deformation retraction is called *nearly stratum-preserving*). If X is a filtered metric space, then it is *weakly stratified* if for each $k > i$, $X_i \subset X_k \cup X_i$ is tame and the projection $\pi : \text{holink}(X_k \cup X_i, X_i) \to X_i$ is a fibration. The filtration of a space X is said to be a *manifold filtration* if each stratum is a manifold (we will always assume without boundary), and X is called a *manifold weakly stratified space* (MWSS) if it is weakly stratified by a manifold filtration. As observed in [19, p. 235], any *locally cone-like* space is manifold weakly stratified, and hence, in particular, stratified PL-pseudomanifolds are manifold weakly stratified.

A *pure* subset Y of a filtered space X is a closed subspace that is the union of connected components of strata, i.e. Y contains each component of X_i that it intersects. For example, each skeleton X^j is a pure subset. According to Hughes [13, Cor. 6.2], for a weakly stratified metric space X with a finite number of strata, if Y is pure in X then $\pi : \text{holink}_s(X, Y) \to Y$ is a stratified fibration.

The mapping cylinder of the holink $\text{holink}(X, Y)$ for a pure subset Y is considered to be filtered as follows: the i skeleton of $\mathcal{M}$ is the union of the i skeleton of B with $\text{holink}_s(X, K)^i \times (0, 1]$.

To conclude this section, we provide the proof of a fundamental fact that we shall need in the sequel:

Proposition 3.4. *An open subset of a manifold weakly stratified space is manifold weakly stratified.*

Proof. The manifold condition on strata is clearly satisfied, since an open subset of a manifold without boundary is a manifold without boundary.

To check the tameness condition, we use Quinn's [20, Lemma 2.5], which states that for X a metric space, $Y \subset X$ is tame if and only if it is locally tame at each point. Here, $Y \subset X$ is *locally tame* at a point $y \in Y$ if there is a neighborhood V of y in X that has a nearly stratum-preserving deformation retraction rel $V \cap Y$ into Y. In our case, if X is a MWSS, let X^i be the skeleta of the filtration and $X_i = X^i - X^{i-1}$ the strata. By definition, each $X_i \subset X_k \cup X_i$ is a tame embedding and hence locally tame. Now suppose that $x \in (X_k \cup X_i) \cap U$. By Quinn's lemma, x has a neighborhood V in $X_k \cup X_i$ with a nearly stratum preserving deformation retraction to X_i. Let us denote this retraction by $R : V \times I \to X_k \cup X_i$, and let us consider $R^{-1}((X_k \cup X_i) \cap U)$. This is an open set in $V \times I$, and it includes $x \times I$ since $x \in U$ and x is stationary under R. It follows from general topology that there is an open neighborhood W of x in V such that $W \times I \subset R^{-1}((X_k \cup X_i) \cap U)$. Therefore, W is a neighborhood of x in $(X_k \cup X_i) \cap U$, and restricting R to $W \times I$ provides a nearly stratum-preserving deformation retraction in $(X_k \cup X_i) \cap U$ into $U \cap X_i$ rel $W \cap X_i \subset U \cap X_i$. Hence, $X_i \cap U$ is locally tame in $(X_k \cup X_i) \cap U$, by which it is tame.

To check the holink fibration condition, recall that by definition each holink evaluation $p : \text{holink}(X_k \cup X_i, X_i) \to X_i$ is a fibration. We need to show that this is true of the restriction $p' : \text{holink}((X_k \cup X_i) \cap U, X_i \cap U) \to X_i \cap U$. So we need to consider the following lifting problem:

$$\begin{array}{ccc} Z \times 0 & \xrightarrow{f} & \text{holink}((X_k \cup X_i) \cap U, X_i \cap U) \\ {\scriptstyle \text{inc.}}\downarrow & & \downarrow{\scriptstyle p'} \\ Z \times I & \xrightarrow{F} & X_i \cap U. \end{array}$$

Let us adjoin to the right of this diagram the inclusions $\text{holink}((X_k \cup X_i) \cap U, X_i \cap U) \hookrightarrow \text{holink}(X_k \cup X_i, X_i)$ and $X_i \cap U \hookrightarrow X_i$. The former is an inclusion since any strict path in $(X_k \cup X_i) \cup U$ with its 0 endpoint in $X_i \cap U$ is certainly also a path in $X_k \cup X_i$ with its 0 endpoint in X_i. So we have a commutative diagram

$$\begin{array}{ccccc} Z \times 0 & \xrightarrow{f} & \text{holink}((X_k \cup X_i) \cap U, X_i \cap U) & \xrightarrow{\text{inc.}} & \text{holink}(X_k \cup X_i, X_i) \\ {\scriptstyle \text{inc.}}\downarrow & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Z \times I & \xrightarrow{F} & X_i \cap U & \xrightarrow{\text{inc.}} & X_i. \end{array}$$

Since the right hand map is a fibration, there is a lift $H : Z \times I \to \text{holink}(X_k \cup X_i, X_i)$ which agrees with f on $Z \times 0$ and such that $pH = F$ (for simplicity, we omit explicit mention of the inclusion maps from the notation). We will now show how to modify H, in fact by a homotopy, to obtain a solution to our initial lifting problem.

Since $H : Z \times I \to \text{holink}(X_k \cup X_i, X_i)$, each image $H(z, s)$ is a path in $X_k \cup X_i$ and we can evaluate it at time t. Hence H induces a map $\bar{H} : Z \times I \times I \to X_k \cup X_i$ given by $\bar{H}(z, s, t) = H(z, s)(t)$. As we vary s, we obtain a homotopy of paths, each of which is strict with 0 endpoint in X_i and, in fact, in $X_i \cap U$ since $\bar{H}(z, s, 0) = H(z, s)(0) = pH(z, s) = F(z, s) \in X_i \cap U$. On the other hand, $\bar{H}(z, 0, t) = H(z, 0)(t) = f(z)(t) \in (X_k \cup X_i) \cap U$ by the definition of f. So $Z \times I \times 0 \cup Z \times 0 \times I \in \bar{H}^{-1}((X_k \cup X_i) \cap U)$, which is an open set and thus contains a neighborhood of $Z \times I \times 0 \cup Z \times 0 \times I$. Our immediate goal is to deformation retract $Z \times I \times I$ into such a neighborhood.

So let us define such a retraction $R : Z \times I \times I \times I \to Z \times I \times I$. We define $\rho : I \times I \to 0 \times I \cup I \times 0$ by thinking of $I \times I$ as being the unit square in the plane and letting $\rho(x, y)$ be the image of (x, y) in $0 \times I \cup I \times 0$ under the radial projection from the point $(2, 2)$. Note that ρ is the identity on $0 \times I \cup I \times 0$. Let $D : Z \times I \times I \to \mathbb{R}$ be the continuous function $D(z, x, y) = d((z, \rho(x, y)), Z \times I \times I - \bar{H}^{-1}((X_k \cup X_i) \cap U))$, i.e. the distance function to the closed set of points in $Z \times I \times I$ that map outside of U under $\bar{H}$. Note also that this function will be positive on all of its domain since $\bar{H}^{-1}((X_k \cup X_i) \cap U)$ is an open set that contains $Z \times I \times 0 \cup Z \times 0 \times I$. For convenience, we group $Z \times I \times I$ together as $Z \times (I \times I)$ so that we can use vector addition and scalar multiplication in the second coordinate, and we define R by

$$R(z, (x, y), t) = \begin{cases} (z, (1-t)\rho(x, y) + t(x, y)), & t \geq D(z, x, y)/2 \\ (z, (1 - \frac{D(z,x,y)}{2})\rho(x, y) + \frac{D(z,x,y)}{2}(x, y)), & t \leq D(z, x, y)/2. \end{cases}$$

To see that this will do, we first observe that R holds fixed all points in $Z \times (0 \times I \cup I \times 0)$. At time 0, (z, x, y) has retracted to $(z, (1 - \frac{D(x,y)}{2})\rho(x, y) + \frac{D(z,x,y)}{2}(x, y))$, whose distance to $(z, \rho(x, y)) \in Z \times (0 \times I \cup I \times 0)$ is $\leq \|[(1 - \frac{D(z,x,y)}{2})p(x, y) + \frac{D(z,x,y)}{2}(x, y)] - p(x, y)\| = \|\frac{D(z,x,y)}{2}(x, y) - \frac{D(z,x,y)}{2}p(x, y)\| = \frac{D(z,x,y)}{2}\|(x, y) - p(x, y)\| \leq \frac{D(z,x,y)}{2}\sqrt{2} < D(z, x, y)$. Hence $\bar{H}(R(z, x, y, 0) \in U$.

So, finally, we observe the following facts: $\bar{H}R : Z \times I \times I \times I \to X_k \cup X_i$ is a homotopy such that $\bar{H}(R(z, s, t, 0) \in (X_k \cup X_i) \cap U$ for all $(z, s, t) \in Z \times I \times I$ and $\bar{H}R(z, s, t, u) = \bar{H}(z, s, t)$ for all $(z, s, t, u) \in Z \times [0 \times I \cup I \times 0] \times I$. In particular, $\bar{H}R(z, s, 0, u) = F(z, s)$ for all $z \in Z$, $s, u \in I$ and $\bar{H}R(z, 0, t, u) = f(z)(t)$ for all $z \in Z$, $t, u \in I$. Lastly, if $t \in (0, 1]$ then $\bar{H}R(z, s, t, u) \in X_k$ because $R(z, s, t, u)$ takes $Z \times I \times (0, 1] \times I$ into $Z \times I \times (0, 1]$ which is in turn mapped to X_k by $\bar{H}$. Now, identifying $\bar{H}R(z, s, t, 0) : Z \times I \times I \to X$ as a map $K : Z \times I \to \text{holink}((X_k \cup X_i) \cap U, X_i \cap U)$ via $K(z, s)(t) = \bar{H}R(z, s, t, 0)$, we obtain the desired solution to the original lifting problem: $pK(z, s) = K(z, s)(0) = \bar{H}R(z, s, 0, 0) = F(z, s)$ and $K(z, 0)(t) = \bar{H}R(z, 0, t, 0) = f(z)(t)$.

□

4 Nearly stratum-preserving deformation retract neighborhoods

We can now introduce our principal objects of study.

Definition 4.1. Let X be a manifold weakly stratified metric space, and let $K \subset X$ be a pure subset. We will call a neighborhood N of K a *nearly stratum-preserving deformation retract neighborhood (NSDRN)* if there exists a nearly stratum-preserving deformation retraction $N \times I \to N$ that retracts N onto K.

Note: all deformation retracts, unless otherwise noted, are "strong" in the sense of being stationary on the subset to which they retract.

Observe that this definition is slightly stronger than what one might consider to be the most natural stratified analogue of the theory of NDRs. In particular, we require that the retraction take place entirely within N as opposed to more generally within X. Nonetheless, NSDRNs occur in many natural geometric settings. For example, a regular neighborhood of a pure subset of a stratified PL pseudomanifold is an NSDRN; see below.

If X is a MWSS and K is a pure subset, let $\mathcal{M} = \mathcal{M}(X, K)$ be the mapping cylinder of the holink evaluation map $p : \text{holink}_s(X, K) \to K$, which takes $w \in \text{holink}_s(X, K)$ to $w(0)$. As a set, we consider $\mathcal{M}$ to be the (disjoint) union of K and $\text{holink}_s(X, K) \times (0, 1]$; often we will also write the elements of K as $(k, 0)$. We endow all mapping cylinders with the *teardrop topology* (see [12, §2] or [24, §3]): the teardrop topology on the mapping cylinder of the map $\pi : Y \to B$ is the topology generated by the basis consisting of open subsets of $Y \times (0, 1]$ and sets of the form $[\pi^{-1}(U) \times (0, \epsilon)] \cup U$, where U is an open set in B. We can then filter this mapping cylinder as follows: Let $\mathcal{M}^i = (X^i \cap K) \cup (\text{holink}_s(X, K)^i \times (0, 1])$. In other words, the stratification on $K \subset M$ is inherited from that of X, while $(w, t) \in \text{holink}_s(X, K) \times (0, 1] \subset M$ lies in the ith stratum if and only if w lies in the ith stratum of X for $0 < t \leq 1$. Up to stratum-preserving homotopy equivalence rel K, all NSDRNs of K in X look like $\mathcal{M}$:

Proposition 4.2. *If X is a MWSS and K is a pure subset, any NSDRN of K is stratum-preserving homotopy equivalent rel K to the mapping cylinder $\mathcal{M}$ of the holink evaluation $p : \text{holink}_s(X, K) \to K$.*

Proof. This proposition is essentially found in the work of Chapman [4] and Quinn [20]. A detailed (and corrected) proof is given in the appendix to [8]. We will not repeat that proof here, but, as they will be important to us later, let us describe the maps involved.

We first dispense with one preliminary step: Suppose that $\mathcal{V}$ is our *NSDRN* of K in X. Then the inclusion $\text{holink}_s(\mathcal{V}, K) \hookrightarrow \text{holink}_s(X, K)$ is a stratum-preserving fiber homotopy equivalence. This follows as in the proof of [20, Lemma 2.4] because any path in $\text{holink}_s(X, K)$ can be retracted along itself until it lies entirely in $\mathcal{V}$. It then follows from [8, Lemma 3.18] that $\mathcal{M}$ is stratum- and cone-preserving homotopy equivalent to the mapping cylinder, M, of the holink evaluation $\text{holink}_s(\mathcal{V}, K) \to K$. So it suffices to show that $\mathcal{V}$ is stratum-preserving homotopy equivalent to M.

We can now define the necessary maps.

Recall ([20, Lemma 2.4] and [13]) that given a map $\delta : K \to (0, \infty)$, the inclusion $\text{holink}_s^{\delta}(\mathcal{V}, K) \hookrightarrow \text{holink}_s(\mathcal{V}, K)$ is a stratum-preserving fiber homotopy equivalence. Furthermore, the proof of this fact comes from constructing a deformation retraction of $\text{holink}_s(\mathcal{V}, K)$ into $\text{holink}_s^{\delta}(V, K)$ by shrinking paths along themselves. Similarly, if $\delta'(x) < \delta(x)$ then $\text{holink}_s^{\delta'}(\mathcal{V}, K) \hookrightarrow \text{holink}_s^{\delta}(\mathcal{V}, K)$ is also a stratum and fiber preserving homotopy equivalence also given by a deformation retraction that shrinks paths and hence remains entirely in $\text{holink}_s^{\delta}(\mathcal{V}, K)$. We define a shrinking map $S : \text{holink}_s(\mathcal{V}, K) \times (0, 1] \to \text{holink}_s(\mathcal{V}, K)$ as follows: on $\text{holink}_s(\mathcal{V}, K) \times [1/2, 1]$, define S as a deformation retraction that shrinks $\text{holink}_s(\mathcal{V}, K)$ into $\text{holink}_s^{1/2}(\mathcal{V}, K)$ (scaled to occur on $[1/2, 1]$ instead of $[0, 1]$); on $[1/3, 1/2]$, define S as the composition of $S(\cdot, 1/2) \times \text{id}_{[1/3,1/2]}$ followed by a deformation retraction from $\text{holink}_s^{1/2}(\mathcal{V}, K)$ into $\text{holink}_s^{1/3}(\mathcal{V}, K)$ (again scaling I to the interval $[1/3, 1/2]$); and so on. So S is stratum and fiber preserving continuous homotopy that shrinks all of the paths in the holink. We can extend S to a continuous map $\bar{S} : M \to M$ as follows: Let $\bar{S}$ be the the identity on the base K, and on $\text{holink}_s(\mathcal{V}, K) \times (0, 1]$ define $\bar{S}$ by $\bar{S}(w, s) = (S(w, s), s)$.

We can now define the stratum-preserving homotopy inverses $f : \mathcal{V} \to M$ and $g : M \to \mathcal{V}$. Given a point $y \in \mathcal{V}$, let r_y be the retraction path of y under the nearly stratum-preserving deformation retraction $r : \mathcal{V} \times I \to \mathcal{V}$ that retracts $\mathcal{V}$ to K. Define f by $f(y) = y$ if $y \in K$ and $f(y) = (r_y, d(y, K))$ if $y \in \mathcal{V} - K$. Here $d(y, K)$ is the distance from y to K. Define g by $g(y) = y$ for $y \in K$ and for $(w, s) \in \text{holink}_s(\mathcal{V}, K) \times (0, 1]$ by $g(w, s) = S(w, s)(s)$; this is the evaluation of the path $S(w, s)$ at the time s. The maps f and g are clearly well-fibered.

It is shown in [8] that f and g are continuous and stratum-preserving homotopy inverses. The stratum-preserving homotopy from gf to the identity is given by $(y, t) \to r(y, t + (1 - t)s')$, where s' is a continuous function $\mathcal{V} \to [0, 1]$ defined so that $gf(y) = r(y, s')$. The homotopy h from fg to the identity is defined by: let $h|K \times I$ be the projection onto K, and let h on $\text{holink}_s(\mathcal{V}, K) \times (0, 1]$ be determined as follows (where p_i is the projection of $\text{holink}_s(\mathcal{V}, K) \times (0, 1]$ onto its ith factor):

$$p_1 h((w,s),u)(t) = \begin{cases} r(w(t),1), & u = 1 \\ r(w(ut+(1-u)s'), \frac{t}{t-u+1}), & 0 \le t \le s'u, 0 \le u < 1 \\ r(w(ut+(1-u)s'), \frac{s'u}{s'u-u+1}(1-\frac{t-s'u}{s'-s'u}) + (u+(1-u)s')(\frac{t-s'u}{s'-s'u}), & \\ & s'u \le t \le s', 0 \le u < 1 \\ r(w(tu+(1-u)s'), u+(1-u)t), & s' \le t \le 1, 0 \le u < 1 \end{cases} \tag{4}$$

For the second coordinate, $p_2 h((w, s), u) = us + (1 - u)d(S(w, s)(s), X)$.

□

Corollary 4.3. *If X is a MWSS and K is a pure subset, any two NSDRNs of K are stratum-preserving homotopy equivalent rel K.*

Definition 4.4. We call a NSDRN N of K in X *cylindrical* if there exists a nearly stratum-preserving deformation retraction to K, $r : N \times I \to N$, so that $r(r(x, t), s) \in r(x, I)$ for all $x \in N$ and $t, s \in I$. In other words, all the points in the deformation path w of x must also retract within w under r (though we impose no condition on the parametrization of their paths compared to the parametrization of the path w).

The examples of NSDRNs we will be most concerned with will be cylindrical. Cylindrical NSDRNs include stratified mapping cylinder neighborhoods (using the the obvious retraction) and open regular neighborhoods of pure subsets of stratified PL pseudomanifolds. For the latter, note that the standard deformation retraction of the neighborhood along join lines (see, e.g., [18, §70]) satisfies the necessary conditions: it is nearly stratum-preserving because each point retracts through its native simplex until time 0, and it is cylindrical because the retractions are all performed along unique straight lines.

5 Main theorem and applications

At this point, we can state our main theorem and provide some applications.

Theorem 5.1. *Suppose X is a manifold weakly stratified space with pure subset K triangulable by a triangulation compatible with the filtration. Let $\mathcal{V}$ be a cylindrical NSDRN of K.*

Let $\mathcal{G}$ me a local coefficient system on the top stratum of $\mathcal{V}$. Then the intersection homology $IH^{\bar{p}}_*(\mathcal{V};\mathcal{G})$ is the abutment of a spectral sequence whose E^2 terms are the homology of K with a PL stratified system of coefficients, $H_p(K;\mathbb{IH}^{\bar{p}}_q(cL;\mathcal{G}))$, where $\mathbb{IH}^{\bar{p}}_q(cL;\mathcal{G})$ is a PL stratified system of coefficients whose fiber over a point $x\in K_j = K\cap X_j$ is the intersection homology of the cone of the fiber L over x of the holink evaluation $\text{holink}_s(X,K_j)\to K_j$ *(with the coefficient system induced by $\mathcal{G}$). Similarly, there is a spectral sequence for the intersection homology $IH^{\bar{p}}_*(\mathcal{V}-K)$ whose E^2 terms are the homology of K with PL stratified coefficients in $\mathbb{IH}^{\bar{p}}_*(L';\mathcal{G})$, whose fiber over $x\in K_j$ is the intersection homology of L', the fiber over X of* $\text{holink}_s((X-K)\cup K_j,K_j)\to K_j$ *(with the coefficient system induced by $\mathcal{G}$). Furthermore, the map induced by inclusion $IH^{\bar{p}}_*(\mathcal{V}-K;\mathcal{G})\to IH^{\bar{p}}_*(\mathcal{V};\mathcal{G})$ gives a map of spectral sequences which on the E^2 terms is determined by the map of stratified systems of coefficients specified on each fiber by the inclusion $L'\hookrightarrow L$, which induces the intersection homology map $\mathbb{IH}^{\bar{p}}_*(L';\mathcal{G})\to\mathbb{IH}^{\bar{p}}_*(L;\mathcal{G})$.*

Remark 5.2. To be precise, the local coefficient systems on L and L' are not restrictions of $\mathcal{G}$, but pullbacks using certain stratum-preserving homotopy equivalences to be discussed below. It is shown in [8] that it is well-defined to work with these pullbacks in the obvious manner via these homotopy equivalences. In particular, modules such as $IH^{\bar{p}}_*(L;\mathcal{G})$ are well-defined independent of fiber over a given connected component of a stratum, and collectively, these modules constitute the fibers of a well-defined system of coefficients over this stratum. Throughout this paper, for simplicity of notation., we blur these distinctions and refer only to $\mathcal{G}$ without making explicit mention of the pullback maps.

