K-Witt bordism in characteristic 2

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Abstract

This note provides a computation of the bordism groups of K-Witt spaces for fields K with characteristic 2. We provide a complete computation for the unoriented bordism groups. For the oriented bordism groups, a nearly complete computation is provided as well a discussion of the difficulty of resolving a remaining ambiguity in dimensions equivalent to $2 \mod 4$. This corrects an error in the $char(K) = 2$ case of the author's prior computation of the bordism groups of K-Witt spaces for an arbitrary field K.

In [1], an n-dimensional K-Witt space, for a field K, is defined[1] to be an oriented compact n-dimensional PL stratified pseudomanifold X satisfying the K-Witt condition that the lower-middle perversity intersection homology group $I^{\bar{m}}H_k(L;K)$ is 0 for each link L^{2k} of each stratum of X of dimension $n-2k-1$, $k>0$. Following the definition of stratified pseudomanifold in [2], X does not possess codimension one strata. Orientability is determined by the orientability of the top (regular) strata. This definition generalizes Siegel's definition in [11] of $\mathbb{Q}$-Witt spaces (called there simply "Witt spaces"). The motivation for this definition is that such spaces possess intersection homology Poincaré duality $I^{\bar{m}}H_i(X;K) \cong \mathrm{Hom}(I^{\bar{m}}H_{n-i}(X;K),K)$.

The author's paper [1] concerns K-Witt spaces and, in particular, a computation of the bordism theory $\Omega_*^{K-\mathrm{Witt}}$ of such spaces. However, there is an error in [1] in the computation of the coefficient groups $\Omega_{4k+2}^{K-\mathrm{Witt}}$ when $\mathrm{char}(K)=2$.

It is claimed in [1] that $\Omega_{4k+2}^{K-\mathrm{Witt}}=0$. When $\mathrm{char}(K)>2$, the null-bordism of a $4k+2$ dimensional K-Witt space X is established in [1] by following Siegel's computation [11] for $\mathbb{Q}$-Witt spaces by first performing a surgery to make the space irreducible and then performing

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[1]There is a minor error in [1] in that Witt spaces are stated to be irreducible, meaning that there is only a single top dimensional stratum. In general, this should not be part of the definition of a K-Witt space; cf. [11]. However, as every K-Witt space of dimension >0 is bordant to an irreducible K-Witt space (see [11, page 1099]), this error does not affect the bordism group computations of [1]. It is not true that every 0-dimensional K-Witt space is bordant to an irreducible K-Witt space, but in this dimension the computations all reduce to the manifold theory and the computations given for this dimension in [1] are also correct if one removes irreducibility from the definition.

a sequence of singular surgeries to obtain a space X' such that $I^{\bar{m}}H_{2k+1}(X';K) = 0$. The K-Witt null-bordism of X is the union of the trace of the surgeries from X to X' with the closed cone $\bar{c}X'$. One performs the singular surgeries on elements $[z] \in I^{\bar{m}}H_{2k+1}(X;K)$ such that $[z]\cdot[z] = 0$, where $\cdot$ denotes the Goresky-MacPherson intersection product [2]. As the intersection product is skew symmetric on $I^{\bar{m}}H_{2k+1}(X;K)$, such a $[z]$ always exists. The error in [1] stems from overlooking that this last fact is not necessarily true in characteristic 2, where skew symmetric forms and symmetric forms are the same thing and so skew-symmetry does not imply $[z]\cdot[z] = 0$.

Corrected computations. To begin to remedy the error of [1], we first observe that it remains true in characteristic 2 that the map[2] $w : \Omega_{4k+2}^{\mathbb{Z}_2-\text{Witt}} \to W(\mathbb{Z}_2)$ is injective, where $W(\mathbb{Z}_2)$ is the Witt group of $\mathbb{Z}_2$ and w takes the bordism class $[X]$ to the class of the intersection form on $I^{\bar{m}}H_{2k+1}(X;\mathbb{Z}_2)$. For $k > 0$, this fact can be proven as it is proven for $w : \Omega_{4j}^{K-\text{Witt}} \to W(K)$, $j > 0$, in [1]: if one assumes that the intersection form on X represents 0 in $W(\mathbb{Z}_2)$ then the intersection form is split, in the language of [7]; see [7, Corollary III.1.6]. And so $I^{\bar{m}}H_{2k+1}(X;\mathbb{Z}_2)$ will possess an isotropic (self-annihilating) element by [7, Lemma I.6.3]. The surgery argument can then proceed[3]. As $W(\mathbb{Z}_2) \cong \mathbb{Z}_2$ (see [7, Lemma IV.1.5]), it follows that $\Omega_{4k+2}^{\mathbb{Z}_2-\text{Witt}}$ is either 0 or $\mathbb{Z}_2$.