In the case of PL stratified pseudomanifolds, this theorem takes a particularly nice geometric form.

Corollary 5.3. *Suppose X is a PL stratified pseudomanifold with pure subset K triangulable by a triangulation compatible with the filtration (and PL structure), and let N be an open regular neighborhood of K. Then the intersection homology $IH^{\bar{p}}_*(N;\mathcal{G})$ is the abutment of a spectral sequence whose E^2 terms are the homology of K with a PL stratified system of coefficients, $H_p(K;\mathbb{IH}^{\bar{p}}_q(cL;\mathcal{G}))$, where $\mathbb{IH}^{\bar{p}}_q(cL;\mathcal{G})$ is a PL stratified system of coefficients whose fiber over a point $x\in K_j$ is the intersection homology of the cone on the link L of x in X. Similarly, there is a spectral sequence for intersection homology $IH^{\bar{p}}_*(N-K;\mathcal{G})$ whose E^2 terms are the homology of K with PL stratified coefficients in $\mathbb{IH}^{\bar{p}}_*(L';\mathcal{G})$ whose fiber over $x\in K_j$ is the intersection homology of L', the link of x in X minus its intersection with K (i.e. $L' = L - L\cap K$). Furthermore, the map induced by inclusion $IH^{\bar{p}}_*(N-K;\mathcal{G})\to IH^{\bar{p}}_*(N;\mathcal{G})$ gives a map of spectral sequences which on the E^2 terms is determined by the map of stratified systems of coefficients specified on each fiber by the inclusion $L'\hookrightarrow L$, which induces the intersection homology map $\mathbb{IH}^{\bar{p}}_*(L';\mathcal{G})\to\mathbb{IH}^{\bar{p}}_*(L;\mathcal{G})$.*

Proof. It follows as in [18, Lemma 70.1] that there is a deformation retraction $N\times I\to N$ that retracts N onto K. It is nearly stratum-preserving, as each point retracts linearly along a join line in its native simplex, remaining in the interior of that simplex until time 0. Hence N is an NSDRN of K in X, and we can apply the Theorem 5.1. That the links have the desired form follows from [8] in which it is shown that for the bottom stratum Y of a PL-stratified pseudomanifold, the fiber of $\text{holink}_s(X,K)$ is stratum-preserving homotopy equivalent to the geometric link L. For higher strata, we need only observe that $\text{holink}_s(X,K_j) = \text{holink}_s((X-$

$K^j) \cup K_j, K_j)$, while K_j is the bottom stratum of $(X - K^j) \cup K_j$ but possesses the same geometric link in this space as it does in X. $\square$

We can immediately apply this corollary to obtain information about intersection Alexander polynomials of non-locally-flat knots. We consider a knot K as the image of a PL embedding $S^{n-2} \hookrightarrow S^n$. We do not assume local-flatness (i.e. disk neighborhood pairs of image points are not necessarily PL unknotted). Following Cappell and Shaneson [3], it is possible to think of such a knot as the singular locus of S^n viewed as a filtered space, i.e. $X^n = S^n$, $X^{n-2} = K$, X^{n-4} =the set of points at which the embedding is not locally flat, and so on, as determined by the strata of the singularities of the embedding. Taking $\mathit{\Gamma}$ as the system of coefficients on $S^n - K$ with stalk $\Gamma = \mathbb{Q}[t, t^{-1}]$, the ring of Laurent polynomials with rational coefficients, and action by an element $\alpha \in \pi_1(S^n - K)$ determined by multiplication by $t^{\text{lk}(\alpha, K)}$, where $\text{lk}(\cdot,\cdot)$ denotes linking number. Then we can define $IH_*^{\bar{p}}(S^n; \mathit{\Gamma})$, which will be a finitely-generated torsion Γ-module, as shown in [7] based upon work in[3]. In case K is locally-flat, $IH_*^{\bar{p}}(S^n; \mathit{\Gamma})$ will be the ordinary Alexander modules. In general, we refer to these as the intersection Alexander modules and define the *intersection Alexander polynomial*, $I\lambda_i^{\bar{p}}$, as the product of the torsion coefficients of the ith intersection Alexander module [7]. These polynomials will be well-defined up to multiplication by ct^k, $c \in \mathbb{Q}$.

We can now apply Corollary 5.3 to prove fairly quickly the following statements. A closely related result was originally proven in [7] by complicated inductions using a special case of Corollary 5.3 in which our pure subset consisted of a single stratum (this special case of the corollary is proven in [8]).

Theorem 5.4. *Let $S^{n-2} \cong K \subset S^n$ be a knot, and let ξ_{iks} denote the sth intersection Alexander polynomial of the geometric link knot $L_{i,k}$ of the kth connected component of the ith stratum S_i^n. Let ζ_{iks} be the corresponding ordinary Alexander polynomial. A prime element $\gamma \in \Gamma$ divides the intersection Alexander polynomial $I\lambda_j^{\bar{p}}$ of K only if $\gamma | \lambda_j$ or γ occurs as a torsion coefficient of either $H_p(K; \mathcal{IH}_q^{\bar{p}}(cL))$ for $p + q = j$ or $H_p(K; \mathcal{IH}_q^{\bar{p}}(L'))$ for $p + q = j - 1$. Furthermore, these last two conditions are possible only if $\gamma | \xi_{iks}$ for some some set of indices i, k, and s such that $j \leq i + s$, $s \leq n - i - 3$, and $q < n - i - 1 - \bar{p}(n - p)$,* or *$\gamma | \zeta_{iks}$ for some set of indices i, k, and s such that $j - 1 \leq i + s$ and $s \leq n - p - 3$.*

The geometric link knots referred to in the theorem are the space pairs of the links of points in K. Since K and S^n are both manifolds, these space pairs will be (possibly knotted) codimension-2 sphere pairs of dimension $< n$. The polynomial λ_j is the ordinary Alexander polynomial of K; these are defined just as for locally-flat knots (see [6]).

Proof of Theorem. Let N be a regular neighborhood of the knot K in S^n. We consider the Mayer-Vietoris sequence

$$\longrightarrow IH_i^{\bar{p}}(N; \mathit{\Gamma}) \oplus IH_i^{\bar{p}}(S^n - K; \mathit{\Gamma}) \longrightarrow IH_i^{\bar{p}}(S^n; \mathit{\Gamma}) \longrightarrow IH_{i-1}^{\bar{p}}(N - K; \mathit{\Gamma}) \longrightarrow .$$

The terms are all torsion Γ-modules. In fact, $IH_i^{\bar{p}}(S^n - K; \mathit{\Gamma})$ is the ordinary Alexander module with polynomial λ_j, and by Corollary 5.3, $IH_i^{\bar{p}}(N; \mathit{\Gamma})$ and $IH_i^{\bar{p}}(N - K; \mathit{\Gamma})$ are given by spectral sequences whose E^2 terms are homology with coefficients in stratified systems. The coefficients for $IH_i^{\bar{p}}(N - K; \mathit{\Gamma})$ have stalks of the form $IH_k^{\bar{p}}(L'; \mathit{\Gamma})$. In this case, $L' = L - L \cap K$ is a link knot complement and so $IH_k^{\bar{p}}(L'; \mathit{\Gamma}) \cong H_k(L'; \mathit{\Gamma})$ is the ordinary Alexander module of the knot and hence torsion. The coefficients for $IH^{\bar{p}}_i(N;\Gamma)$ have stalks of the form $IH^{\bar{p}}_k(cL;\Gamma)$. Since the ith intersection homology of a cone cZ of dimension m is 0 if $i \geq m-1-\bar{p}(m)$ and identical to $IH^{\bar{p}}_i(Z)$ otherwise [15], $IH^{\bar{p}}_k(cL;\Gamma)$ is either 0 or the intersection Alexander module of the link knot pair L. Hence, it is torsion in either case. Thus all E^2 terms of the spectral sequence are Γ torsion, which implies that all terms in the spectral sequence and its abutment are torsion (see [7]).

It is shown in [6] and [7] that the torsion polynomial corresponding to a term in a long exact sequence of torsion modules must factor as a product of polynomials that divide the neighboring terms. Hence γ can appear in $I\lambda^{\bar{p}}_j$ if and only if it divides either λ_j or the torsion polynomial of either $IH^{\bar{p}}_j(N;\Gamma)$ or $IH^{\bar{p}}_{j-1}(N-K;\Gamma)$. But all of the terms in the filtration of $IH^{\bar{p}}_j(N;\Gamma)$ or $IH^{\bar{p}}_{j-1}(N_K;\Gamma)$ at the E^∞ stage of the spectral sequence are quotients of submodules of the E^2 terms and cannot have torsion coefficients that the E^2 terms do not have. Similarly, this restriction carries on to $IH^{\bar{p}}_j(N;\Gamma)$ and $IH^{\bar{p}}_{j-1}(N-K;\Gamma)$ themselves (see [7]). Since it is the $E^2_{p,q}$ terms with $p+q=j$ that contribute to $IH^{\bar{p}}_j(N;\Gamma)$ and those with $p+q=j-1$ that contribute to $IH^{\bar{p}}_{j-1}(N-K;\Gamma)$, this completes the first part of the theorem.

The more specific statements concerning the Alexander polynomials and intersection Alexander polynomials of the links comes from observing that $H_p(K;\mathcal{IH}^{\bar{p}}_q(cL)$ is the homology of a chain complex whose pth chain group is the direct sum of modules of the form $IH_q(cL)$, where the links L are links in S^n of points in the interiors of simplices of dimension p. This will be shown in greater detail below in the course of the proof of the main theorem. But such points can only lie in strata S^n_l for $l \geq p$. Hence, only these torsion terms can exist under taking homology (against using the rule for quotients of submodules of torsion modules). Furthermore, we use the following facts: For a knot $S^{m-2} \subset S^m$, both ordinary and intersection Alexander polynomials are trivial (similar to 1) in dimensions greater than $m-2$ [16],[7]; the link knot of the stratum of dimension i in S^n has link knot $S^{m-2} \subset S^m$ with $m = n-i-1$; and, as we have already noted, the ith intersection homology of a cone of dimension m is 0 if $i \geq m-1-\bar{p}(m)$.

□

Corollary 5.5. *The prime $\gamma \in \Gamma$ divides $I\lambda^{\bar{p}}_j$ only if $\gamma|\lambda_j$ or γ occurs as a factor of the ordinary or intersectional Alexander polynomial of one of the link knots of K.*

Further statements along the lines of this theorem can be obtained by continuing to relate the intersection Alexander polynomials of the links back to their ordinary Alexander polynomials and the polynomials of *their* links. Similarly, we can rule out certain torsion terms from occurring in an intersection Alexander polynomials if we know that it does not occur in the ordinary Alexander polynomial and if every occurrence of the corresponding torsion in $IH^{\bar{p}}_i(N-K;\Gamma)$ is mapped isomorphically to the corresponding torsion term in $IH^{\bar{p}}_i(N;\Gamma)$ in the Mayer-Vietoris sequence. It is also possible to formulate bounds on the power to which a prime element can divide the intersection Alexander polynomial based upon the powers to which it occurs in the ordinary Alexander polynomials and in the homologies of K with stratified coefficients. We will not pursue these computations here, but refer the reader to [7] for similar results proven by the original inductive methods.

6 Local behavior and intersection homology

In this section, we begin the series of lemmas and propositions which lead up to the proof of Theorem 5.1. We start by studying the local behavior in the (relative) neighborhoods of simplices of K. Our first goal is to show that these neighborhoods possess retractions which will allow us to use Proposition 4.2 to show that they are stratum-preserving homotopy equivalent to mapping cylinders of stratified fibrations. This will allow us to compute the intersection homology of these neighborhoods and of certain relative pairs of them.

Let K' denote the barycentric subdivision of a fixed triangulation of K. We will always denote *interiors* of simplices by lower case Greek letters (σ, τ, etc.) and closed simplices with bars ($\bar{\sigma}$, $\bar{\tau}$, etc.). We abuse notation and identify a simplex τ and its underlying space $|\tau| \in K$. For an open simplex σ of K, let $\mathcal{S}$ be the open star of σ in K: $\mathcal{S} = \cup\{\tau \in K \mid \bar{\tau} \cap \sigma \neq \emptyset\}$. Let S denote the open star of σ in K': $S = \cup\{\tau \in K' \mid \bar{\tau} \cap \sigma \neq \emptyset\}$, and let $\bar{S}$ be the closed star of σ in K': $\bar{S} = \cup\{\bar{\tau} \in K' \mid \bar{\tau} \cap \sigma \neq \emptyset\}$; this is equal to the union of all *closed* simplices in K' that include the barycenter of σ as a vertex. Note that S is a regular neighborhood of σ in $\mathcal{S}$.

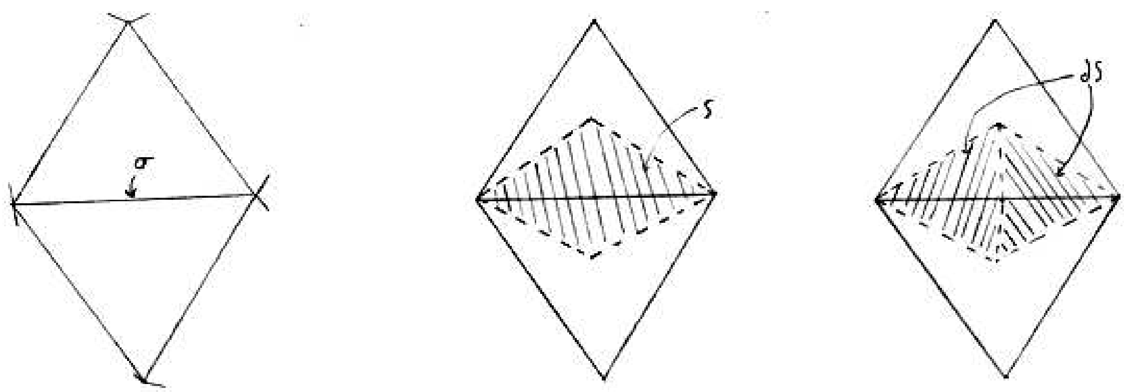


Figure 1: Examples of the sets S and ∂S (defined below)

Let U be the mapping cylinder of the restriction of $p : \text{holink}_s(X, K) \to K$ to $p^{-1}(S)$. Note that $p^{-1}(S) = \text{holink}_s((X - K) \cup S, S)$ and that this restricted map is also a stratified fibration. We stratify U as for mapping cylinders above (i.e. U is stratified by the restriction of the stratification of the mapping cylinder of $p : \text{holink}_s(X, K) \to K$).

Assume at first that we have a σ such that $\sigma \subset K_j = K \cap X_j$ and $\dim(\sigma_j) = \dim(K_j)$. Note that since K is a pure subset, it is a union of connected component of manifold strata of X. Thus K_j is a manifold of some dimension, not necessarily j. If K_j is not connected, we use $\dim(K_j)$ to mean the dimension of the connected component containing σ.

Lemma 6.1. *Let U be the mapping cylinder of the restriction of $p : \text{holink}_s(X, K) \to K$ to $p^{-1}(S)$. There exists a nearly stratum-preserving deformation retraction $R : U \times I \to U$ that retracts U onto σ.*

Proof. We first construct a nearly stratum-preserving cylindrical deformation retraction of S to σ. In fact, consider $\bar{S}$ as a complex with triangulation induced by K', and note that $\bar{\sigma}$

is a full subcomplex of $\bar{S}$. Let A be the union of the simplices in $\bar{S}$ that do not intersect $\bar{\sigma}$. Then there is a deformation retraction of $\bar{S} - A$ to $\bar{\sigma}$. Such a retraction can be constructed as in [18, Lemma 70.1] by considering each simplex in $\bar{S}$ as the join of a simplex in $\bar{\sigma}$ with a simplex in A and then retracting linearly along the join lines. Now, we need only note that S is a subset of $\bar{S} - A$ that contains σ and that all retraction lines for points in S remain in S. This is because the points whose retraction lines intersect $\bar{S} - S$ are those that lie in the join of a simplex in $\partial\bar{\sigma}$ with a simplex in A, but all such points will lie in $\bar{S} - S$ to begin with. This deformation retraction $\mathfrak{r} : S \times I \to S$ is nearly stratum-preserving since the triangulation respects the filtration of X and every point remains in its native open simplex until time 0 of the homotopy (since the retraction is linear along the join lines). It is also cylindrical since all points retract along their unique join lines.

We now extend $\mathfrak{r}$ to the mapping cylinder U. We will use the fact that the restriction $p : p^{-1}(S) \to S$ is a stratified fibration since it is the restriction of the stratified fibration $p : \text{holink}_s(X, K) \to K$ ([13]). We cannot lift $\mathfrak{r}$ directly because $\mathfrak{r}$ is not stratum-preserving, it is only *nearly* stratum-preserving. So we do the following: Consider the stratified homotopy lifting problem given by the projection $p^{-1}(S) \times 1 \to p^{-1}(S)$ and the homotopy $\mathfrak{r}(p \times \text{id}_{[1/2,1]}) : p^{-1}(S) \times [1/2, 1] \to S$. Since $\mathfrak{r}$ is stratum-preserving except at time 0, this lifting problem has a stratum-preserving solution $\mathcal{R}_1$. Next, consider the stratified homotopy lifting problem given by $\mathcal{R}_1(\cdot, 1/2) : p^{-1}(S) \times 1/2 \to p^{-1}(S)$ and $\mathfrak{r}(p \times \text{id}_{[1/3,1/2]}) : p^{-1}(S) \times [1/3, 1/2] \to S$. Again we have a stratified solution $\mathcal{R}_2$, and clearly we can think of this as continuing the homotopy of $\mathcal{R}_1$. Continuing indefinitely in this manner, we obtain a stratum-preserving homotopy $\mathcal{R}$. This partial homotopy from the identity of $p^{-1}(S)$ covers $\mathfrak{r}$ for time in $(0, 1]$.

Now we can define $R : U \times I \to U$ as follows, recalling that U is the mapping cylinder of the $p : p^{-1}(S) \to S$ and that we abuse notation by representing all points in the base S in the form $(w, 0)$ to maintain consistency with the notation for points in the rest of the mapping cylinder. Define

$$R((w,s),t) = \begin{cases} (\mathfrak{r}(w,t), 0), & s = 0, \\ (\mathcal{R}(w,t), st), & s > 0, t > 0, \\ (\mathfrak{r}(p(w), 0), 0), & s > 0, t = 0. \end{cases}$$

So we employ the retraction $\mathcal{R}$ for time $(0, 1]$ while simultaneously pushing down the mapping cylinder. At time 0, we extend continuously and everything collapses into σ. It is evident from the teardrop topology on the mapping cylinder that R is continuous. □

Corollary 6.2. *Let $\mathcal{V}$ be a cylindrical NSDRN of K in X with deformation retraction $r : \mathcal{V} \times I \to \mathcal{V}$ that retracts $\mathcal{V}$ onto K at time 0. Let $\tilde{S} = \{x \in \mathcal{V} \mid r(x, 0) \in S\}$, where S continues to denote the union of $\{\tau \in K' \mid \bar{\tau} \cap \sigma \neq \emptyset\}$. Then there is a nearly stratum-preserving deformation retraction of $\tilde{S}$ to σ in V.*

Proof. Note that because $\mathcal{V}$ is cylindrical, if $x \in V$ then so are all of the point $r(x, I)$. In particular, $\tilde{S}$ is a connected neighborhood of S in X, and, in fact, we see that there is a nearly stratum-preserving strong deformation retraction of $\tilde{S}$ to S in $\tilde{S}$ given by the restriction of r. To show that we in fact have a retraction to σ, we will employ the preceding lemma.

First, we note that $\tilde{S}$ is stratum-preserving homotopy equivalent to the mapping cylinder of the holink evaluation $\pi : \text{holink}_s(\tilde{S}, S) \to S$. Let us call this mapping cylinder M. Note that $M \subset U$, where U is the mapping cylinder of the previous lemma. This follows as in Proposition 4.2, since S is a pure subset of $\tilde{S}$, which as an open subset of an MWSS is a an MWSS itself by Proposition 3.4. Let $\phi : \tilde{S} \to M$ and $\psi : M \to \tilde{S}$ be the homotopy inverses. Then there is a stratum-preserving homotopy from the identity of V to $\psi\phi V$ and this is simply a homotopy along the retraction lines of r ([8, App.]). We consider this as a stratum-preserving homotopy $\tilde{S} \times [1,2] \to \tilde{S}$, which is the identity at time 2 and $\psi\phi$ at time 1. We now employ the deformation retraction $R : U \times I \to U$ of the last lemma. Consider the map $\tilde{S} \times [0,1] \to \tilde{S}$ given by $\psi R(\phi \times \mathrm{id}_{[0,1]})$. At time 1 this map is $\psi\phi$ which we have already homotoped from the identity. Then this map on time $[0,1]$ gives the remainder of the nearly stratum-preserving retraction to σ. (Note that $\psi|_\sigma = \mathrm{id}_{\sigma}$.).

Thus concatenating the two retractions of the last paragraph gives us the desired nearly stratum-preserving strong deformation retraction from $\tilde{S}$ to σ.

□

Corollary 6.3. *The set $\tilde{S}$ of the previous corollary is stratum-preserving homotopy equivalent to the mapping cylinder of the holink projection $p : \mathrm{holink}_s(X,\sigma) \to \sigma$.*

Proof. Since σ is a pure subset of the MWSS $\tilde{S}$ and an NSDRN by the last corollary, it follows from Proposition 4.2 that $\tilde{S}$ is stratum-preserving homotopy equivalent to the mapping cylinder of the holink evaluation $\mathrm{holink}_s(\tilde{S},\sigma) \to \sigma$. But by choosing an appropriate $\delta : \sigma \to (0,\infty)$, we see that we have the following string of stratum-preserving *fiber* homotopy equivalences: $\mathrm{holink}_s(\tilde{S},\sigma) \sim_{spfhe} \mathrm{holink}_s^\delta(\tilde{S},\sigma) \sim_{spfhe} \mathrm{holink}_s^\delta((X-K)\cup S,\sigma) \sim_{spfhe} \mathrm{holink}_s^\delta(X,\sigma) \sim_{spfhe} \mathrm{holink}_s(X,\sigma)$. The middle expression $\mathrm{holink}_s^\delta((X-K)\cup S,\sigma) \sim_{spfhe} \mathrm{holink}_s^\delta(X,\sigma)$ is due to the fact that any path ending in σ but lying outside of it until time 0 cannot possibly approach σ except by passing through $(X-K)\cup S$. □

In our further applications it will be useful to have a slightly stronger version of the above retraction. Note that, as is, we have little control over the path of the deformation retraction. We would like to be able to say that the paths of certain points do not roam too far afield from what one would expect of a nice deformation retraction. In particular, let $\mathfrak{R} : S \times I \to S$ continue to denote the cylindrical nearly stratum-preserving deformation retraction of S to σ, and let B' and C' be the intersection with σ of two open regular neighborhoods in $\bar{\sigma}$ of $\partial\bar{\sigma}$ and such that $\bar{C}' \subset B'$. Define B to be the set of $x \in S$ such that $\mathfrak{r}(x) \in B'$ and similarly for C. Since the retraction $\mathfrak{R}$ is cylindrical, these will be regular neighborhoods of $\partial\bar{\sigma}$ in $S \cup \partial\bar{\sigma}$. Again, let U be the mapping cylinder of the restriction of $p : \mathrm{holink}_s(X,K) \to K$ to $p^{-1}(S)$, and recall that this restricted map is also a stratified fibration. In the following lemma we use the notation $\tilde{Z}$ for $Z \subset S$ to mean the set of elements x of $\mathcal{V}$ such that $r(x,0) \in Z$.

Lemma 6.4. *There exists a nearly stratum-preserving strong deformation retraction $\mathfrak{R}$ of U to σ such that if w is the retraction path of a point under $r : \tilde{S} \times I \to \tilde{S}$ and $w(0) \in C$ then $\mathfrak{R}((w,s),I)$ lies in the mapping cylinder of $p : \mathrm{holink}_s(\tilde{B},C) \to C$. Hence, the evaluation $\mathfrak{R}((w,s),u)(t)$ lies in $\tilde{B}$ in the cylindrical neighborhood $\tilde{S}$ of S in X. (Note that we continue to use $(w,0)$ to refer to points in the mapping cylinder U that lie in the base S so that the statement holds for these points as well.)*

Proof. We modify our construction of Lemma 6.1. For the contraction of the base S, we continue to use the same retraction $\mathfrak{r}$, though we add some time so that we obtain a slight extension which we again denote $\mathfrak{r} : S \times [0,2] \to S$. We define $\mathfrak{r}$ on $S \times [1,2]$ as the projection to S (so nothing moves in S in the time interval $[1,2]$). Consider $p_\delta : \text{holink}_s^\delta(\tilde{S}, S) \to S$, where $\delta : S \to (0,\infty)$ is defined so that if $x \in C$ then $\delta(x) < d(x, \widetilde{S-B})$. Since $\bar{C} \subset B$ and B is open in S, such a δ exists.

Now the inclusions $\text{holink}_s^\delta(\tilde{S}, S) \hookrightarrow \text{holink}_s(\tilde{S}, S)$ and $\text{holink}_s(\tilde{S}, S) \hookrightarrow p^{-1}(S)$ are fiber homotopy equivalences, the former by [20] and the latter by similar methods, since all paths in $X - K$ ending in S can be shrunk into $\tilde{S}$ (it is crucial here that S is open in K). Let $G : p^{-1}(S) \to \text{holink}_s^\delta(\tilde{S}, S)$ be the homotopy inverse so that, if $i : \text{holink}_s^\delta(\tilde{S}, S) \hookrightarrow p^{-1}(S)$ is the composite inclusion, ir is stratum-preserving fiber homotopic to the identity. Now define $\mathfrak{R}$ on time $[1,2]$ so that on the first component of $p^{-1}(S) \times (0,1] \subset U$, it is the reverse stratum-preserving fiber homotopy that retracts $p^{-1}(S)$ into $\text{holink}_s^\delta(\tilde{S}, S)$, taking w to $G(w)$. On the second component and on the base, $\mathfrak{R}$ is constant for $t \in [1,2]$.

Now, let $\mathcal{R}$ continue to denote the retraction of Lemma 6.1. Define $\mathfrak{R}$ for $t \in [0,1]$ by

$$\mathfrak{R}((w,s),t) = \begin{cases} (\mathfrak{r}(w,t),0), & s = 0, \\ (G(\mathcal{R}(w,t)), st), & s > 0, t > 0, \\ (\mathfrak{r}(p(w),0),0), & s > 0, t = 0. \end{cases}$$

The whole homotopy is well-defined, since at time $t = 1$, $(G(\mathcal{R}(w,t)), st) = (G(w), s)$. Continuity at time $t = 0$ follows just as for the previous lemma.

Lastly, to see that our desired condition is satisfied, let us suppose that w is a retraction path of r such that $w(0) \in C$. Then under $\mathfrak{R}$ on the time interval $[1,2]$, (w,s) stays fixed in the s coordinate while w shrinks along itself to $G(w)$. Since $w(I)$ lies in $\tilde{C}$, the same is true of each path in the homotopy on $[1,2]$. On the remainder of the homotopy, $G(\mathcal{R}(w,t))$ lies in the image of G, which is in $\text{holink}_s^\delta(\tilde{S}, S)$, and furthermore, $p(G(\mathcal{R}(w,t)))$ (for time $(0,1]$) always lies in the $\mathfrak{r}$ retraction path of $w(0)$. This latter path always lies in C, and so $G(\mathcal{R}(w,t))(I) \subset \tilde{B}$ by the choice of δ. □

Corollary 6.5. *Let $\mathcal{V}$ be a cylindrical NSDRN of K in X with deformation retraction $r : \mathcal{V} \times I \to \mathcal{V}$. Let $\tilde{S} = \{x \in \mathcal{V} \mid r(x,0) \in S\}$, where S continues to denote the union of $\{\tau \in K' \mid \bar{\tau} \cap \sigma \neq \emptyset\}$. Let B and C be as defined above. Then there is a nearly stratum-preserving strong deformation retraction of $\tilde{S}$ to σ in $\tilde{S}$ such that if $x \in \tilde{C}$ then the retraction path of x remains in $\tilde{B}$.*

Proof. The proof follows as for Corollary 6.2 with some extra notes and using the last lemma instead of Lemma 6.1:

First, we note that $\tilde{S}$ is stratum-preserving homotopy equivalent to the mapping cylinder of the holink evaluation $\pi : \text{holink}_s(\tilde{S}, S) \to S$. Let us call this mapping cylinder M. This follows again as in our previous arguments (e.g. Corollary 6.3) since S is a pure subset of $\tilde{S}$, which as an open subset of an MWSS is an MWSS itself by Proposition 3.4. Let $\phi : \tilde{S} \to M$ and $\psi : M \to \tilde{S}$ be the homotopy inverses. Then there is a stratum preserving homotopy from the identity of V to $\psi\phi V$, and this is simply a homotopy along the retraction lines of r (see [8, App.]). Hence, points in $\tilde{C}$ homotop within $\tilde{C}$. We consider this as a stratum-preserving homotopy $V \times [1,2] \to V$ which is the identity at time 2 and $\psi\phi$ at time 1. We now employ the deformation retraction of the last lemma. Consider the map $\tilde{S} \times [0,1] \to \tilde{S}$ given by $\psi\mathfrak{R}(\phi \times \text{id}_{[0,1]})$. At time 1 this map is $\psi\phi$, which we have already homotoped from the identity. Then this map on time $[0,1]$ gives the remainder of the nearly stratum-preserving retraction to σ. Since, for $x \in \tilde{S}$, $\phi(x)$ is represented in the first coordinate of the mapping cylinder by the retraction line of x via r (or the stationary path if $x \in S$), the points $\mathfrak{R}(\phi(x) \times t)$, $0 \le t \le 1$, lie in the mapping cylinder of the restriction $p : \mathrm{holink}_s(\tilde{B}, C) \to C$ by the construction of the last lemma. Hence $\psi\mathfrak{R}(\phi(x) \times t) \in \tilde{B}$ for all $0 \le t \le 1$.

Thus, concatenating the two retractions of the last paragraph together gives us the desired nearly stratum-preserving strong deformation retraction from V to σ. □

Our next goal is to generalize the framework of the last set of lemmas slightly to the case where $\sigma \subset X_j$ and $\dim(\sigma) < \dim(X_j)$. In this case, let $s = X_j \cap S$, where S continues to denote $S = \cup\{\tau \in K' \mid \bar{\tau} \cap \sigma \neq \emptyset\}$. Since the triangulation of K respects the stratification of X, every open simplex $\tau \in K'$ either lies in X_j or does not intersect it at all. Therefore s will also be a union of of simplices: $s = \cup\{\tau \in K' \mid \bar{\tau} \cap \sigma \neq \emptyset, \tau \subset X_j\}$. Also, observe that s must be open in X_j. In fact, it will be an open regular neighborhood of σ in the open manifold $X_j \cap S$.

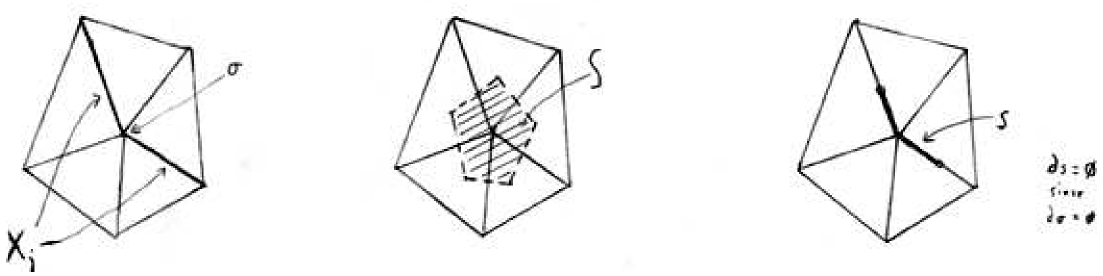


Figure 2: Examples of the sets S and s

Just as we retracted S to σ above, we can also retract S to s and use this as the basis for a retraction of $\tilde{S}$ to s. In Lemma 6.1, we defined the retraction of S to σ by using the retraction of $\bar{S} - A$ to $\bar{\sigma}$ along join lines, where A is the union of the simplices τ in $\bar{S}$ such that $\bar{\tau} \cap \bar{\sigma} = \emptyset$. Similarly, there exists such a retraction $\bar{S} - A'$ to $\bar{s}$, where A' is the union of the simplices in $\bar{S}$ that do not intersect $\bar{s}$, the closure of s in $\bar{S}$. Again, this restricts to a retraction of S to s. This will also be a cylindrical retraction.

The remainder of the proofs of the previous lemma and corollaries then proceed with obvious minor modifications. In particular, in Lemma 6.4, the sets B' and C' will be the intersections of open regular neighborhoods in $\bar{s}$ of $\partial\bar{s}$ with s. Then B and C will be the sets of points in S that retract onto B' and C'. We obtain the following statement:

Corollary 6.6. *Let $\mathcal{V}$ be a cylindrical NSDRN of K in X with deformation retraction $r : \mathcal{V} \times I \to \mathcal{V}$. Let $\tilde{S} = \{x \in \mathcal{V} \mid r(x,0) \in S\}$, where S continues to denote the union of $\{\tau \in K' \mid \bar{\tau} \cap \sigma \neq \emptyset\}$. Let s, B, and C be as defined above. Then there is a nearly stratum-preserving strong deformation retraction of $\tilde{S}$ to s in $\tilde{S}$ such that if $x \in \tilde{C}$, then the retraction path of x remains in $\tilde{B}$.*

Corollary 6.7. *The set $\tilde{S}$ of the previous corollary is stratum-preserving homotopy equivalent to the mapping cylinder of the holink projection $p : \mathrm{holink}_s(X, s) \to s$.*

At this point, we are ready to proceed to some intersection homology computations.

Remark 6.8. In the Theorem 5.1, we allow $\mathcal{V}$ to have its own local system of coefficients $\mathbb{G}$. However, in the following sections leading up to the proof, we primarily use intersection homology with standard $\mathbb{Z}$ coefficients, simply to keep these proofs from getting overly complex. However, everything adapts directly to intersection homology with local coefficients. The procedure for extending the proofs is the same as that employed in [8]: for every homotopy arguments that we make, we must also lift that homotopy to a coefficient homotopy in the covering space of local coefficients over the top stratum. Since such covering spaces are fibrations and since all of our homotopies will be stratum-preserving, there is no serious difficulty in making these extensions, there is simply the requirement of some extra bookkeeping.

For σ an open simplex in K, we let $\partial\bar{\sigma}$ be the standard boundary of the closed simplex $\bar{\sigma}$, but we let $\partial\sigma$ stand for $\sigma - \hat{\sigma}$. Note that $\partial\sigma$ can also be described as the union of the open simplices ξ in K' such that $\xi \subset \bar{\sigma}$ and $\bar{\xi} \cap \partial\bar{\sigma} \neq \emptyset$. Also, let $\breve{\sigma}$ be the union of open simplices $\xi \in K'$ such that $\bar{\xi} \cap \bar{\sigma} \neq \emptyset$, and let $\partial\breve{\sigma}$ be the union of open simplices $\xi \in K'$ such that $\bar{\xi} \cap \partial\bar{\sigma} \neq \emptyset$. For the NSDRN $\mathcal{V}$ of K in X, let $\breve{\sigma} = \{x \in \mathcal{V} \mid r(x,0) \in \breve{\sigma}\}$, and let $\partial\breve{\sigma} = \{x \in \mathcal{V} \mid r(x,0) \in \partial\breve{\sigma}\}$. We denote by $\partial\sigma''$ the union of open simplices ξ in the second barycentric subdivision K'' such that $\xi \subset \sigma$ and $\bar{\xi} \cap \partial\bar{\sigma} \neq \emptyset$. Let $\partial S = \cup\{\xi \in K' \mid \xi \subset S, \bar{\xi} \cap \partial\bar{\sigma} \neq \emptyset\}$, and let $\partial S'' = \cup\{\xi \in K'' \mid \xi \subset S, \bar{\xi} \cap \partial\bar{\sigma} \neq \emptyset\}$; note that $\partial S \cap \sigma = \partial\sigma$ and $\partial S'' \cap \sigma = \partial\sigma''$.

Again, let us begin with the case where $\sigma \subset X_l$ and $\dim(\sigma) = \dim(X_l)$. This will then be immediately generalized to the case $\dim(\sigma) < \dim(X_l)$.

Proposition 6.9. *Let $\mathcal{V}$ be a cylindrical NSDRN of K in X with retraction r. Suppose $\sigma \subset X_l$ and $dim(\sigma) = dim(X_l) = j$. Let $L_{\hat{\sigma}}$ be the fiber of the stratified holink $p : \text{holink}_s(X,\sigma) \to \sigma$ over the barycenter $\hat{\sigma}$ of σ. Then $IH_i^{\bar{p}}(\breve{\sigma}, \partial\breve{\sigma}) \cong IH_{i-j}^{\bar{p}}(cL_{\hat{\sigma}})$, where cZ denotes the open cone on Z.*

Proof. A note about the following proof: All homotopies of intersection chains will be stratum-preserving. Analogously to ordinary homology, stratum-preserving homotopies of intersection chains induce intersection homologies; this follows, just as for ordinary homology, by triangulating the trace of the homotopy and keeping coefficients fixed (see [8] for details). We often abuse notation in this way by speaking of homotoping a chain. Of course by this we mean that we can homotop the geometric carrier of the chain while also pulling along the fixed coefficient of each simplex. Again, so long as the homotopies are stratum-preserving, this induces intersection homologies just as for ordinary singular homology theory.

Let S continue to denote the open star of σ in K', $S = \cup\{\tau \in K' \mid \bar{\tau} \cap \sigma \neq \emptyset\}$, and ∂S the union of the τ in S such that $\bar{\tau} \cap \partial\bar{\sigma} \neq \emptyset$. Then we can perform an excision $IH_i^{\bar{p}}(\breve{\sigma}, \partial\breve{\sigma}) \cong IH_i^{\bar{p}}(\tilde{S}, \partial\tilde{S})$. (Recall that for $Z \subset K$, $\tilde{Z} = \{x \in \mathcal{V} \mid r(x,0) \in Z\}$; also, we simplify the notation $\widetilde{\partial S}$ to simply $\partial\tilde{S}$.) In particular, we excise all points lying in $\breve{\sigma}$ that retract under $r(\cdot,0)$ to points in open simplices in $\breve{\sigma}$ whose closures intersect $\bar{\sigma}$ but not σ. So we seek to compute $IH_i^{\bar{p}}(\tilde{S}, \partial\tilde{S})$.

It follows from [8] and Corollary 6.3 that S is stratum-preserving homotopy equivalent to $\sigma \times cL_{\hat{\sigma}}$. This follows from the fact that $p : \text{holink}_s(X,\sigma)$ is a stratified fibration by Hughes [13], and it is shown in [8] that, at least with unfiltered base spaces such as σ, these obey certain stratified versions of the usual properties of fibrations. In particular, over contractible bases they are stratum-preserving fiber homotopy equivalent to products, and these homotopy equivalences can be taken to be the identity over a point, for instance $\hat{\sigma}$ (hence they are *strong* fiber homotopy equivalence) In particular, all fibers are stratum-preserving homotopy equivalent. The triviality can then easily be extended to the mapping cylinder, in which the fiber of $\hat{\sigma}$ is $cL_{\hat{\sigma}}$. However, this is insufficient to ensure that everything works out in the pair, i.e. we do not know a priori that $(\tilde{S}, \partial\tilde{S})$ is stratum-preserving homotopy equivalent to the product of a pair. Hence this proposition will require more work.

It will suffice to prove the proposition with $\text{holink}_s(\tilde{S}, \sigma)$ in place of $\text{holink}_s(X, \sigma)$ since, as noted in the proof of Corollary 6.3, the two spaces are stratum-preserving *fiber* homotopy equivalent. This implies that the mapping cylinders are stratum- and cone-preserving homotopy equivalent, where the cones here are the inverse images of points under the obvious retraction of the mapping cylinder to σ (see [8]). Let M denote the mapping cylinder of $p : \text{holink}_s(\tilde{S}, \sigma) \to \sigma$, and let $P : M \to \sigma$ be the obvious projection of the mapping cylinder to σ. We wish to show that $IH_i^{\bar{p}}(\tilde{S}, \partial\tilde{S}) \cong IH_i^{\bar{p}}(P^{-1}(\sigma), P^{-1}(\partial\sigma))$, since then by the stratum-preserving fiber homotopy equivalences mentioned above, this will be isomorphic to $IH_i^{\bar{p}}(\sigma \times cL_{\hat{\sigma}}; \partial\sigma \times cL_{\hat{\sigma}})$. From here we can conclude via the Künneth theorem for intersection homology, since $(\sigma, \partial\sigma)$ is an unstratified manifold pair (see [15] and [8]).