This argument does not hold for $4k+2 = 2$ as in this case the dimensions are not sufficient to guarantee that every middle-dimensional intersection homology class is representable by an irreducible element, which is necessary for the surgery argument; see [11, Lemma 2.2]. However, all 2-dimensional Witt spaces must have at worst isolated singularities, and so in particular such a space must have the form $X \cong (\amalg S_i)/\sim$, where the S_i are closed oriented surfaces and the relation $\sim$ glues them together along various isolated points. But then X is bordant to $\amalg S_i$. This can be seen via a sequence of pinch bordisms as defined by Siegel [11, Section II] that pinch together the regular neighborhoods of sets of points of $\amalg S_i$. To see that the bordism is via a Witt space, it is only necessary to observe that the link of the interior cone point in each such pinch bordism will be a wedge of S^2s, and it is easy to compute that $I^{\bar{m}}H_1(\vee_i S^2;K) = 0$ for any K. But now, since all closed oriented[4] surfaces bound, $\Omega_2^{\mathbb{Z}_2-\text{Witt}} = 0$. This special case was also over-looked in [1], though this argument holds for any field K and is consistent with the claim of [1] that $\Omega_2^{K-\text{Witt}} = 0$ for all K.

Thus we have shown that $w : \Omega_{4k+2}^{\mathbb{Z}_2-\text{Witt}} \to W(\mathbb{Z}_2) \cong \mathbb{Z}_2$ is an injection for $k \geq 0$, trivially

[2]Recall from [1, Corollary 4.3] that the bordism groups depend only on the characteristic of the field, so for characteristic 2 it suffices to consider $K = \mathbb{Z}_2$.

[3]There is one other possible complication due to characteristic 2 that must be checked but that does not provide difficulty in the end: For characteristic not equal to 2, every split form is isomorphic to an orthogonal sum of hyperbolic planes [7, Lemma I.6.3], and this appears to be used in the proof of Theorem 4.4 of [11], which is heavily referenced in [1]. For characteristic 2, one can only conclude that a split form is isomorphic to one with matrix $\begin{pmatrix} 0 & I \\ I & A \end{pmatrix}$ for some matrix A. However, a detailed reading of the proof of [11, Theorem 4.4, particularly page 1097] reveals that it is sufficient to have a basis $\{\alpha, \beta, \gamma_1, \dots, \gamma_{2m}\}$ such that $\alpha\cdot\alpha = \alpha\cdot\gamma_i = 0$ for all i and $\alpha\cdot\beta = 1$, and this is certainly provided by a form with the given matrix.

[4]Recall that $\mathbb{Z}_2$-Witt spaces are assumed to be $\mathbb{Z}$-oriented, though see below for more on orientation considerations

so for $k = 0$. Unfortunately, the question of surjectivity of w in dimensions $4k+2$ is more complicated and not yet fully resolved. We can, however, make the following observation: if X is a $\mathbb{Z}_2$-Witt space of dimension $4k-2$, then[5] $w([X \times \mathbb{C}P^2]) = w([X])$. So if there is a non-trivial element of $\Omega_{4k-2}^{\mathbb{Z}_2-\text{Witt}}$, then there is a non-trivial element of $\Omega_{4k+2}^{\mathbb{Z}_2-\text{Witt}}$.

Putting this together with the computations from [1] of $\Omega_*^{K-\text{Witt}}$ in dimension $\not\equiv 4k+2$ mod 4 (which remain correct), we have the following theorem:

Theorem 1. *For a field K with char$(K) = 2$, $\Omega_*^{K-Witt} = \Omega_*^{\mathbb{Z}_2-Witt}$, and for*[6] *$k \geq 0$,*

1. $\Omega_0^{K-Witt} \cong \mathbb{Z}$,
2. *for $k > 0$, $\Omega_{4k}^{K-Witt} \cong \mathbb{Z}_2$, generated by $[\mathbb{C}P^{2k}]$,*
3. $\Omega_{4k+3}^{K-Witt} = \Omega_{4k+1}^{K-Witt} = 0$,
4. *Either*
   (a) $\Omega_{4k+2}^{K-Witt} = 0$ *for all k, or*
   (b) *there exists some $N > 0$ such that $\Omega_{4k+2}^{K-Witt} = 0$ for all $k < N$ and $\Omega_{4k+2}^{K-Witt} \cong \mathbb{Z}_2$ for all $k \geq N$.*

We will provide below some further discussion of the difficulties of deciding which case of (4) holds after discussing unoriented bordism.