Let $f : \tilde{S} \to M$ and $g : M \to \tilde{S}$ be the inverse stratum-preserving homotopy equivalences built around the nearly stratum-preserving deformation retraction $\mathfrak{R} : \tilde{S} \times I \to \tilde{S}$ of Corollary 6.5. We choose B' and C' as follows: We choose B' to be $\sigma - \hat{\sigma} = \partial\sigma$ and C' so that $\overline{\partial S''} \subset C$, $\bar{C}' \subset B'$. Note that $B \subset \partial S$ since if $x \in S$ retracts to $\sigma - \hat{\sigma}$ under $\mathfrak{r}$, then x lies in the interior of a simplex of K' that has a face $\tau \subset \sigma$ with $\bar{\tau} \cap \partial\bar{\sigma} \neq \emptyset$. But such a simplex lies in ∂S by definition. On the other hand, $\partial S \subset B$, since if x lies in the interior of a simplex of ∂S, then x retracts along a join line of that simplex to a point in $\sigma - \hat{\sigma}$. Hence $B = \partial S$. The choice of C is possible because $\overline{\partial S''} \subset B = \partial S$, as can be seen by considering the retraction of S to σ along join lines of simplices. By our construction of this retraction in Lemma 6.5, any point starting in $\tilde{C}$ retracts along a path in $\tilde{B}$. Furthermore, since f and g are stratum-preserving, this will allow us to define intersection homology maps based upon them.

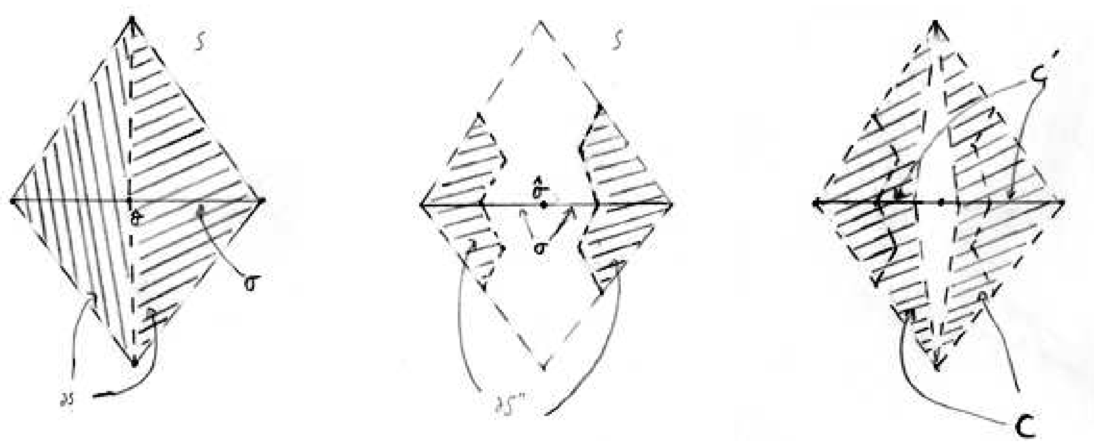


Figure 3: Examples of ∂S, $\partial S''$, C', and C

We first need to find appropriate relative chain maps induced by f and g. To do this, we first modify the domain of g_* slightly by certain isomorphisms. First, instead of the base pair $(\sigma, \partial\sigma)$ as above, let us consider the pair $(\sigma, \partial\sigma'')$. Hence in this pair there is a positive distance from the subset to the barycenter of σ. The inclusion $(\sigma, \partial\sigma'') \hookrightarrow (\sigma, \partial\sigma)$ induces an inclusion $(P^{-1}(\sigma), P^{-1}(\partial\sigma'')) \hookrightarrow (P^{-1}(\sigma), P^{-1}(\partial\sigma))$. Since $\partial\sigma$ deformation retracts to $\partial\sigma''$ in σ, the same is true for P^{-1} of these sets by using that $p : \text{holink}_s(\tilde{S}, \sigma)$ is a stratified fibration and extending this to the mapping cylinder (see [8]; alternatively, we can use the trivialization of M). Hence $IH_i^{\bar{p}}(P^{-1}(\sigma), P^{-1}(\partial\sigma)) \cong IH_s^{\bar{p}}(P^{-1}(\sigma), P^{-1}(\partial\sigma''))$. Furthermore, we have the stratum-preserving *fiber* homotopy equivalence $\text{holink}_s^{\delta}(\tilde{S}, \sigma) \hookrightarrow \text{holink}_s(\tilde{S}, \sigma)$, which also induces an isomorphism of the intersection homologies of the pairs of the corresponding mapping cylinders. This time, we choose a δ so that for $x \in \partial\sigma''$, $\delta(x) \leq d(x, \tilde{S} - \tilde{C})$, in other words so that all paths ending in $\partial\sigma''$ must lie in $\tilde{C}$. We can clearly find such a δ. Let P_δ denote the projection to σ of the mapping cylinder of the projection p restricted to $\text{holink}_s^{\delta}(\tilde{S}, \sigma)$. So we have an isomorphism $\theta : IH_i^{\bar{p}}(P^{-1}(\sigma), P^{-1}(\partial\sigma)) \to IH_i^{\bar{p}}(P_\delta^{-1}(\sigma), P_\delta^{-1}(\partial\sigma''))$ induced by these two prior maps. From the definitions, the map g restricted to the pair $(P_\delta^{-1}(\sigma), P_\delta^{-1}(\partial\sigma''))$ gives a well-defined map of pairs into $(\tilde{S}, \partial\tilde{S})$. So $g_*\theta$ is a homomorphism from $IH_i^{\bar{p}}(P^{-1}(\sigma), P^{-1}(\partial\sigma))$ to $IH_i^{\bar{p}}(\tilde{S}, \partial\tilde{S})$.

Next, we need to find an appropriate map of pairs induced by f. By a lemma to be proven immediately below, the inclusion $i : (\tilde{S}, \partial\tilde{S}'') \hookrightarrow (\tilde{S}, \partial\tilde{S})$ induces an isomorphism ν on intersection homology. The map f restricted to this first pair induces a homomorphism $f_* : IH_i^{\bar{p}}(\tilde{S}, \partial\tilde{S}'') \to IH_i^{\bar{p}}(P^{-1}(\sigma), P^{-1}(\partial\sigma))$.

Our goal now is to see that the chain maps f_* and g_* defined in the previous two paragraphs are isomorphisms. We will first show that $f_*\nu^{-1}g_*\theta$ is onto from which it will follow that f_* is onto. So let us choose an element $[\xi]$ of $IH_i^{\bar{p}}(P^{-1}(\sigma), P^{-1}(\partial\sigma))$. Under the isomorphism θ, $[\xi]$ maps to an element of $IH_i^{\bar{p}}(P_\delta^{-1}(\sigma), P_\delta^{-1}(\partial\sigma''))$. This can be represented by a chain ξ whose boundary lies in $P_\delta^{-1}(\partial\sigma'')$. Since θ^{-1} is induced by inclusions, this chain also represents our original homology element $[\xi]$. Note that the points in $\partial\xi$ are represented by paths that end in $\partial\sigma''$, so using the compactness of ξ and its boundary and the openness of $\partial\tilde{S}''$ in $\tilde{S}$, we can find a chain homotopic to ξ so that $g(\partial\xi) \subset \partial\tilde{S}''$. In fact, we can do this by simply pushing ξ down the mapping cylinder until it lies sufficiently close to the base σ. So this homotopy then induces an intersection homology to another chain representing both $\theta([\xi])$ and $[\xi]$ and whose relative image under g lies in $(\tilde{S}, \partial\tilde{S}'')$. Let us abuse notation and rename this new chain ξ. Then the image $g(\xi)$ (with appropriate coefficient) represents both $g_*\theta([\xi])$ and $\nu^{-1}g_*\theta([\xi])$, since the isomorphism ν is induced by inclusion. Thus $f_*\nu^{-1}\phi g_*([\xi]) \in IH_i^{\bar{p}}(P^{-1}(\sigma), P^{-1}(\partial\sigma))$ is represented by $fg(\xi)$, and we claim that its intersection homology class is exactly $[\xi]$, which is represented by ξ.

Now, we know that the chain $fg(\xi)$ will be stratum-preserving homotopic to ξ since fg is stratum-preserving homotopic to the identity. We must see that $\partial\xi$ remains in $P_\delta^{-1}(\partial\sigma)$ throughout the homotopy. This will allow us to conclude that $[\xi]$ and $(fg)_*[\xi]$ are relatively intersection homologous. To see this, we make explicit use of equation (4) which gives the homotopy from fg to the identity (adapting, of course, the notation to the current context). In particular, we look at the homotopies of points in the mapping cylinder: If $x \in \sigma$, then x remains fixed. If (w, s), $s > 0$, is a point in $\partial\xi$ not in the base of the mapping cylinder, then the homotopy to $fg(w, s)$ is simply linear in the second coordinate, while in the first coordinate, we sweep through a set of paths whose endpoints $(t = 0)$ are the endpoints of the retraction lines (under $\mathfrak{R}$) of the points w. But by the choice of δ, all points in w lie in $\tilde{C}$, and such points retract in paths in $\tilde{B}$. Hence their endpoints lie in $B' = \partial\sigma$. Therefore, by definition, the homotopy of (w, s) keeps it in $P^{-1}(\partial\sigma)$, and we can conclude that $f_*\nu^{-1}g_*\theta$ is onto (in fact the identity isomorphism) so that f_* is surjective and g_* is injective (since θ and ν are isomorphisms).

Next, we need to show that f_* is injective. We begin by considering the map gf on a chain representing an element of $IH^{\bar{p}}_s(\tilde{S}, \partial\tilde{S}'')$ and hence, by inclusion, an element of $IH^{\bar{p}}_s(\tilde{S}, \partial\tilde{S})$. Recall that we have chosen the nearly stratum-preserving deformation retraction of $\tilde{S}$ to σ to be the retraction $\mathfrak{R}$ of Lemma 6.5 above, where we choose B and C so that $\overline{\partial S''} \subset C$, $\bar{C} \subset B'$. Now consider the stratum-preserving homotopy from gf to the identity. We show that under this homotopy $\partial\xi$ remains in $\partial\tilde{S}$: We know that this homotopy simply moves points along their retraction lines, and by our construction of this retraction in Lemma 6.5, any point starting in $\tilde{C}$ retracts along a path in $\tilde{B}$. So the composition gf induces a map $(gf)_* : IH^{\bar{p}}_i(\tilde{S}, \partial\tilde{S}'') \to IH^{\bar{p}}_i(\tilde{S}, \partial\tilde{S})$. Furthermore, using the above homotopy, the image of ξ under this map is relatively homologous to ξ in $IH^{\bar{p}}_i(\tilde{S}, \partial\tilde{S})$. Thus the map $(gf)_*$ is equal to the isomorphism ν induced by inclusion.

Now suppose that ξ is a chain representing $[\xi] \in IH^{\bar{p}}_s(\tilde{S}, \partial\tilde{S}'')$ and that $f_*([\xi]) = 0$, i.e. the intersection chain image of ξ in $(P^{-1}(\sigma), P^{-1}(\partial\sigma))$ relatively bounds. Let η be an intersection chain in $(P^{-1}(\sigma), P^{-1}(\partial\sigma))$ whose boundary is carried by $f(\xi)$ together with another piece, ζ, with $\zeta \subset P^{-1}(\partial\sigma)$. Then the boundary of $g(\eta)$ (with appropriate coefficients) will consist of $gf(\xi)$, which represents $\nu([\xi])$, and a chain carried by $g(\zeta)$. If $g(\zeta)$ lies in $\partial\tilde{S}$, it will will follow that $\nu([\xi]) = 0$ and hence $[\xi] = 0$. To ensure this, we modify η slightly. Since $\partial\eta$ is compact and lies in $P^{-1}(\partial\sigma)$ and since $\partial\sigma \subset \partial\tilde{S}$, which is open in $\tilde{S}$, we can can homotop η down the mapping cylinder M so that the end of this homotopy is a chain η' such that the points $g(x)$ for $x \in \partial\eta$ lie entirely in $\partial\tilde{S}$. Since we simply push down the cylinder, fibers of M are preserved and η' will thus be relatively intersection homologous to η. The image of η' under g now satisfies the desired condition. Furthermore, the image under g of this homotopy of η restricted to $f(\partial\xi)$ lies entirely in $\partial\tilde{S}$, since this image will simply be a retraction along paths and we have already noted that, for x in $f(\partial\xi)$, the path of x is a retraction path that lies entirely in $\tilde{B} \subset \partial\tilde{S}$. Using this homotopy, we can construct a relative homology from ξ to its image under the composition of f, the restriction of the homotopy of η to η', and g. So this image is relatively homologous to ξ but relatively bounded by $g(\eta')$. Hence $[\xi] = 0$ and f_* is injective.

Hence, f_* is an isomorphism, and thus so is g_*, since we have already established that $f_*\nu^{-1}g_*\theta$, ν, and θ are isomorphisms.

□

Lemma 6.10. *The inclusion $i : (\tilde{S}, \partial\tilde{S}'') \hookrightarrow (\tilde{S}, \partial\tilde{S})$ induces an isomorphism on intersection homology.*

Proof. We first prove surjectivity. Let ξ be a chain representing an element $[\xi]$ of $IH^{\bar{p}}_i(\tilde{S}, \partial\tilde{S})$. We will find a stratum-preserving homotopy in $\partial\tilde{S}$ that takes $\partial\xi$ into $\partial\tilde{S}''$. This will induce an intersection homology of $[\xi]$ to an element of $IH^{\bar{p}}_i(\tilde{S}, \partial\tilde{S}'')$.

Let M be the mapping cylinder of $p : \text{holink}_s(X, S) \to S$, and let $f : \tilde{S} \to M$ and $g : M \to \tilde{S}$ be the stratum-preserving homotopy inverses (see Proposition 4.2). Let $h : \partial\xi \times I \to S$ be a homotopy in ∂S from $r(\partial\xi, 0) = pf(\partial\xi)$ into $\partial S''$ and that is stationary for time $[0, 1/3] \cup [2/3, 1]$. This exists since ∂S weakly deformation retracts into $\partial S''$. In fact, we can retract each simplex, ζ of ∂S linearly towards the simplex $\bar{\zeta} \cap \partial\bar{\sigma}$ (in the boundary of it closure) along the join lines to the faces of $\bar{\zeta}$ that do not intersect $\partial\bar{\sigma}$. At some time > 0, we will have retracted ∂S completely into $\partial S''$. We will cover this retraction with one of $\partial\xi$. Consider the deformation retraction ρ parametrized over $[2/3, 1]$ of $p^{-1}(\partial S)$ to $\text{holink}_s^\delta((X - K) \cup \partial S, \partial S)$, where δ is chosen so that $\delta(x) < d(x, X - \partial\tilde{S})$ for x in an open regular neighborhood T of $\partial S''$ in ∂S such that $\overline{\partial S''} \subset T$, $\bar{T} \subset \partial S$, and $h(\partial\xi, I) \subset T$. So composing the extension of ρ to the mapping cylinder M with $f \times \text{id}_{[2/3,1]}$ gives a homotopy of $\partial\xi$ on time $[2/3, 1]$ on which p is constant and the image at time $2/3$ is in the mapping cylinder of $\text{holink}_s^\delta((X - K) \cup \partial S, \partial S) \subset \text{holink}_s(\partial\tilde{S}, \partial S)$. Since $\partial\tilde{S}$ is a MWSS as an open subset of one, this latter holink is a stratified fibration, so we can cover $h|\partial\xi \times [1/3, 2/3]$ with a homotopy that extends from $\rho(\cdot, 2/3)f$ on ∂S (for points in the mapping cylinder, we leave the second coordinate constant and homotop the first according to either h, if $x \in S$, or according to the solution of the stratified lifting problem in the first coordinate for points in $f(\partial\xi - \partial\xi \cap S)$). So at this point, we have a homotopy from $f|\partial\xi$ to a map that takes $\partial\xi$ into the mapping cylinder of $\text{holink}_s^\delta((X - K) \cup \partial S, \partial S)$ and furthermore into the fibers over $\partial S''$. Finally, on time $[0, 1/3]$, we apply the retraction of this mapping cylinder so that the end of our homotopy lies in $g^{-1}(\partial\tilde{S}'')$. Since $\partial\xi$ is compact and $\partial S''$ is open in ∂S, this is possible by again retracting $\text{holink}_s((X - K) \cup \partial S, \partial S)$ to $\text{holink}_s^\delta((X - K) \cup \partial S, \partial S)$ for a suitably small δ, which can be allowed to depend on $\partial\xi$. Altogether, we now have a homotopy $H : \partial\xi \times I \to M$ from $f|\partial\xi$ to a map whose composition with g is in $\partial\tilde{S}''$. So we consider gH. By our constructions, this homotopy is stratum-preserving and has its image in $\partial\tilde{S}$ at all times. So it only remains to demonstrate a stratum-preserving homotopy $\partial\xi \times I \to \tilde{S}$ parametrized over $[1, 2]$ which is the inclusion of $\partial\xi$ at time 2 and gf at time 1 and which also lies within $\partial\tilde{S}$ at all times. But we can obtain this by the standard stratum-preserving homotopy which demonstrated that g and f are stratum-preserving homotopy inverses (the last desired condition is satisfied since the homotopy keeps each point within its cylinder fiber for the deformation retract r, and this preserves fibers over $\tilde{S}$ by definition).

Injectivity is proven similarly. If ξ is a chain representing an element $[\xi]$ of $IH_i^{\bar{p}}(\tilde{S}, \partial\tilde{S}'')$ and ξ maps to 0 under the inclusion into $IH_i^{\bar{p}}(\tilde{S}, \partial\tilde{S})$, then there is a chain η in $IC_{i+1}^{\bar{p}}$ whose boundary consists of ξ plus possibly another component whose support lies in $\partial\tilde{S}$. But using the methods of the last paragraph, we can homotop η into a chain whose boundary lies entirely in $\partial\tilde{S}''$ and we can do this in such a way that $\partial\eta$ remains in $\partial\tilde{S}$ throughout the homotopy. Furthermore, we can ensure that $\partial\xi$ remains in $\partial\tilde{S}''$ throughout the homotopy. This can be done by composing the homotopy H of the previous paragraph (though adapted to $\partial\eta$ instead of $\partial\xi$) with a further shrinking, α, of the holink to $\text{holink}_s^\delta((X - K) \cup \partial S, \partial S)$, where δ is taken so that for x in the compact set $pH(\partial\xi) \subset \partial S''$, $\delta(x) < d(x, \tilde{S} - \partial\tilde{S}'')$. This homotopy αH satisfies all of the desired conditions when sent back into $\tilde{S}$ via g. And again, we can easily join the inclusion of $\partial\eta$ to $g\alpha H$ by a homotopy on time $[1, 2]$ via a contraction along retraction paths of r. The homotopy of the boundary of η thus induces a homology of ξ to another chain representing the same element of $IH_i^{\bar{p}}(\tilde{S}, \partial\tilde{S}'')$, but now clearly this element must be zero since it is a relative boundary of the homotoped η. Hence ξ represented the zero element to begin with, and the homomorphism on intersection homology induced by inclusion $(\tilde{S}, \partial\tilde{S}'') \hookrightarrow (\tilde{S}, \partial\tilde{S})$ is injective, hence an isomorphism.

□

Recall the definition of s as $s = X_j \cap S = \cup\{\tau \in K' \mid \bar{\tau} \cap \sigma \neq \emptyset, \tau \subset X_j\}$. Let

$\partial s = s \cap \partial S = \cup\{\tau \in s | \bar{\tau} \cap \partial \bar{\sigma} \neq \emptyset\}$.

Proposition 6.11. *Let $\mathcal{V}$ be a cylindrical NSDRN of K in X with retraction r. Suppose $\sigma \subset X_k$ and $dim(\sigma) < dim(X_k)$. Let L be the fiber of the stratified holink $p : \text{holink}_s(X, s) \to s$, and let j be the dimension of $\sigma \in K$. Then $IH^{\bar{p}}_i(\breve{\sigma}, \partial\breve{\sigma}) \cong IH^{\bar{p}}_{i-j}(cL_{\hat{\sigma}})$, where cZ denotes the open cone on Z.*

Proof. The proof is the same as that of the last proposition except that in this case $\tilde{S}$ is homotopy equivalent to $s \times cL$, where L is the fiber of $\text{holink}_s(X, s) \sim_{sphe} \text{holink}_s(\tilde{S}, s)$, and the base in the Künneth theorem for intersection homology is $(s, \partial s)$, where we define ∂s most simply by $\partial S \cap s$. Note that the inclusion $(\sigma, \partial\sigma) \hookrightarrow (s, \partial s)$ is a homotopy equivalence of pairs since $(s, \partial s)$ deformation retracts onto $(\sigma, \partial\sigma)$ by restricting the retraction that takes S onto σ. So the dimensions work out correctly in the Künneth theorem. The remainder of the proof then involves the obvious modifications, e.g. replacing σ and $\partial\sigma$ with s and ∂s, where appropriate, and replacing $\partial\sigma''$ with $\partial s''$, the intersection of s with $\partial S''$.

We make the following additional notes concerning the necessary modification: 1) ∂s (weakly) deformation retracts into $\partial s''$ within ∂s (we can again use linear retractions of ∂s towards $\partial\bar{\sigma}$), and 2) we choose $B' = \partial s$ and by reasoning analogous to the previous case, there exists a C' such that $\overline{\partial s''} \subset C$, $\bar{C} \subset \partial s$.

□

Corollary 6.12. *If $\bar{\tau}$ is a j-face of the m-simplex $\bar{\sigma}$ and both open simplices σ and τ lie in the same stratum X_k, then $IH^{\bar{p}}_i(\breve{\sigma}, \partial\breve{\sigma}) \cong IH^{\bar{p}}_{i-(m-j)}(\breve{\tau}, \partial\breve{\tau})$.*

Proof. By the above proposition, $IH^{\bar{p}}_i(\breve{\sigma}, \partial\breve{\sigma}) \cong IH^{\bar{p}}_{i-m}(cL_{\hat{\sigma}})$, where $L_{\hat{\sigma}}$ is the fiber of $\text{holink}_s(X, s)$ over $\hat{\sigma}$ and s is an open subset of the stratum X_k (if $\dim(\sigma) = \dim(X_k)$, then $\sigma = s$). Similarly, $IH^{\bar{p}}_i(\breve{\tau}, \partial\breve{\tau}) \cong IH^{\bar{p}}_{i-j}(cL_{\hat{\tau}})$, where $L_{\hat{\tau}}$ is the fiber of $\text{holink}_s(X, t)$ over $\hat{\tau}$ and t is also an open subset of X_j defined for τ analogously as s is for σ. Hence, in particular, $t \cap s \neq \emptyset$ and $s \cup t \subset X_k$ is unfiltered. Therefore, the fiber in each case is stratum-preserving homotopy equivalent to the fiber of $\text{holink}_s(X, s \cup t)$ (all such fibers are stratum-preserving homotopy equivalent since $s \cup t$ is connected and $\text{holink}_s(X, s \cup t)$ is a stratified fibration over an unstratified base; see [8]). In particular then, $IH^{\bar{p}}_{i-(m-j)}(\breve{\tau}, \partial\breve{\tau}) \cong IH^{\bar{p}}_{i-j-(m-j)}(cL) \cong IH^{\bar{p}}_{i-m}(cL) \cong IH^{\bar{p}}_i(\breve{\sigma}, \partial\breve{\sigma})$. □

We shall denote the isomorphisms of Propositions 6.9 and 6.11 by $\Phi_\sigma : IH^{\bar{p}}_i(\breve{\sigma}, \partial\breve{\sigma}) \cong IH^{\bar{p}}_{i-j}(cL_{\hat{\sigma}})$. Let us note that these maps are independent of the choices involved. In fact, the only choices are the that of strong stratified trivialization and of δ. The choice of strong stratified trivialization is irrelevant since, just as for ordinary fibrations, any two such are fiber- (and stratum-) preserving homotopic (see [23] and [8]). As for the choice of δ, this is irrelevant because for any two choices δ_1 and δ_2 satisfying the requirements of the above propositions, there is a continuous δ_3 satisfying the requirements with $\delta_3 < \delta_2$ and $\delta_3 < \delta_1$.

Then we obtain the following diagram.

$$\begin{array}{ccccc}
IH_i^{\bar{p}}(P^{-1}(s),P^{-1}(\partial s)) & \xrightarrow{=} & IH_i^{\bar{p}}(P^{-1}(s),P^{-1}(\partial s)) & \xrightarrow{=} & IH_i^{\bar{p}}(P^{-1}(s),P^{-1}(\partial s)) \\
\text{inc.}\uparrow & & \text{inc.}\uparrow & & \text{inc.}\uparrow \\
IH_i^{\bar{p}}(P_{\delta_1}^{-1}(s),P_{\delta_1}^{-1}(\partial s'')) & \xleftarrow{\text{inc.}} & IH_i^{\bar{p}}(P_{\delta_3}^{-1}(s),P_{\delta_1}^{-1}(\partial s'')) & \xrightarrow{\text{inc.}} & IH_i^{\bar{p}}(P_{\delta_2}^{-1}(s),P_{\delta_2}^{-1}(\partial s'')) \\
g_*\downarrow & & g_*\downarrow & & g_*\downarrow \\
IH_i^{\bar{p}}(\tilde{S},\partial\tilde{S}) & \xleftarrow{=} & IH_i^{\bar{p}}(\tilde{S},\partial\tilde{S}) & \xrightarrow{=} & IH_i^{\bar{p}}(\tilde{S},\partial\tilde{S})
\end{array}.$$

All boxes clearly commutes, and each map is an isomorphism. Thus, the composite map from $IH_i^{\bar{p}}(\tilde{S},\partial\tilde{S})$ to $IH_i^{\bar{p}}(P^{-1}(s),P^{-1}(\partial s))$ is independents of the choice of δ.

7 The NSDRN $\mathcal{V}$ induces a stratified system of coefficients on K

We now that show that a pure subset K of X that possesses a cylindrical NSDRN and a PL structure compatible with the stratification of X can be endowed with a certain natural PL stratified system of local coefficients. The homology modules of K with this system of coefficients will constitute the E^2 terms of our spectral sequence.

We continue with the convention of the previous section whereby we work explicitly only with intersection homology with $\mathbb{Z}$ coefficients. However, we continue to note that all statements hold with local coefficients by following the methods for extension in [8].

Again, we endow K with a given triangulation and denote by K^i the subcomplex triangulating $K\cap X^i$. We assume that the simplices of K have been oriented once and for all and that this is the orientation used in all chain complexes discussed below. Over the stratum $K_i = K^i - K^{i-1}$, we have the stratified fibration $p_i : \text{holink}_s(X, K_i) = \text{holink}_s(X - K^{i-1}, K_i) \to K_i$. This is the restriction of the stratified fibration $\pi_i : \text{holink}_s(X, K^i) \to K^i$ to $\pi_i^{-1}(K_i)$. This restriction is also a stratified fibration over an unfiltered base, and hence it induces a bundle of coefficients (i.e. a bundle of modules or a locally-constant sheaf) $\mathcal{IH}_*^{\bar{p}}(L)$ over K_i. Here $L = p_i^{-1}(x) = \text{holink}_s((X - K^i)\cup x, x)$, for any x in K_i, is the stratified fiber of p_i, and the stalk of the coefficient system is $IH_*^{\bar{p}}(L)$. If K_i is not connected, then this stalk depends on the choice of connected component. The bundle of coefficients is induced just as for an ordinary fibration by using homotopy lifting over paths; see [8] for details. Similarly, by Proposition 3.3, the open mapping cylinder $M_i = M_{p_i}$ of p_i is also a stratified fibration over K_i with fiber cL, and it induces a local system of coefficients $\mathcal{IH}_*^{\bar{p}}(cL)$.

Our first goal is to show that the following diagram commutes, where $\bar{\tau}$ is a $k-1$ face of the k-simplex $\bar{\sigma}$, τ and σ like in the same stratum K_j, and $\psi_{\sigma\tau}$ is the map on intersection homology induced in the bundle of coefficients by a path in $\sigma\cup\tau$:

$$\begin{array}{ccc}
IH_{i-k}^{\bar{p}}(cL_{\hat{\sigma}}) & \xrightarrow{\psi_{\sigma\tau}} & IH_{i-k}^{\bar{p}}(cL_{\hat{\tau}}) \\
\Phi_\sigma\uparrow & & \Phi_\tau\uparrow \\
IH_i^{\bar{p}}(\breve{\sigma},\partial\breve{\sigma}) & \xrightarrow{(-1)^{\sigma\tau}\pi_\tau\partial_*} & IH_{i-1}^{\bar{p}}(\tilde{\tau},\partial\tilde{\tau}).
\end{array} \tag{5}$$

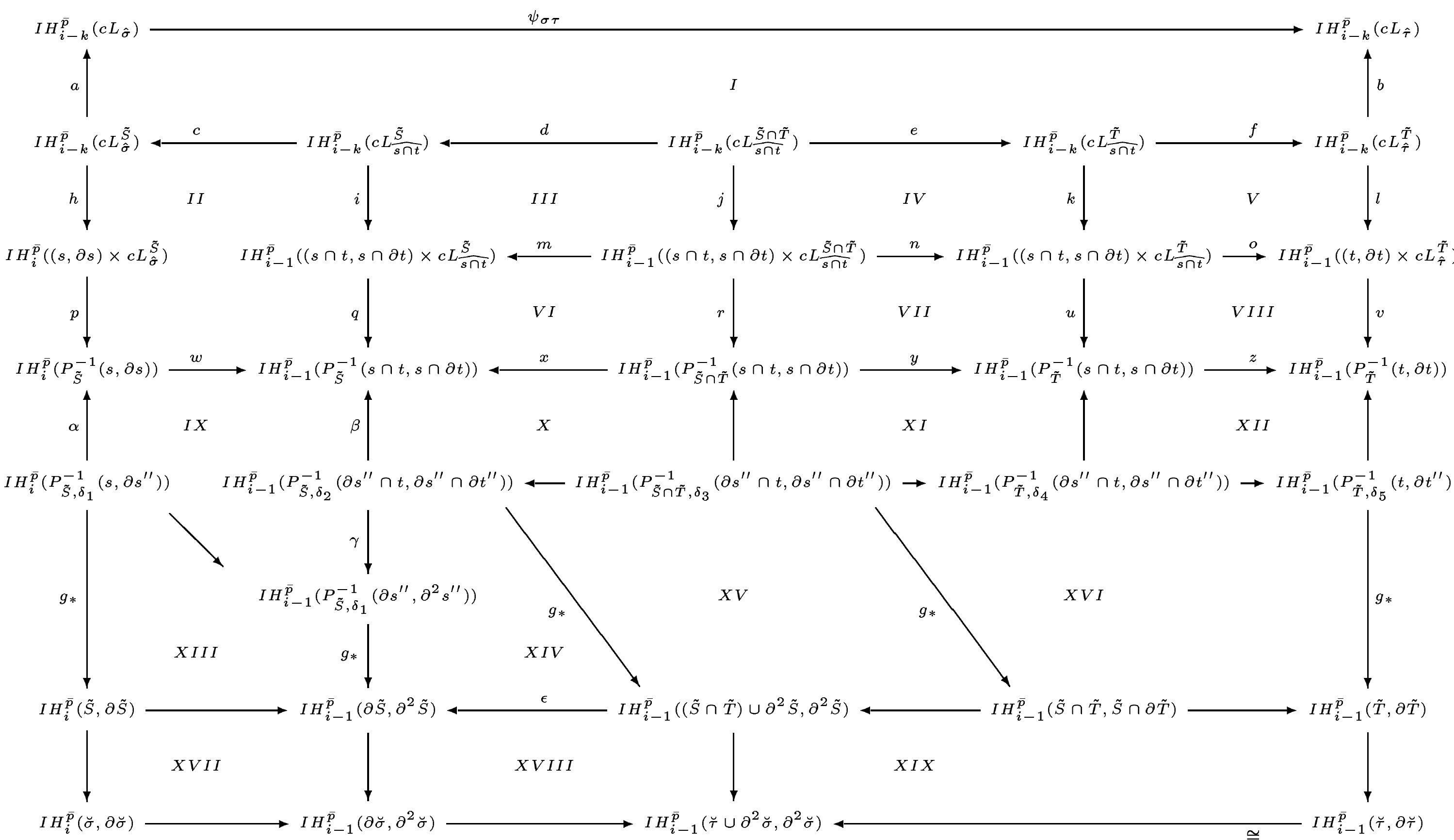
Recall the definition of Φ_σ as given at the end of Section 6; Φ_τ is defined similarly. The bottom horizontal map is defined as the following composition: From the long exact sequence of the triple, there is a boundary map $\partial_* : IH^{\bar{p}}_i(\breve{\sigma}, \partial\breve{\sigma}) \to IH^{\bar{p}}_{i-1}(\partial\breve{\sigma}, \partial^2\breve{\sigma})$, where $\partial^2\breve{\sigma}$ is defined as $\cup\{\breve{\xi} \mid \bar{\xi}$ is a $k-2$ face of $\bar{\sigma}\}$. The group $IH^{\bar{p}}_{i-1}(\partial\breve{\sigma}, \partial^2\breve{\sigma})$ is isomorphic to the direct sum $\oplus_\zeta IH^{\bar{p}}_{i-1}(\breve{\zeta}, \partial\breve{\zeta})$, where the sum is taken over $k-1$ faces ζ of σ. We can then compose this boundary morphism with the projection π_τ to the summand $IH^{\bar{p}}_{i-1}(\breve{\tau}, \partial\breve{\tau})$. Finally, define $(-1)^{\sigma\tau}$ to be the sign that $\bar{\tau}$ takes in the boundary formula for $\bar{\sigma}$ treated as an elementary k-chain (with $\mathbb{Z}$ coefficients). This sign is determined by our fixed choice of orientation for each simplex in K.

Lemma 7.1. *Diagram* (5) *commutes.*

Proof. Unfortunately, the definitions of Φ_σ and Φ_τ involve restrictions to the holinks $\text{holink}_s(\tilde{S}, s)$ and $\text{holink}_s(\tilde{T}, t)$, so to compare the two, we will need to fill out the diagram with some intermediate stages. First, let us be more explicit about the maps Φ_σ. By our constructions in Section 6, they are the compositions of the following isomorphisms (we have added decorations of $\tilde{S}$ to be clear about which space the holink paths are take in; similar equations hold for τ and $\tilde{T}$; undecorated spaces and maps assume paths in X):

$$\begin{aligned}
IH^{\bar{p}}_{i-k}(cL_{\hat{\sigma}}) &\cong IH^{\bar{p}}_{i-k}(cL^{\tilde{S}}_{\hat{\sigma}}) && \text{stratified h.e.}\\
&\cong IH^{\bar{p}}_{i}((s, \partial s) \times cL^{\tilde{S}}_{\hat{\sigma}}) && \text{Künneth theorem}\\
&\cong IH^{\bar{p}}_{i}(P^{-1}_{\tilde{S}}(s), P^{-1}_{\tilde{S}}(\partial s)) && \text{strong stratified fiber h.e.}\\
&\cong IH^{\bar{p}}_{i}(P^{-1}_{\tilde{S},\delta}(s), P^{-1}_{\tilde{S},\delta}(\partial s'')) && \text{the map } \theta\\
&\cong IH^{\bar{p}}_{i}(\tilde{S}, \partial\tilde{S}) && \text{the map } g_*\\
&\cong IH^{\bar{p}}_{i}(\breve{\sigma}, \partial\breve{\sigma}) && \text{excision}
\end{aligned}$$

So, we will explore the following diagram whose outside edge is diagram (5) (the bottom edge has also been expanded but still represents the same map as the bottom of (5)).



It will be useful to adopt the following notations: $\widehat{s\cap t}$ = some base point chosen in $s\cap t$. For convenience, we choose $\widehat{s\cap t}\in\sigma$. $L^{\tilde S}_{\widehat{s\cap t}}$ is the fiber over of $\widehat{s\cap t}$ of the stratified fibration $\text{holink}_s(\tilde S, s\cap t)\to s\cap t$, similarly for the other cL's. We define $\partial^2 s''$ as the intersection of $\partial s''$ with all second derived neighborhoods of the $k-2$ skeleton of $\bar\sigma$. The columns of this diagram then follow the same pattern of isomorphisms as Φ_σ, while most of the horizontal maps are induced by inclusions. We will describe the remaining notations, maps, and groups in the following proofs of commutativity of the numbered polygons. To cut down on the already cumbersome notation, we sometimes confuse notation for maps on intersection homology with the geometric maps that induce them (i.e. $f=f_*$).

Let us show that each numbered subdiagram commutes:

I. The maps a, b, d and e are induced by inclusions, while $\psi_{\sigma\tau}$, c, and f are induced, respectively, by extending the inclusions of the fibers $cL_{\hat\sigma}$, $cL^{\tilde S}_{\widehat{s\cap t}}$, and $cL^{\tilde T}_{\widehat{s\cap t}}$ into the respective mapping cylinders of the stratified fibrations $\text{holink}_s(X,X_j)\to X_j$, $\text{holink}_s(\tilde S, s\cap t)\to s\cap t$, and $\text{holink}_s(\tilde T, s\cap t)\to s\cap t$ into homotopies over paths in the unstratified contractible sets $\sigma\cup\tau$, σ, and t. (Recall that we have shown that the mapping cylinder of a stratified fibration also gives a stratified fibration). Hence it is clear that bfe and $\psi_{\sigma\tau}acd$ both take intersection cycles in $IH^{\bar p}_{i-k}(cL^{\tilde S\cap\tilde T}_{\widehat{s\cap t}})$ to intersection cycles in $IH^{\bar p}_{i-k}(cL_{\hat\tau})$ that are homotopic in the mapping cylinders of the stratified fibrations $\text{holink}_s(X,X_j)\to X_j$ to the original cycle, by homotopies over two different paths in $\sigma\cup\tau$. But $\sigma\cup\tau$ is contractible and so the two paths are homotopic rel endpoints and the two intersection cycles must be stratum-preserving homotopic and hence intersection homologous. This follows just as in the theory of ordinary fibrations. For details in the case of a stratified fibration over an unfiltered base space (as is the current situation), see [8, §4] .

II. The maps h and i are isomorphisms by the Künneth theorem: note that our retraction of s to σ takes $(s\cap t, s\cap\partial t)$ to $(\sigma\cap t,\sigma\cap\partial t)$, which is homology equivalent to (D^{k-1}, S^{k-2}). The maps and p and q are strong stratum-preserving fiber homotopy equivalences with the fibers over the respective base points being the ones preserved (see [8] for a detailed proof of the existence of these homotopy equivalences). The map w is a composition of two maps. The first is $(-1)^{\sigma\tau}$ times the boundary map in the long exact sequence of the triple $(P^{-1}_{\tilde{S}}(s), P^{-1}_{\tilde{S}}(\partial s), P^{-1}_{\tilde{S}}(\partial^2 s))$, where $\partial^2 s$ is the intersection of s with the first derived neighborhood in K' of the $k-2$ skeleton of the the k simplex $\bar{\sigma}$ (just as ∂s is the intersection of s with the first derived neighborhood of the $k-1$ skeleton of σ). In other words, if $\bar{\sigma}^{k-2}$ is the simplicial $k-2$ skeleton of $\bar{\sigma}$, then $\partial^2 s=\cup\{\xi\in s \mid \bar{\xi}\cap\bar{\sigma}^{k-2}\neq\emptyset\}$. Now, for each $k-1$ face $\bar{\xi}$ of $\bar{\sigma}$, define $x=\cup\{v\in K'\mid v\subset s, \bar{v}\cap\xi\neq\emptyset\}$ and $\partial x=\cup\{v\in x\mid \bar{v}\cap\partial\bar{\xi}\neq\emptyset\}$. Note that for $\xi=\tau$, $x=s\cap t$ and $\partial x=s\cap\partial t$. It is clear from the definitions of these sets and from excision that $IH^{\bar{p}}_*(P^{-1}_{\tilde{S}}(\partial s), P^{-1}_{\tilde{S}}(\partial^2 s))\cong\oplus_\xi IH^{\bar{p}}_*(P^{-1}_{\tilde{S}}(x,\partial x))$, where the sum is over $k-1$ faces $\bar{\xi}$ of $\bar{\sigma}$. We can then project from this direct sum down to $IH^{\bar{p}}_*(P^{-1}_{\tilde{S}}(s\cap t, s\cap\partial t))$, which is the summand corresponding to τ. The map w is the (signed) boundary map followed by this projection. That this square commutes follows from some minor modifications to the proof of the main theorem in [8].

III and IV. These squares commute due to the naturality of the Künneth theorem since d and m are compatible inclusions, as are e and n. See also item II.

V. As noted above, the map f is induced by extending the inclusion of the fiber to a homotopy over a path. The map o is induced by the same homomorphism on the fiber plus the homotopy equivalence of the pairs $(t,\partial t)$ and $(s\cap t, s\cap\partial t)$. Hence, the commutativity of this square follows from this homotopy equivalence and the naturality of the Künneth theorem.