Remark. Independent of the existence or value of N in condition (4) of the theorem, the computations from [1, Section 4.5] of $\Omega_*^{K-\text{Witt}}(\cdot)$ as a generalized homology theory on CW complexes continue to hold and to imply that for char$(K) = 2$,

$$\Omega_n^{K-\text{Witt}}(X) = \Omega_n^{\mathbb{Z}_2-\text{Witt}}(X) \cong \bigoplus_{r+s=n} H_r(X; \Omega_s^{\mathbb{Z}_2-\text{Witt}}).$$

Unoriented bordism. Given the motivation to recognize spaces that possess a form of Poincaré duality, it seems reasonable to consider K-Witt spaces that are K-*oriented*. This has no effect when char$(K) \neq 2$, in which case K-orientability is equivalent to $\mathbb{Z}$-orientability as considered in [1]. But when char$(K) = 2$, all pseudomanifolds are $\mathbb{Z}_2$-orientable, which is equivalent to being K orientable, and the Poincaré duality isomorphism $I^{\bar{m}}H_k(X;K) \cong \text{Hom}(I^{\bar{m}}H_{n-k}(X;K), K)$ holds for all such compact pseudomanifolds satisfying the K-Witt condition.

If we allow K-Witt spaces and K-Witt bordism using K-orientations, then for char$(K) = 2$ we are essentially talking about unoriented bordism[7], so to clarify the notation, let us denote the resulting bordism groups by $\mathcal{N}_*^{K-\text{Witt}}$. These groups can be computed as follows:

[5] Recall that the Künneth theorem holds within a single perversity when one term is a manifold, so we can compute the intersection forms of such product spaces in the usual way; see e.g. [6].

[6] Since these are geometric bordism groups, they vanish in negative degree.

[7] One could also define unoriented bordism groups of unoriented compact PL pseudomanifolds satisfying the K-Witt condition with char$(K) \neq 2$, but it is not clear how to study such groups by the present techniques, as there is no reason to expect that $I^{\bar{m}}H_*(X;K)$ would satisfy Poincaré duality for such a space X.

Theorem 2. *For a field K with char$(K) = 2$ and for $i \geq 0$,*

$$\mathcal{N}_i^{K-Witt} \cong \begin{cases} \mathbb{Z}_2, & i \equiv 0 \mod 2, \\ 0, & i \equiv 1 \mod 2. \end{cases}$$

Since writing [1], the author has discovered that this theorem is also provided without detailed proof by Goresky in [4, page 498]. We provide here the details:

Proof. It continues to hold that the local Witt condition depends only on the characteristic of K for the reasons provided in [1], so we may assume $K = \mathbb{Z}_2$. To see that $\mathcal{N}_n^{\mathbb{Z}_2-\text{Witt}} = 0$ for n odd, we simply note that X bounds the closed cone $\bar{c}X$, which is a $\mathbb{Z}_2$-Witt space. The map $w : \mathcal{N}_{2k}^{\mathbb{Z}_2-\text{Witt}} \to W(\mathbb{Z}_2) \cong \mathbb{Z}_2$ is onto for each $k > 0$, as the intersection pairing on the $\mathbb{Z}_2$-coefficient middle-dimensional homology of the real projective space $\mathbb{R}P^{2k}$ corresponds to the generator of $W(\mathbb{Z}_2)$ represented by the matrix $\langle 1 \rangle$. Furthermore, w is injective for $k > 1$ as in the preceding surgery argument, which does not rely on whether or not X is oriented, only on the existence of the intersection pairing over $\mathbb{Z}_2$. In dimension 0, we have unoriented manifold bordism of points, so $\mathcal{N}_0^{\mathbb{Z}_2-\text{Witt}} \cong \mathbb{Z}_2$. Finally, as in the argument above for $\Omega_2^{\mathbb{Z}_2-\text{Witt}}$, the group $\mathcal{N}_2^{\mathbb{Z}_2-\text{Witt}}$ must be generated by closed surfaces (now not necessarily oriented), so $\mathcal{N}_2^{\mathbb{Z}_2-\text{Witt}}$ is a quotient of the unoriented manifold bordism group $\mathcal{N}_2 \cong \mathbb{Z}_2$; thus $\mathcal{N}_2^{\mathbb{Z}_2-\text{Witt}}$ must be isomorphic to $\mathbb{Z}_2$ as w maps $\mathbb{R}P^2$ onto the non-trivial element of $W(\mathbb{Z}_2) \cong \mathbb{Z}_2$. □