VI and VII. We demonstrate VI; VII is equivalent. The maps m and x are induced by inclusions; r and q are induced by strong stratum-preserving fiber homotopy equivalences, each of which is stationary over $\widehat{s\cap t}$. So qm and xr are both stratum- and fiber-preserving and agree over the basepoint $\widehat{s\cap t}$ and on the base $s\cap t$. Since $(s\cap t,\widehat{s\cap t})$ is an NDR pair, $((s\cap t)\times I, [(s\cap t)\times\dot{I}]\cup\widehat{s\cap t}\times I)$ is a DR pair. Thus, $(s\cap t,\widehat{s\cap t})\times cL^{\tilde{S}\cap\tilde{T}}_{\widehat{s\cap t}}$ is a stratum-preserving NDR pair and $((s\cap t)\times I, [(s\cap t)\times\dot{I}]\cup\widehat{s\cap t}\times I)\times cL^{\tilde{S}\cap\tilde{T}}_{\widehat{s\cap t}}$ is a stratum-preserving DR pair, by taking the product of the deformation retraction on the base with the identity map on $cL^{\tilde{S}\cap\tilde{T}}_{\widehat{s\cap t}}$. Let $H:(s\cap t)\times cL^{\tilde{S}\cap\tilde{T}}_{\widehat{s\cap t}}\times I\to s\cap t$ be the projection to $s\cap t$, and define $\tilde{H}:[(s\cap t)\times\dot{I}]\cup\widehat{s\cap t}\times I\to P^{-1}_{\tilde{S}}(s\cap t)$ by xr on one end of the homotopy, qm on the other end, and $zr=qm$ on any point over $\widehat{s\cap t}\times I$. This sets up a stratified lifting extension problem to the stratified fibration $P^{-1}_{\tilde{S}}(s\cap t)\to s\cap t$, and by [8, Lemma 3.5], this has a solution which provides a stratum preserving homotopy from qm to xr. It is clear that this homotopy will also be fiber preserving, and hence it induces an isomorphism of the relative intersection homology groups.

VIII. The map z is induced by inclusion; o is induced by the product of inclusion with the map $cL^{\tilde{T}}_{\widehat{s\cap t}}\to cL^{\tilde{T}}_{\hat{\tau}}$ induced by extending the inclusion of the fiber over a path in t in the fibration $P^{-1}_{\tilde{T}}(t)\to t$; and u and v are induced by strong stratum-preserving fiber homotopy equivalences. For u, the stationary fiber is over $\widehat{s\cap t}$, while for v it is that over $\hat{\tau}$. As in the proof that VI and VII commute, we show that there is a fiber- and stratum-preserving homotopy from vo to zu over the stationary homotopy induced by projection to $s\cap t$ followed by inclusion $s\cap t\hookrightarrow t$. In fact, just as in the previous proof, it suffices to find a stratum-preserving fiber homotopy from $zu(cL^{\tilde{T}}_{\widehat{s\cap t}})$ to $vo(cL^{\tilde{T}}_{\widehat{s\cap t}})$ over $\widehat{s\cap t}$, and then we can again use [8, Lemma 3.5] in the same manner to extend to the desired homotopy. On $cL^{\tilde{T}}_{\widehat{s\cap t}}$, xu acts simply as the inclusion. The map vo acts as follows: Suppose that γ is the path from $\widehat{s\cap t}$ to $\hat{\tau}$ used to define the map on the fiber factor of o, in other words define a stratum-preserving homotopy $h_1 : cL^{\tilde{T}}_{\widehat{s\cap t}}\times I\to P^{-1}_{\tilde{T}}(t)$ over γ extending the inclusion of $cL^{\tilde{T}}_{\widehat{s\cap t}}$ into the stratified fibration $P^{-1}_{\tilde{T}}(t)\to t$; this homotopy extension exists precisely because $P^{-1}_{\tilde{T}}(t)\to t$ is a stratified fibration, and the image of $cL^{\tilde{T}}_{\widehat{s\cap t}}\times 1$ in $cL^{\tilde{T}}_{\hat{\tau}}$ is precisely the image in the fiber factor of the map o. Consider now the homotopy $h_2 : cL^{\tilde{T}}_{\widehat{s\cap t}}\times I\to P^{-1}_{\tilde{T}}(t)$ given by the composition of $\bar{o}\times \mathrm{id}_I$ with the strong stratified trivialization $v: t\times cL^{\tilde{T}}_{\hat{\tau}}\to P^{-1}_{\tilde{T}}(t)$ restricted to $\gamma^{-1}\times cL^{\tilde{T}}_{\hat{\tau}}$ (we use γ^{-1} to indicate the path γ with the opposite orientation). It is clear that adjoining the two homotopies gives a homotopy over the path $\gamma*\gamma^{-1}$ from the inclusion of $cL^{\tilde{T}}_{\widehat{s\cap t}}$ to $vo(cL^{\tilde{T}}_{\widehat{s\cap t}})$. But since $\gamma*\gamma^{-1}$ is nullhomotopic in an unstratified base space, we can use the stratum-preserving versions of the usual homotopy lifting and extension theorems (see [8]) to conclude that the restrictions of xu and vo to $cL^{\tilde{T}}_{\widehat{s\cap t}}$ are fiber homotopic.

Before proceeding to boxes IX-XII, let us be explicit about the definitions of the continuous functions δ_i which are defined from the various subsets of K to $(0,\infty)$. Recall that δ_1 was defined above so that for a point $x\in\partial s''$, $\delta_1(x)<d(x,\tilde{S}-\partial\tilde{S})$. Then α, induced by inclusion, is the inverse of the map that we previously denoted θ. Similarly, we require the other δ_i defined so that for $x\in\partial s''\cap\partial t''$, $\delta_2(x)$, $\delta_3(x)$, and $\delta_4(x)$ are all $<d(x,\tilde{S}\cap\tilde{T}-\tilde{S}\cap\partial\tilde{T})$, and for $x\in\partial t''$, $\delta_5(x)<d(x,\tilde{T}-\partial\tilde{T})$. In fact, it is clear that we can find a single continuous function, δ, defined on $s\cup t$ that satisfies all of these requirements, and we can consider each δ_i as a restriction of δ. We also can and will assume that δ satisfies $\delta(x)<d(x,\partial\tilde{S}-\partial^2\tilde{S})$ for all x in $\partial^2 s''$ and that $\delta(x)<d(x,\tilde{S}-[(\tilde{S}\cap\tilde{T})\cup\partial^2\tilde{S}])$ for $x\in\partial s''\cap t$. That all of the vertical maps in this row are isomorphisms follows as in the proof of Propositions 6.9 and 6.11. This is because the inclusions $\partial s''\cap t\hookrightarrow s\cap t$ and $\partial s''\cap\partial t''\hookrightarrow s\cap\partial t$ are homotopy equivalences. In fact, $\partial s''\cap t$ deformation retracts to $\partial\sigma''\cap t$ while $s\cap t$ deformation retracts to $\sigma\cap t$, $\partial s''\cap\partial t''$ deformation retracts to $\partial\sigma''\cap\partial t''$, and $s\cap\partial t$ deformation retracts to $\sigma\cap\partial t$. Then our claimed homotopy equivalences follow from the clear homotopy equivalences $\partial\sigma''\cap t\hookrightarrow\sigma\cap t$ and $\partial\sigma''\cap\partial t''\hookrightarrow s\cap\partial t$. Hence, it follows that $P^{-1}_{\tilde{S}}(\partial s''\cap t)\hookrightarrow P^{-1}_{\tilde{S}}(s\cap t)$ and $P^{-1}_{\tilde{S}}(\partial s''\cap\partial t'')\hookrightarrow P^{-1}_{\tilde{S}}(\partial s\cap\partial t)$ are stratum-preserving homotopy equivalences, and similarly for $P^{-1}_{\tilde{T}}$ and $P^{-1}_{\tilde{S}\cap\tilde{T}}$. Along with fiberwise shrinkings to $P^{-1}_{\tilde{S},\delta}$, etc., these induce the intersection homology isomorphisms of the pairs.

IX. The pentagon IX should really be viewed as the following composite of rectangles, where each ξ represents a $k-1$ faces of the k-simplex σ, and for each ξ, x and ∂x represent the subsets as defined in II and $\partial x''=\cup\{v\in K''\mid v\subset x,\bar{v}\cap\partial\bar{\xi}\neq\emptyset\}$ (recall also that

$s \cap x = t$ when $\xi = \tau$):

$$\begin{array}{ccccc}
IH_i^{\bar{p}}(P_{\tilde{S}}^{-1}(s,\partial s)) & \xrightarrow{(-1)^{\sigma\tau}\partial_*} & IH_{i-1}^{\bar{p}}(P_{\tilde{S}}^{-1}(\partial s,\partial^2 s)) \cong \oplus_\xi IH_{i-1}^{\bar{p}}(P_{\tilde{S}}^{-1}(x,\partial x)) & \xrightarrow{\text{projection}} & IH_{i-1}^{\bar{p}}(P_{\tilde{S}}^{-1}(s\cap t, s\cap\partial t)) \\
\alpha\uparrow & & \uparrow & & \uparrow\beta \\
IH_i^{\bar{p}}(P_{\tilde{S},\delta}^{-1}(s,\partial s'')) & \xrightarrow{(-1)^{\sigma\tau}\partial_*} & IH_{i-1}^{\bar{p}}(P_{\tilde{S},\delta}^{-1}(\partial s'',\partial^2 s'')) \cong \oplus_\xi IH_{i-1}^{\bar{p}}(P_{\tilde{S},\delta}^{-1}(\partial s''\cap x,\partial s''\cap\partial x'')) & \xleftarrow[\gamma]{\text{projection}} & IH_{i-1}^{\bar{p}}(P_{\tilde{S},\delta}^{-1}(\partial s''\cap t,\partial s''\cap\partial t'')).
\end{array}$$

The left-most horizontal maps are the boundary maps in the long exact sequences of the triples $P_{\tilde{S}}^{-1}(s,\partial s,\partial^2 s)$ and $P_{\tilde{S},\delta}^{-1}(s,\partial s'',\partial^2 s'')$. The first square then commutes by the naturality of homology theories since it is induced by inclusion maps. The isomorphisms of the middle terms to direct sums is apparent from the definitions and excision, and it is clear that the middle vertical map preserves the direct sum decomposition. The commutativity of the right-hand square is thus clear as the rightmost map is simply that induced on the corresponding summand.

X-XII. All maps are induced by inclusions, hence commutativity is immediate.

For XIII-XVI, all maps labeled g_* can be considered the appropriate restrictions of the modified evaluation map $g : M \to X$ of Proposition 4.2, where M is the mapping cylinder of $\text{holink}_s(X,K_j) = \text{holink}_s(X - K^{j-1}, K_j) \to K_j$.

XIII. This quadrangle commutes by the naturality of homology since the diagonal and horizontal maps are $(-1)^{\sigma\tau}$ times the boundary maps of the respective triples $P_{\tilde{S},\delta}^{-1}(s,\partial s'',\partial^2 s'')$ and $(\tilde{S},\partial\tilde{S},\partial^2\tilde{S})$. Note that g is a well-defined map of triples by our choice of δ.

XIV. Recall that $IH_{i-1}^{\bar{p}}(P_{\tilde{S},\delta}^{-1}(\partial s'',\partial^2 s'')) \cong \oplus_\xi IH_{i-1}^{\bar{p}}(P_{\tilde{S},\delta}^{-1}(\partial s''\cap x,\partial s''\cap\partial x''))$, and similarly, using excisions, we can write $IH_{i-1}^{\bar{p}}(\partial\tilde{S},\partial^2\tilde{S})$ as a number of compatible direct sums. Let $\Xi = \cup\{v \in K' \mid \bar{v}\cap\xi \neq \emptyset\}$, $\partial\Xi = \cup\{v \in \Xi \mid \bar{v}\cap\partial\bar{\xi} \neq \emptyset\}$, $\tilde{\Xi} = \{z \in \mathcal{V} \mid r(z,0) \in \Xi\}$, and $\partial\tilde{\Xi} = \{z \in \mathcal{V} \mid r(z,0) \in \partial\Xi\}$ (so if $\xi = \tau$, then $\tilde{\Xi} = \tilde{T}$ and $\partial\tilde{\Xi} = \partial\tilde{T}$). Then $IH_{i-1}^{\bar{p}}(\partial\tilde{S},\partial^2\tilde{S}) \cong \oplus_\xi(\tilde{S}\cap\tilde{X},\tilde{S}\cap\partial\tilde{X}) \overset{\cong}{\hookrightarrow} \oplus_\xi IH_{i-1}^{\bar{p}}((\tilde{S}\cap\tilde{X})\cup\partial^2\tilde{S},\partial^2\tilde{S})$ (note also that $\partial\tilde{S}\cap\tilde{X} = \tilde{S}\cap\tilde{X}$ and $\partial\tilde{S}\cap\partial\tilde{X} = \tilde{S}\cap\partial\tilde{X}$). Observe that $g : P_{\tilde{S},\delta}^{-1}(\partial s''\cap t,\partial s''\cap\partial t'')) \to ((\tilde{S}\cap\tilde{X})\cup\partial^2\tilde{S},\partial^2\tilde{S})$ and $g : P_{\tilde{S},\delta}^{-1}(\partial s'',\partial^2 s'')) \to (\partial\tilde{S},\partial^2\tilde{S})$ are well-defined by our choice of δ. Then, since γ and ϵ are inclusions of corresponding summands, it is clear that XIV commutes.

XV. Both horizontal maps are inclusions (the lower one inducing an excision isomorphism), and since inclusion commutes with g, this quadrangle commutes.

XVI. Again, all horizontal maps are inclusions, which commute with g.

XVII. This commutes by naturality of the boundary maps in the long exact sequences of the triples $(\tilde{S},\partial\tilde{S},\partial^2\tilde{S})$ and $(\breve{\sigma},\partial\breve{\sigma},\partial^2\breve{\sigma})$. Both horizontal maps are $(-1)^{\sigma\tau}$ times the respective boundary map ∂_*. The vertical maps are induced by inclusion.

XVIII. The vertical maps are inclusions while the horizontal maps are projections onto or inclusions of direct summands. Since the left-hand vertical maps preserves summands and the right-hand map is associated to corresponding summands, the square commutes.

XIX. This diagram commutes because all maps are induced by inclusions. N.B. the bottom map is an excision isomorphism.

We have thus shown that all of the subdiagrams commute. We are now free to reverse, if necessary, any of the arrows representing maps that are isomorphisms. This includes specifically all arrows on the far right and left, the bottom-rightmost horizontal arrow, and, since the first four rows of vertical arrows are all isomorphisms, so are all horizontal arrows in the top five rows. That the outer rectangle of the diagram commutes now follows from some elementary graphic manipulation.

□

Next, we will show how to construct the coefficient homomorphisms $\phi_{\sigma\tau} : IH_j^{\bar{p}}(L_{\hat{\sigma}}) \to IH_j^{\bar{p}}(L_{\hat{\tau}})$, where σ is any open simplex of K and τ is an open face of $\bar{\sigma}$. Again, let $\check{\sigma}$ denote the open first derived neighborhood of σ in the barycentric subdivision K'. In other words, $\check{\sigma}$ is the union of all open simplices ξ of K' such that $\bar{\xi} \cap \bar{\sigma} \neq \emptyset$. Let $\partial\check{\sigma}$ be the union of the $\check{\xi}$ for all ξ in $\partial\bar{\sigma}$. Let $\breve{\sigma} = \{x \in \mathcal{V} \mid r(x,0) \in \check{\sigma}\}$ and $\partial\breve{\sigma} = \{x \in \mathcal{V} \mid r(x,0) \in \partial\check{\sigma}\}$.

If σ has dimension $k = i - j$, recall that we have constructed in Propositions 6.9 and 6.11 isomorphisms $\Phi_\sigma : IH_i^{\bar{p}}(\breve{\sigma}, \partial\breve{\sigma}) \cong IH_j^{\bar{p}}(cL_{\hat{\sigma}})$. We will regard these isomorphisms as canonical and show how they induce a consistent set of maps $\phi_{\sigma\tau}$.

First, consider the case where σ is a simplex of dimension k and τ is a face of σ of dimension $k-1$. From the long exact sequence of the triple, there is a boundary map $\partial_* : IH_i^{\bar{p}}(\breve{\sigma}, \partial\breve{\sigma}) \to IH_{i-1}^{\bar{p}}(\partial\breve{\sigma}, \partial^2\breve{\sigma})$, where $\partial^2\breve{\sigma}$ is defined, as above, as $\cup\{\breve{\xi} \mid \bar{\xi}$ is a $k-2$ face of $\bar{\sigma}\}$. The group $IH_{i-1}^{\bar{p}}(\partial\breve{\sigma}, \partial^2\breve{\sigma})$ is isomorphic to the direct sum $\oplus_\zeta IH_{i-1}^{\bar{p}}(\breve{\zeta}, \partial\breve{\zeta})$, where the sum is taken over $k-1$ faces ζ of σ. We can then compose this boundary morphism with the projection π_τ to the summand $IH_{i-1}^{\bar{p}}(\breve{\tau}, \partial\breve{\tau})$. Finally, define $(-1)^{\sigma\tau}$ to be the sign that $\bar{\tau}$ takes in the boundary formula for $\bar{\sigma}$ treated as an elementary k-chain (with $\mathbb{Z}$ coefficients). This sign is determined by our fixed choice of orientation for each simplex in K.

We can now define the homomorphism $\phi_{\sigma\tau}$ to be the homomorphism for which the following diagram commutes:

$$\begin{array}{ccc} IH_i^{\bar{p}}(\breve{\sigma}, \partial\breve{\sigma}) & \xrightarrow{(-1)^{\sigma\tau}\pi_\tau\partial_*} & IH_i^{\bar{p}}(\breve{\tau}, \partial\breve{\tau}) \\ {\scriptstyle\Phi_\sigma}\downarrow{\scriptstyle\cong} & & {\scriptstyle\Phi_\tau}\downarrow{\scriptstyle\cong} \\ IH_j^{\bar{p}}(cL_{\hat{\sigma}}) & \xrightarrow{\phi_{\sigma\tau}} & IH_j^{\bar{p}}(cL_{\hat{\tau}}). \end{array} \tag{6}$$

This implies that if $x \in IH_j^{\bar{p}}(cL_{\hat{\sigma}})$, then

$$\partial_*\Phi_\sigma^{-1}(x) = \sum_\xi (-1)^{\sigma\tau}\Phi_\tau^{-1}\phi_{\sigma\tau}(x), \tag{7}$$

where the sum is taken over all $k-1$ faces ξ of $\bar{\sigma}$. So, in particular, the correspondence of signs with faces in this boundary formula behaves just as for the correspondence in the chain complex $C_*(K)$.

Notice that if σ and τ lie in the same stratum of X, then by Lemma 7.1, $\phi_{\sigma\tau} = \psi_{\sigma\tau}$.

To define $\phi_{\sigma\tau}$ when $\dim(\tau) = l < k-1$, let $\bar{\sigma} > \bar{\xi}^{k-1} > \bar{\xi}^{k-2} > \cdots > \bar{\xi}^{l+1} > \bar{\tau}$, where the $\bar{\xi}$ form a chain of faces of $\bar{\sigma}$ from $\bar{\sigma}$ to $\bar{\tau}$, one of each dimension between k and l. We set $\phi_{\sigma\tau} = \phi_{\xi^{l+1}\tau}\phi_{\xi^{l+1}\xi^{l+1}}\cdots\phi_{\xi^{k-1}\xi^{k-2}}\phi_{\sigma\xi^{k-1}}$.

Lemma 7.2. *The homomorphism $\phi_{\sigma\tau}$ does not depend on the choice of the chain of $\bar{\xi}$'s.*

Proof. We induct on the length of chain $k-l$. If $k-l=1$, then $\phi_{\sigma\tau}$ is given by our basic definition; there is no ambiguity of chain. Next suppose $k-l=2$ so that we must compare two chains $\bar{\sigma} > \bar{\xi} > \bar{\tau}$ and $\bar{\sigma} > \bar{\zeta} > \bar{\tau}$, where $\bar{\xi}$ and $\bar{\zeta}$ are distinct simplices of dimension $k-1 = l+1$. In fact, there are only two such possibilities: if $\bar{\sigma}$ is a simplex represented by $[v_0, \ldots, v_k]$ for vertices v_i, let us assume, without loss of generality, that $\bar{\tau} = \pm[v_0, \ldots, v_{k-2}]$ (where the sign is chosen to conform with the fixed orientations in K). Then the only possibilities for the middle simplex in the chain are $\bar{\xi} = \pm[v_0, \ldots, v_{k-2}, v_{k-1}]$ and $\bar{\zeta} = \pm[v_0, \ldots, v_{k-2}, v_k]$.

Now, using equation (7) and the fact that $\bar{\xi}$ and $\bar{\zeta}$ are the only $k-1$ faces of σ that have $\bar{\tau}$ as a face, we see that the summand corresponding to $\bar{\tau}$ under the map $\partial_*\partial_*$ of any element $x \in IH_i^{\bar{p}}(\breve{\sigma}, \partial\breve{\sigma})$ is given by $0 = \Phi_\tau^{-1}[(-1)^{\sigma\xi}(-1)^{\xi\tau}\phi_{\xi\tau}\phi_{\sigma\xi} + (-1)^{\sigma\zeta}(-1)^{\zeta\tau}\phi_{\zeta\tau}\phi_{\sigma\zeta}](x)$. But since ∂^2 is always 0 for chain complexes, it follows that $(-1)^{\sigma\xi}(-1)^{\xi\tau}$ and $(-1)^{\sigma\zeta}(-1)^{\zeta\tau}$ must have opposite signs. Therefore, we conclude that $\phi_{\xi\tau}\phi_{\sigma\xi} = \phi_{\zeta\tau}\phi_{\sigma\zeta}$.

Next suppose that $k-l = m > 2$ and that we have shown independence of choice of $\bar{\xi}$'s for all $k-l < m$. Again let $\bar{\sigma} = [v_0, \ldots, v_k]$, and, without loss of generality, let $\tau = \pm[v_0, \ldots, v_l]$. Let us consider two different chains $\bar{\sigma} > \bar{\xi}^{k-1} > \bar{\xi}^{k-2} > \cdots > \bar{\xi}^{l+1} > \bar{\tau}$ and $\bar{\sigma} > \bar{\zeta}^{k-1} > \bar{\zeta}^{k-2} > \cdots > \bar{\zeta}^{l+1} > \bar{\tau}$. The corresponding homomorphisms are $\phi_{\xi^{l+1}\tau}\phi_{\xi^{l+2}\xi^{l+1}}\cdots\phi_{\xi^{k-1}\xi^{k-2}}\phi_{\sigma\xi^{k-1}}$ and $\phi_{\zeta^{l+1}\tau}\phi_{\zeta^{l+2}\zeta^{l+1}}\cdots\phi_{\zeta^{k-1}\zeta^{k-2}}\phi_{\sigma\zeta^{k-1}}$. If $\bar{\xi}^{l+1} = \bar{\zeta}^{l+1}$, we can conclude using the induction assumption since $\phi_{\sigma\xi^{l+1}} = \phi_{\sigma\zeta^{l+1}}$ is independent of the chains used. So assume, without loss of generality, $\bar{\xi}^{l+1} = \pm[v_0, \ldots, v_l, v_r]$ and $\bar{\zeta}^{l+1} = \pm[v_0, \ldots, v_l, v_s]$ for some $r < s$. Then there is a unique $l+2$ face $\bar{\eta} = \pm[v_0, \ldots, v_l, v_r, v_s]$ that has both $\bar{\xi}^{l+1}$ and $\bar{\zeta}^{l+1}$ as faces. Since $\phi_{\sigma\zeta^{l+1}}$ and $\phi_{\sigma\xi^{l+1}}$ are independent of chain by the induction assumption, we can assume that $\phi_{\sigma\xi^{l+1}} = \phi_{\eta\xi^{l+1}}\phi_{\sigma\eta}$ and $\phi_{\sigma\zeta^{l+1}} = \phi_{\eta\zeta^{l+1}}\phi_{\sigma\eta}$. Then $\phi_{\xi^{l+1}\tau}\phi_{\xi^{l+2}\xi^{l+1}}\cdots\phi_{\xi^{k-1}\xi^{k-2}}\phi_{\sigma\xi^{k-1}} = \phi_{\xi^{l+1}\tau}\phi_{\sigma\xi^{l+1}} = \phi_{\xi^{l+1}\tau}\phi_{\eta\xi^{l+1}}\phi_{\sigma\eta}$, and similarly, the map using $\bar{\zeta}$'s is $\phi_{\zeta^{l+1}\tau}\phi_{\eta\zeta^{l+1}}\phi_{\sigma\eta}$. But by our base step of $k-l=2$, we know that $\phi_{\zeta^{l+1}\tau}\phi_{\eta\zeta^{l+1}} = \phi_{\xi^{l+1}\tau}\phi_{\eta\xi^{l+1}}$. Hence we can conclude that $\phi_{\sigma\tau}$ is independent of the choice of chain of $\bar{\xi}$'s. □

Combining this definition of $\phi_{\sigma\tau}$ with Lemma 7.1 above, it is clear that this construction provides a well-defined PL-stratified system of groups over K with the given triangulation.

Next, we show that we have a PL-stratified system of coefficients on the space $|K|$.

For two simplices ξ and ζ, possibly in different triangulations but lying within in the same stratum and within the same closed simplex of some triangulation, we continue to let $\psi_{\xi\zeta} : IH_*^{\bar{p}}(cL_{\hat{\xi}}) \to IH_*^{\bar{p}}(cL_{\hat{\zeta}})$ denote the map on intersection homology induced by using the stratified lifting property to extend the the inclusion of the fiber $cL_{\hat{\xi}}$ to a homotopy over a path from $\hat{\xi}$ to $\hat{\zeta}$ which lies entirely in one stratum and within the containing closed simplex.

Proposition 7.3. *Let L be a subdivision of K. Let σ_K be a k-simplex of K whose interior lies in the jth stratum K_j, and let σ_L be a k-simplex of L such that $\sigma_L \subset \sigma_K$. Suppose also that there is a $k-1$ face τ_L of σ_L that lies within a $k-1$ face τ_K of σ_K and that the interior of both τ_k and τ_L lie in a lower stratum K_l, $l < j$. Then $\phi_{\sigma_L\tau_L}\psi_{\sigma_K\sigma_L} = \psi_{\tau_K\tau_L}\phi_{\sigma_K\tau_K}$.*

Proof. Choosing a dimension for our system of local-coefficients, both maps are homomorphisms from $IH_{i-k}^{\bar{p}}(cL_{\hat{\sigma}_K})$ to $IH_{i-k}^{\bar{p}}(cL_{\hat{\tau}_L})$, where $L_{\hat{\sigma}_K}$ is the fiber over $\hat{\sigma}_K$ of the stratified fibration $\text{holink}_s(X, K_j) \to K_j$ and $L_{\hat{\sigma}_K}$ is the fiber over $\hat{\tau}_L$ of the stratified fibration $\text{holink}_s(X, K_l) \to K_l$ (here we recall that we can think of the filtration K^* as a filtration of spaces apart from its association with the simplicial complex K; the simplicial complex L is compatible with this filtration, and we do not need to bother with defining a filtration L^*). We also recall that the maps ϕ are defined by appealing to the maps Φ and applying the boundary maps of triples (see (6)), while the maps ψ are induced in the bundles of coefficients by the fiber maps induced by lifts over appropriate paths.

Without loss of generality, we will assume throughout the proof that σ_K and σ_L have consistent orientations and similarly for τ_K and τ_L. We are free to choose the orientations this way. We also use subscripts K and L to indicate in respect to which triangulation a set is defined.

So let $[\zeta]$ be an element of $IH^{\bar{p}}_{i-k}(cL_{\hat{\sigma}_K})$. We choose a chain ζ to represent $[\zeta]$.

Now let us find a useful chain that represents $\Phi^{-1}_{\sigma_K}$ in $IH^{\bar{p}}_i(\breve{\sigma}_K, \partial\breve{\sigma}_K)$. Once again recall that $\Phi^{-1}_{\sigma_K}$ is defined by the following sequence of isomorphisms:

$$\begin{aligned} IH^{\bar{p}}_{i-k}(cL_{\hat{\sigma}_K}) &\cong IH^{\bar{p}}_{i-k}(cL^{\tilde{S}}_{\hat{\sigma}_K}) && \text{stratified h.e.} \quad (8)\\ &\cong IH^{\bar{p}}_i((s_K, \partial s_K)\times cL^{\tilde{S}_K}_{\hat{\sigma}_K}) && \text{Künneth theorem} \quad (9)\\ &\cong IH^{\bar{p}}_i(P^{-1}_{\tilde{S}_K}(s_K), P^{-1}_{\tilde{S}_K}(\partial s_K)) && \text{strong stratified fiber h.e.}\\ &\cong IH^{\bar{p}}_i(P^{-1}_{\tilde{S}_K,\delta}(s_K), P^{-1}_{\tilde{S}_K,\delta}(\partial s''_K)) && \text{the map } \theta\\ &\cong IH^{\bar{p}}_i(\tilde{S}_K, \partial\tilde{S}) && \text{the map } g_*\\ &\cong IH^{\bar{p}}_i(\breve{\sigma}_K, \partial\breve{\sigma}_K) && \text{excision.} \end{aligned}$$

Since the stratum-preserving homotopy equivalence of the first isomorphism is induced by inclusion, we can assume that ζ lies in $cL^{\tilde{S}_K}_{\hat{\sigma}_K}$.

To find a representative after the next isomorphism, let us consider the following generator for $IH^{\bar{p}}_i(s_K, \partial s_K) \cong H_i(s_K, \partial s_K) \cong H_i(\sigma_K, \partial\sigma_K)$, by homotopy equivalence of these pairs which all lie in the same stratum. As a representative singular simplex generator, let us consider a linear k-simplex η_K that is similar to σ_K but slightly smaller. We can locate η_K with faces parallel to those of σ_K and nearly (but not necessarily exactly) concentric to σ_K. We place η_K in such a way that its boundary is close enough to the the boundary of $\bar{\sigma}_K$ so that $\partial\eta_K$ lies within $\partial s''_K$ and so that $\partial\eta_K \cap \sigma_L \subset \partial s''_L$. We also require that the boundary of the $k-1$ face of η_K parallel to τ_K has its boundary contained in $\partial\tilde{T}''_K$. Here s_K and s_L are the respective s subcomplexes in appropriate subdivisions of K and L and similarly for $\partial s''_K$, etc. Furthermore, we choose η_K close enough to σ_K so that for each $k-2$ simplex μ of η_K, μ lies within $\breve{\xi}$, where ξ is the corresponding $k-2$ simplex of σ_K. We also can assume that η_K is oriented consistently with σ_K. At this point, let us also define η_L to be the analogue of η_K in σ_L, i.e. a k-simplex nearly concentric to and similar to σ_L, shrunken in slightly but so that $\partial\eta_L \subset \partial s''_L$ and so that it satisfies a condition on the $k-2$ simplices analogous to that for η_K and also analogous conditions for the boundary of the facet parallel to τ_L. We will also assume (by slightly readjusting η_K if necessary) that $\eta_L \subset \eta_K$ and that the $k-1$ face of η parallel to τ_L is a subset of the $k-1$ face of η_K that lies parallel to τ_K.

Now, a representation of $[\zeta]$ after the first two isomorphisms of (8) is given by $\eta_K \times \zeta$. To progress to a chain representing an element of $IH^{\bar{p}}_i(P^{-1}_{\tilde{S}_K,\delta}(s_K), P^{-1}_{\tilde{S}_K,\delta}(\partial s''_K))$, it suffices to apply the inverse of the strong stratified trivialization to $\eta_K \times \zeta$ and then shrink fiberwise to obtain a chain in the appropriate $\text{holink}^{\delta}_s(\tilde{S}_K, s_K)$. Since $\partial\eta_K \subset \partial s''_K$, this shrinking is all that is required. We are free to choose a δ with the following convenient properties:

1. For $x \in \partial s''_K$, $\delta(x) < d(x, \tilde{S}_K - \partial\tilde{S}_K)$.

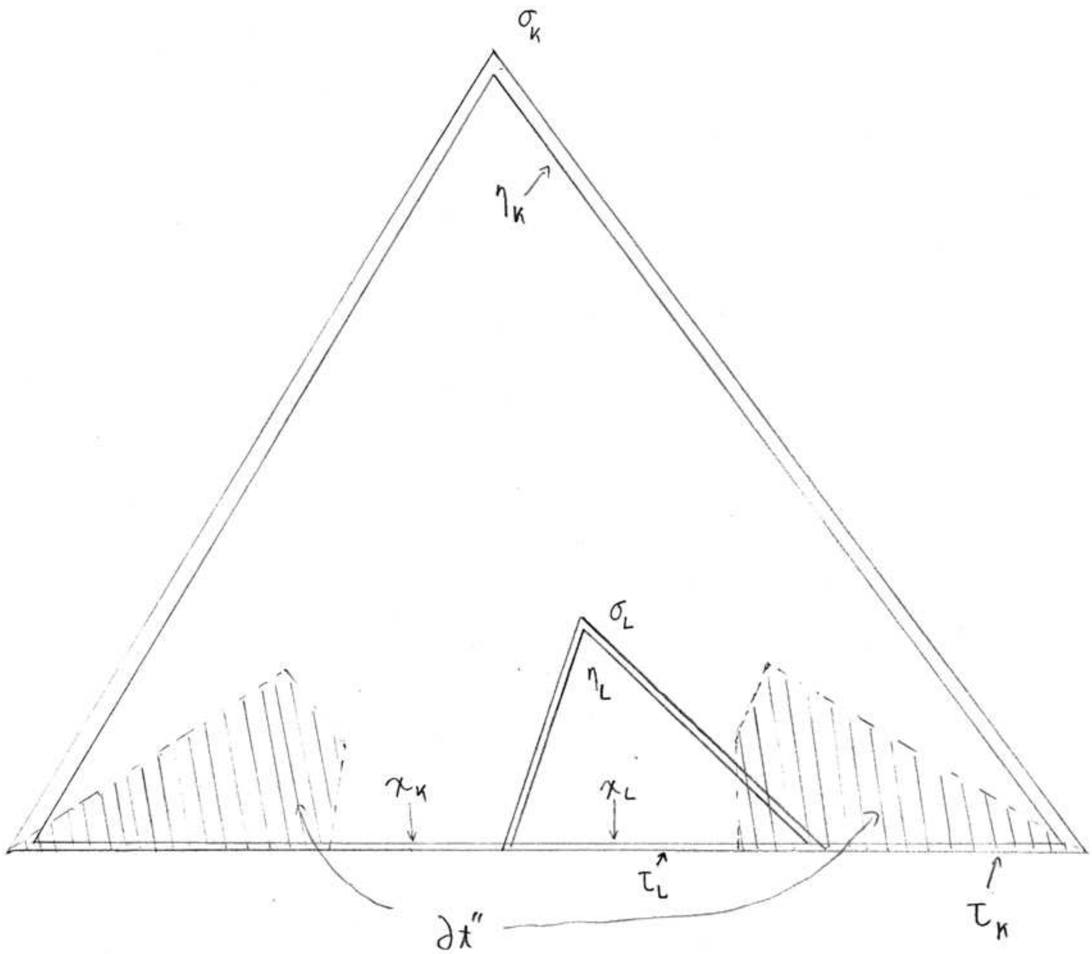


Figure 4: Diagram showing η_k and η_L, illustrating some of the imposed conditions

2. For $x \in \partial s''_L$, $\delta(x) < d(x, \tilde{S}_L - \partial \tilde{S}_L)$.

3. If $x \in \eta_L$, then $\delta(x) < d(x, X - \tilde{S}_L)$ (note that η_L is compact in the open set $\tilde{S}_L$).

4. If μ_K is the $k-1$ face of η_K parallel to the $k-1$ face ξ_K of σ_K and $x \in \mu_K$, then $\delta(x) < d(x, X - \breve{\xi}_K)$ (note that we have chosen η_K close enough to σ_K so that $\mu_K \subset \breve{\xi}_K$, and similar considerations hold for the following items).

5. If μ_L is the $k-1$ face of η_L parallel to the $k-1$ face ξ_L of σ_L and $x \in \mu_L$, then $\delta(x) < d(x, X - \breve{\xi}_L)$.

6. If μ_K is the $k-2$ face of η_K corresponding to the $k-2$ face ξ of σ_K and $x \in \mu_K$, then $\delta(x) < d(x, X - \breve{\xi}_K)$.

7. If μ_L is the $k-2$ face of η_L corresponding to the $k-2$ face ξ of σ_L and $x \in \mu_L$, then $\delta(x) < d(x, X - \breve{\xi}_L)$.

Let us denote the image of $\eta_K \times \zeta$ in $P^{-1}_{\tilde{S}_K,\delta}(s_K)$ by $\lambda(\eta_k \times \zeta)$. Then finally $g\lambda(\eta_K \times \zeta)$ gives us a chain representing $\Phi^{-1}_{\sigma_K}([\zeta])$ (since the last isomorphism in (8) is an excision).

We observe that the restriction of $g\lambda$ to $\eta_L \times \zeta$ gives a representative chain for $\Phi^{-1}_{\sigma_L}(\psi_{\sigma_K\sigma_L}(\zeta))$: It suffices to show that $\Phi_{\sigma_L}(g_*\lambda_*[\eta_L \times \zeta])$ is homologous to $[\zeta_1] \in IH^{\bar{p}}_{i-k}(cL_{\hat{\sigma}_L})$. So let us reverse the order of the isomorphisms in the equivalent version of (8) for L. We can first peel away g_*, since the restriction of λ to $\eta_L \times \zeta$ yields a chain in $IH^{\bar{p}}_i(P^{-1}_{\tilde{S}_L,\delta}(s_L), P^{-1}_{\tilde{S}_L,\delta}(\partial s''_L))$, by our choice of δ. This chain pushes into $IH^{\bar{p}}_i(P^{-1}_{\tilde{S}_L}(s_L), P^{-1}_{\tilde{S}_L}(\partial s_L))$ by inclusion. Note that, at this stage, the chain under consideration still behaves nicely with respect to fibers, i.e. λ maps $\eta_L \times \zeta$ fiber-wise over η_L. Next, we reverse the strong stratum-preserving trivialization over s_L to obtain a chain in $IH^{\bar{p}}_i((s_L, \partial s_L) \times cL^{\tilde{S}_L}_{\hat{\sigma}_L})$. Since this is also a strong fiber-preserving map, this chain must be stratum-preserving fiber homotopic to $\eta_L \times \zeta_2$, where ζ_2 is the image of $\hat{\sigma}_L \times \zeta$ under λ. This follows from the stratified versions of the properties of fiber-preserving maps of fibrations and since $\hat{\sigma}_L$ is an NDR of η_L (see [8]). So, we see that ζ_2 represents $\Phi_{\sigma_L}(g_*\lambda_*[\eta_L \times \zeta])$ in $IH^{\bar{p}}_{i-k}(cL_{\hat{\sigma}_L})$, and we must see that it represents $\psi_{\sigma_K\tau_K}([\zeta])$. But $\zeta_2 \subset cL_{\hat{\sigma}_L}$ is the result of a fiberwise shrinking of the image of $\hat{\sigma}_L \times \zeta$ under the strong stratum-preserving homotopy equivalence from $s_K \times cL^{\tilde{S}_K}_{\hat{\sigma}_K}$ to $P^{-1}_{\tilde{S}_K}(s_K)$. The restriction of this homotopy equivalence to the product of ζ with a path in σ_K from $\hat{\sigma}_K$ to $\hat{\sigma}_L$ gives a homotopy H in $P^{-1}_{\tilde{S}_K}(s_K)$ from $\hat{\sigma}_K \times \zeta$ to a chain in $cL_{\hat{\sigma}_K}$ that is stratum-preserving homotopic to ζ_2. But by inclusion, these can be considered as homotopies in the mapping cylinder of the stratified fibration $\text{holink}_s(X, K_j) \to K_j$, in which case by definition the chain on the other end of the homotopy H from $\hat{\sigma}_K \times \zeta$ represents $\psi_{\sigma_K\sigma_L}([\zeta])$. And again, this is stratum-preserving homotopy equivalent to ζ_2 within the fiber. Hence, we can conclude that indeed the restriction of $g\lambda$ to $\eta_L \times \zeta$ gives a representative chain for $\Phi^{-1}_{\sigma_L}(\psi_{\sigma_K\sigma_L}(\zeta))$.

Next, let χ_K be the $k-1$ face of η_K parallel to τ_K, and define χ_L similarly for η_L and τ_L. Consider the restriction of $g\lambda$ to the product $\chi_L \times \zeta$. From the definitions and our choice of δ, this chain represents the projection of $\partial_*\Phi^{-1}_{\sigma_K}([\zeta])$ to the summand corresponding to τ_K of $IH^{\bar{p}}_{i-1}(\partial\breve{\sigma}_K, \partial^2\breve{\sigma}_K) \cong \oplus_{\xi_K} IH^{\bar{p}}_{i-1}(\breve{\xi}_K, \partial\breve{\xi}_K)$ (where again the sum is taken over $k-1$ faces ξ_K of σ_K). So $\Phi_{\tau_K}(g_*\lambda_*[\chi_K \times \zeta])$ is $(-1)^{\sigma_K\tau_K}\phi_{\sigma_K\tau_K}([\zeta])$ by (6). And similarly, $\Phi_{\tau_L}(g_*\lambda_*[\chi_L \times \zeta]) = (-1)^{\sigma_L\tau_L}\phi_{\sigma_L\tau_L}\psi_{\sigma_K\sigma_L}([\zeta])$. Hence, since $(-1)^{\sigma_K\tau_K} = (-1)^{\sigma_L\tau_L}$ by our choice of orientations, it remains only to show that $\Phi_{\tau_L}(g_*\lambda_*[\chi_L \times \zeta])$ is also equal to $(-1)^{\sigma_L\tau_L}\psi_{\tau_K\tau_L}\phi_{\sigma_K\tau_K}([\zeta]) = \psi_{\tau_K\tau_L}\Phi_{\tau_L}(g_*\lambda_*[\chi_L \times \zeta])$. So it suffices to show that the chain representing $g_*\lambda_*[\chi_L \times \zeta]$ also represents $\Phi^{-1}_{\tau_L}((-1)^{\sigma_L\tau_L}\psi_{\tau_K\tau_L}\phi_{\sigma_K\tau_K}([\zeta]))$. So, essentially, we must study the relationship between $g_*\lambda_*(\chi_K \times \zeta)$ and $g_*\lambda_*(\chi_L \times \zeta)$.

To proceed with this last part of the proof, it will be necessary to further modify both the placements of η_K and η_L and the function δ we used above. However, we will simply be adding restrictions which we could have imposed earlier, so there is no difficulty and the first part of the proof remains valid. First, we can ensure, by repositioning η_K, that the points in $\partial\chi_K$ have their retraction paths contained entirely in $\partial\tilde{T}''_K$. The retraction paths we are referring to are those which retract $\tilde{T}_K$ to t_K as above. This requirement can be obtained by creating η_K so that $\partial\chi_K$ is sufficiently close to τ_K. Next, we make δ small enough that all points of $g\lambda(\partial\chi_K \times \zeta)$ also have their retraction paths entirely in $\partial\tilde{T}''_K$. This is possible by continuity since by shrinking δ we can make the points in $g\lambda(\partial\chi_K \times \zeta)$ arbitrarily close to the corresponding points of $\partial\chi_K$ in σ_K (after all, each point in $\partial\chi_K \times \zeta$ represents a path whose base point is the corresponding point in $\partial\chi_K$, and by shrinking δ we shrink the paths).

Similarly, we can assure that all points in the image of $\partial\chi_L \times \zeta$ under $g\lambda$ have their $\tilde{T}_L$ to t_L retraction paths contained in $\partial\tilde{T}''_L$. Furthermore, we can even arrange that all points in the image of $\partial\chi_L \times \zeta$ under $g\lambda$ have their $\tilde{T}_K$ to t_K retraction paths contained in $\partial\tilde{T}''_L$ and all points in the image of $\chi_L \times \zeta$ under $g\lambda$ have their $\tilde{T}_K$ to t_K retraction paths contained in $\tilde{T}_L$. Again, we use the fact that both deformation retractions are continuous and that we can place both χ's arbitrarily close to τ_K and τ_L, which are stationary points of the retraction.

Let $f_K : \tilde{T}_K \to P^{-1}_{\tilde{T}_K}(t_K)$ and $f_L : \tilde{T}_L \to P^{-1}_{\tilde{T}_L}(t_L)$ be the maps of Corollary 6.7 corresponding to τ_K and τ_L. With these choices for η's and χ's, we see that $f_K(g\theta(\partial\chi_K \times \zeta))$ lies entirely within $P^{-1}_{\tilde{T}_K}(\partial t''_K)$ (recall that this is the mapping cylinder of the holink evaluation on the set of paths in $\tilde{T}_K$ whose endpoints lie in $\partial t''_K$, which by definition is $\partial T''_K \cap t$). Similarly $f_L(g\lambda(\partial\chi_K \times \zeta))$ lies entirely within $P^{-1}_{\tilde{T}_L}(\partial t''_L)$.

So let us consider $f_K(g\lambda(\chi_K \times \zeta))$, which we can think of as lying in the stratified fibration given by the mapping cylinder of $\text{holink}_s(X, K_l) \to K_l$. In particular, this image lies over t_K, so we can consider the restriction of the stratified fibration over t_K. Similarly, $f_K(g\lambda(\chi_L \times \zeta))$ lies over t_L. Now we note that $f_K(g\lambda(\chi_K \times \zeta))$ maps a product $(\chi_K \times \zeta)$ into a stratified fibration, and in particular into a fibration over t_K in such a way that $\partial\chi_L \times \zeta$ get mapped into the part of the stratified fibration over $\partial t''_L$. We know that $(t_K, \partial t''_K)$ is homotopy equivalent to (D^{k-1}, S^{k-2}), and it is clear from our constructions and choice of small δ that the projection to $\partial t''_K$ of $\partial\chi_K \times *$ (for some basepoint $*$ of ζ) must be a degree ± 1 map of spheres (the sign depending on the consistency of orientation of σ_K and τ_K). Similarly for $\partial\chi_L \times *$ and $(t_L, \partial t''_L)$. It then follows from the standard theorem for fibrations and their stratified analogues [8] that $f_K(g\lambda(\chi_K \times \zeta))$ is stratum-preserving homotopic, by a homotopy H, to a stratum-preserving map that also preserves fibers (so that for each $x \in \chi_K$, $x \times \zeta$ gets mapped into a single fiber of the stratified fibration). Furthermore, we can arrange this to be a small homotopy in such a way that the image of $\chi_K \times \zeta \times I$ remains in $P^{-1}_{\tilde{T}_K}(t_K)$ and the image of $\partial\chi_K \times \zeta \times I$ remains in $P^{-1}_{\tilde{T}_K}(\partial t''_K)$. We can in fact do more: we can arrange for the restriction of the homotopy to $\chi_L \times \zeta$ to remain in $P^{-1}_{\tilde{T}_L}(t_L)$ and the restriction to $\partial\chi_L \times \zeta$ to stay in $P^{-1}_{\tilde{T}_L}(\partial t''_L)$. Let us call the end result of this homotopy $h(\chi_K \times \zeta)$. This can all be done by letting H be the lifting of a suitably chosen homotopy on the base. Then we can begin by lifting to a homotopy of $f_K(g\lambda(\chi_L \times \zeta))$ in the stratified fibration $P^{-1}_{\tilde{T}_L}(t_L)$. This homotopy can then be extended to all of $f_K(g\lambda(\chi_K \times \zeta))$ in the stratified fibration $P^{-1}_{\tilde{T}_K}(t_K)$ by stratified lifting extension (see [8]).

Now $h(\chi_K \times \eta)$ is a relative cycle in $IH^{\bar{p}}_{i-1}(P^{-1}_{\tilde{T}_K}(t_K), P^{-1}_{\tilde{T}_K}(\partial t''_K))$. Appropriately shrinking and applying $g_{\tau *}$ (where $g_\tau : P^{-1}_{\tilde{T}_K}(t_K) \to \tilde{T}_K$ is the stratum-preserving homotopy inverse to f_K) will give us back a chain intersection homologous to $g\lambda(\chi_K \times \zeta)$ in $(\tilde{T}_K, \partial\tilde{T}_K)$, since g_τ and f_K are stratum-preserving homotopy inverses and we can apply g_τ to (a shrinking of) the entire homotopy H to realize this homology. On the other hand, by applying to $h(\chi_K \times \zeta)$ strong trivialization and the Künneth theorem (the remaining steps in Φ_{τ_K}), we return to the element of $IH^{\bar{p}}_{i-k}(cL_{\hat{\tau}_K})$ that represents $(-1)^{\sigma_K\tau_K}\phi_{\sigma_K\tau_K}([\zeta])$. But then just as in the earlier part of our proof (the seventh paragraph, to be exact) g_τ of an appropriate fiber-shrinking of $h(\chi_L \times \zeta)$ must represent $\Phi^{-1}_{\tau_L}[(-1)^{\sigma_L\tau_L}\psi_{\tau_K\tau_L}\phi_{\sigma_K\tau_K}([\zeta])]$. On the other hand, we already know that $\Phi_{\tau_L}(g_*\lambda_*[\chi_L \times \zeta]) = (-1)^{\sigma_L\tau_L}\phi_{\sigma_L\tau_L}\psi_{\sigma_K\sigma_L}([\zeta])$, so it only remains to show that $g\lambda[\chi_L \times \zeta]$ is intersection homologous to g_τ of (an appropriate shrinking) of $h(\chi_L \times \zeta)$. We have already observed this fact with K's in place of L's. So now we just restrict, having observed that there is no trouble in choosing our homotopy H relative its starting points so that its image on $(\chi_L \times \zeta, \partial\chi_L \times \zeta) \times I$ lies entirely in the preimage under g_τ of the pair $(\tilde{T}_L), \partial\tilde{T}_L)$.

This complete the proof. □

8 Completion of the proof of the main theorem

We are now equipped with all the tools necessary to complete the proof of Theorem 5.1.

Proof. We first consider the spectral sequence for $IH_*^{\bar{p}}(\mathcal{V};\mathcal{G})$.

Let ρ stand for the projection $r(\cdot,0):\mathcal{V}\to K$. We filter $\mathcal{V}$ by open sets. It is unfortunate that we deal with two distinct concepts commonly referred to as filtrations. As our purpose is to construct spectral sequences, we will briefly refer to this filtration by open sets as an *SS-filtration.* We will also use the term *simplicial skeleton* to refer to the skeleton of a polyhedron in the standard sense of skeleta of simplicial complexes. Consider the triangulation of K, and let B^j be the neighborhood of the simplicial s-skeleton, $K(s)$, of K given by the union of the open stars in K' of the simplices in $K'\cap|K(s)|$. In other words, B^j is the union of the sets S associated to the simplices σ in K of dimensions $\leq j$. These sets provide an SS-filtration of K and induces an SS-filtration on $\mathcal{V}$ by $J^s=\rho^{-1}(B^j)$:

$$\begin{array}{ccccccccc}
\mathcal{V}=J\supset\cdots\supset & J^j & \supset & J^{s-1}\supset\cdots\supset & J^0 & \supset & \emptyset \\
\big\downarrow & \big\downarrow & & \big\downarrow & \big\downarrow & & \big\downarrow \\
K=B\supset\cdots\supset & B^s & \supset & B^{s-1}\supset\cdots\supset & B^0 & \supset & \emptyset.
\end{array}$$

These are both SS-filtrations by open sets, and thus we can define the singular intersection chain complexes $IC_i^{\bar{p}}(J^s;\mathcal{G})$ and filter $IC_i^{\bar{p}}(\mathcal{V};\mathcal{G})$ by $F_sIC_*^{\bar{p}}(\mathcal{V};\mathcal{G})=\operatorname{im}(IC_*^{\bar{p}}(J^s;\mathcal{G})\to IC_*^{\bar{p}}(\mathcal{V};\mathcal{G}))$. This is obviously an increasing and exhaustive SS-filtration. The general theory of spectral sequences of SS-filtrations (see [17]) then provides a first quadrant spectral sequence with E^1 term

$$\begin{aligned}
E^1_{p,q}&=H_{p+q}(F_pIC_*(\mathcal{V};\mathcal{G})/F_{p-1}IC_*(\mathcal{V};\mathcal{G}))\\
&=H_{p+q}(IC_*(J^p,J^{p-1});\mathcal{G})=IH_{p+q}(J^p,J^{p-1};\mathcal{G})
\end{aligned}$$

and boundary map, d^1, given by the boundary map of the long exact intersection homology sequence of the triple (J^p,J^{p-1},J^{p-2}). (Note that the long exact sequence of the triple for intersection homology can be defined just as for the standard case by using quotients of the intersection chain complexes, so long as both subsets are open (see [15]).) The spectral sequence abuts to $H_*(IC_*^{\bar{p}}(\mathcal{V};\mathcal{G}))=IH_*^{\bar{p}}(\mathcal{V};\mathcal{G})$.

It remains to identify the E^2 terms of this spectral sequences. In particular, we need to show that $E^1_{*,q}$ is isomorphic to the chain complexes with the desired PL-stratified system of local coefficients.

Recall the above definitions: σ^i = the interior of an i dimensional simplex of K; $\breve{\sigma} = \cup\{\xi \in K' \mid \bar{\xi} \cap \bar{\sigma} \neq \emptyset\}$; $\partial\breve{\sigma} = \cup\{\xi \in K' \mid \bar{\xi} \cap \partial\bar{\sigma} \neq \emptyset\}$; $\breve{\sigma} = \{x \in \mathcal{V} \mid r(x,0) \in \breve{\sigma}\}$; and $\partial\breve{\sigma} = \{x \in \mathcal{V} \mid r(x,0) \in \partial\breve{\sigma}\}$. We claim that $H_i(IC^{\bar{p}}_*(J^p, J^{p-1}); \mathcal{G}) \cong \oplus_\alpha IH^{\bar{p}}_i(\breve{\sigma}^p_\alpha, \partial\breve{\sigma}^p_\alpha; \mathcal{G})$ for all i, where the sum is taken over all p-simplices in K. The claim is obvious for $p = 0$. For $p > 0$, we proceed by induction on the simplices. In particular, order the simplices of each dimension by placing an ordering on the indices α, and let A^p_j equal the union of the first j of the $\breve{\sigma}^p_\alpha$. We will show by induction that $IH^{\bar{p}}_i(A^p_j, A^p_j \cap J^{p-1}; \mathcal{G}) = \oplus_{\alpha \leq j} IH^{\bar{p}}_i(\breve{\sigma}^p_\alpha, \partial\breve{\sigma}^p_\alpha; \mathcal{G})$ for all j, which will imply the claim since $A^p_j \subset A^p_{j+1}$ for all j, $\cup A^p_j = J^p$, and $J^p \cap J^{p-1} = J^{p-1}$. The formula obviously holds for $j = 1$. For $j > 1$, consider the Mayer-Vietoris sequence ([15])

$$\begin{aligned}
\to IH^{\bar{p}}_i(A^p_j \cap \breve{\sigma}^p_{j+1}, A^p_j \cap J^{p-1} \cap \breve{\sigma}^p_{j+1}; \mathcal{G}) \\
\to IH^{\bar{p}}_i(A^p_j, A^p_j \cap J^{p-1}; \mathcal{G}) \oplus IH^{\bar{p}}_i(\breve{\sigma}^p_{j+1}, \partial\breve{\sigma}^p_{j+1}; \mathcal{G}) \\
\to IH^{\bar{p}}_i(A^p_{j+1}, A^p_{j+1} \cap J^{p-1}; \mathcal{G}) \to .
\end{aligned}$$

Now, we have $A^p_j \cap \breve{\sigma}^p_{j+1} \subset J^{p-1}$: In the base K, the intersection of $\cup^j_{i=0}\bar{\sigma}^p_i$ and $\bar{\sigma}^p_{j+1}$ must lie in a lower dimensional skeleton and the intersection of their neighborhoods will lie in the neighborhood of this lower dimensional skeleton, hence in B^{p-1}. We can then consider the inverse images of these sets under the projection ρ. Therefore, the intersection terms of the Mayer-Vietoris sequence are $IH^{\bar{p}}_i(A^p_j \cap \breve{\sigma}^p_{j+1}, A^p_j \cap J^{p-1} \cap \breve{\sigma}^p_{j+1}; \mathcal{G}) = 0$. This suffices to prove the claim by a possibly infinite induction. Note, by the way, that it is in order to employ such Mayer-Vietoris sequences that we must use the open sets $\breve{\sigma}$ and their inverse images under ρ instead of simply the closed simplices, because Mayer-Vietoris sequences for intersection homology require open sets [15].

But now, by employing the isomorphisms Φ_σ, $\oplus_\alpha IH^{\bar{p}}_i(\breve{\sigma}^p_\alpha, \partial\breve{\sigma}^p_\alpha; \mathcal{G}) \cong \oplus_\alpha IH^{\bar{p}}_{i-p}(cL_{\hat{\sigma}^p_\alpha}; \mathcal{G})$. But this is isomorphic to the dimension p chain group of K with coefficients in $\mathbb{IH}^{\bar{p}}_q(cL; \mathcal{G})$. To see that this in fact induces a chain map to the chain complex with stratified system of coefficients, we need only observe that we have already shown (in fact by our definition of $\phi_{\sigma\tau}$) that the boundary maps of the system of coefficients are compatible with the boundary maps given by long exact sequences of triples $(\breve{\sigma}, \partial\breve{\sigma}, \partial^2\breve{\sigma})$.

This completes the proof for $IH^{\bar{p}}_*(\mathcal{V}; \mathcal{G})$.

For $IH^{\bar{p}}_*(\mathcal{V} - K; \mathcal{G})$, the proof is exactly the same, but in all steps here and in the lemmas and propositions above, we must leave out K. This does not provide any difficulty, however, as the intersections of K with all sets we have considered are pure subsets. Hence all stratum-preserving maps, homotopies, and homotopy equivalence that we have used can be replaced with their restrictions off of K, and these new maps remain well-defined and stratum-preserving. Furthermore, stratum-preserving homotopy equivalences remain so, as no map we have considered interchanges components of strata. For the relevant homotopy links, paths must still be allowed to end in K, but we no longer allow paths completely contained in K. So, for example, $\text{holink}_s(X, K)$ remains the same, but instead of its mapping cylinder, we consider the mapping cylinder minus its base, K. Of course this is just stratum-preserving homotopy equivalent to $\text{holink}_s(X, K)$. For sets like $\text{holink}_s(X, s)$ and $\text{holink}_s(\tilde{S}, s)$, we replace them with $\text{holink}_s((X-K)\cup s, s)$ and $\text{holink}_s((\tilde{S}-(\tilde{S}\cap K))\cup s, s)$.

The mapping cylinder of the holink evaluations of these sets are then respectively stratum-preserving homotopy equivalent to $(X-K)\cup s$ and $[\tilde{S}-(\tilde{S}\cap K)]\cup s$ by restricting the maps of Proposition 4.2. Therefore, $\tilde{S}-(\tilde{S}\cap K)$ is stratum-preserving homotopy equivalent to $\text{holink}_s((X-K)\cup s, s)$ and to $\text{holink}_s((\tilde{S}-K)\cup s, s)$ by removing the base of the mapping cylinder and retracting. Similarly, it follows that if σ has dimension k and lies in the stratum X_l of dimension j, then $IH_i^{\bar{p}}(\breve{\sigma}-\breve{\sigma}\cap K, \partial\breve{\sigma}-\partial\breve{\sigma}\cap K;\mathcal{G})\cong IH_{i-j}^{\bar{p}}(L';\mathcal{G})$, where L' is the fiber of $\text{holink}((X-K)\cup X_l, X_l)\to X_l$ over $\hat{\sigma}$. There is no cone, since again we must cut out the base which lies in K, allowing us to retract back to the original holink fiber.

The last statement of the theorem is clear since all maps are induced by the inclusion of $\mathcal{V}-K$ into $\mathcal{V}$.

□

References

[1] Markus Banagl, Sylvain Cappell, and Julius Shaneson, *Computing twisted signatures and l-classes of stratified spaces*, submitted.

[2] Sylvain Cappell and Julius Shaneson, *The mapping cone and cylinder of a stratified map*, Prospects in topology (Princeton, NJ), Ann. of Math. Stud., vol. 138, Princeton Univ. Press, 1994, pp. 58–66.

[3] Sylvain E. Cappell and Julius L. Shaneson, *Singular spaces, characteristic classes, and intersection homology*, Annals of Mathematics **134** (1991), 325–374.

[4] T. A. Chapman, *Concordances of Hilbert cube manifolds and tubular neighborhoods of finite-dimensional manifolds*, Geometric Topology (James C. Cantrell, ed.), New York: Academic Press, 1970, pp. 581–596.

[5] A. Borel et. al., *Intersection cohomology*, Progress in Mathematics, vol. 50, Birkhauser, Boston, 1984.

[6] Greg Friedman, *Alexander invariants of non-locally-flat knots*, submitted.

[7] ———, *Intersection Alexander polynomials*, submitted.

[8] ———, *Stratified fibrations and the intersection homology of the regular neighborhoods of bottom strata*, submitted.

[9] ———, *Polynomial invariants of non-locally-flat knots*, Ph.D. thesis, New York University, New York, NY, 2001.

[10] Mark Goresky and Robert MacPherson, *Intersection homology theory*, Topology **19** (1980), 135–162.

[11] ———, *Intersection homology II*, Invent. Math. **72** (1983), 77–129.

[12] Bruce Hughes, *Stratifications of teardrops*, Fundamenta Mathematicae **161** (1999), 305–324.

[13] ———, *Stratified path spaces and fibrations*, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), 351–384.

[14] Henry C. King, *Intersection homology and homology manifolds*, Topology **21** (1982), 229–234.

[15] ———, *Topological invariance of intersection homology without sheaves*, Topology Appl. **20** (1985), 149–160.

[16] Jerome Levine, *Polynomial invariants of knots of codimension two*, Ann. of Math **84** (1966), no. 2, 537–554.

[17] John McCleary, *User's guid to spectral sequences*, Publish or Perish, Inc., Wilmington, Delaware, 1985.

[18] James R. Munkres, *Elements of algebraic topology*, Addison-Wesley, Reading, MA, 1984.

[19] Frank Quinn, *Intrinsic skeleta and intersection homology of weakly stratified sets*, Geometry and topology (Athens, GA, 1985), Lecture Notes in Pure and Appl. Math., vol. 105, Dekker, New York, 1987, pp. 225–241.

[20] ———, *Homotopically stratified sets*, J. Amer. Math. Soc. **1** (1988), 441–499.

[21] David A. Stone, *Stratified polyhedra*, Lecture Note in Mathematics, vol. 252, Spring-Verlag, Berlin-Heidelberg-New York, 1972.

[22] Shmuel Weinberger, *The topological classification of stratified spaces*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1994.

[23] George W. Whitehead, *Elements of homotopy theory*, Springer-Verlag, New York, 1978.

[24] Bruce Hughes, Laurence R. Taylor, Shmuel Weinberger, Bruce Williams, *Neighborhoods in stratified spaces with two strata*, Topology **39** (2000), 873–919.