Remark. An even simpler version of the argument of [1] implies that as a generalized homology theory

$$\mathcal{N}_n^{K-\text{Witt}}(X) \cong \bigoplus_{r+s=n} H_r(X; \mathcal{N}_s^{K-\text{Witt}})$$

for char$(K) = 2$, as in this case one no longer needs a separate argument to handle the odd torsion that can arises in $H_n(X; \Omega_0^{K-\text{Witt}})$ as a result of $\Omega_0^{K-\text{Witt}} \cong \mathbb{Z}$ not being 2-primary.

Further discussion of oriented bordism. We next provide some results that demonstrate the difficulty of determining which case of item (4) of Theorem 1 holds.

We will first see that $w([M]) = 0$ for any $\mathbb{Z}$-oriented manifold: Since dimension mod 2 is the only invariant[8] of $W(\mathbb{Z}_2)$, this is a consequence of the following lemma, recalling that for a manifold, $I^{\bar{m}}H_*(M) = H_*(M)$.

Lemma. *Let M be a closed connected $\mathbb{Z}$-oriented manifold of dimension $4k+2$. Then* $\dim(H_{2k+1}(M; \mathbb{Z}_2)) \equiv 0 \mod 2$.

Proof. By the universal coefficient theorem,

$$H_{2k+1}(M; \mathbb{Z}_2) \cong (H_{2k+1}(M) \otimes \mathbb{Z}_2) \oplus (H_{2k}(M) * \mathbb{Z}_2),$$

[8] As observed in the proof of [7, Lemma III.3.3], rank mod 2 yields a homomorphism $W(F) \to \mathbb{Z}_2$ for any field F. Since we know that $W(\mathbb{Z}_2) \cong \mathbb{Z}_2$ and that $\langle 1 \rangle$, which has rank 1, is a generator of $W(F)$ (it is certainly non-zero, using [7, Lemma I.6.3 and Lemma III.1.6]), it follows that rank mod 2 determines the isomorphism.

where the asterisk denotes the torsion product. Let $T_*(M)$ denote the torsion subgroup of $H_*(M)$, and let $T^2_*(M)$ denote $T_*(M) \otimes \mathbb{Z}_2 \cong T_*(M) * \mathbb{Z}_2$; the isomorphism follows from basic homological algebra because $T_*(M)$ is a finite abelian group. $T^2_*(M)$ is a direct sum of $\mathbb{Z}_2$ terms. Then $H_{2k+1}(M) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2^B \oplus T^2_{2k+1}(M)$, where B is the $2k+1$ Betti number of M, and $H_{2k}(M) * \mathbb{Z}_2 \cong T^2_{2k}(M)$. Thus $H_{2k+1}(M;\mathbb{Z}_2) \cong \mathbb{Z}_2^B \oplus T^2_{2k+1}(M) \oplus T^2_{2k}(M)$. Since M is a closed $\mathbb{Z}$-oriented manifold, there is a nondegenerate skew-symmetric intersection form on $H_{2k+1}(M;\mathbb{Q})$, and so B is even. Since M is a closed $\mathbb{Z}$-oriented manifold, the nonsingular linking pairing $T_{2k+1}(M) \otimes T_{2k}(M) \to \mathbb{Q}/\mathbb{Z}$ gives rise to an isomorphism $T_{2k+1}(M) \cong \mathrm{Hom}(T_{2k}(M), \mathbb{Q}/\mathbb{Z})$, and since $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_n$, it follows that $T_{2k+1}(M) \cong T_{2k}(M)$. Therefore $T^2_{2k+1}(M) \cong T^2_{2k}(M)$. Thus $H_{2k+1}(M;\mathbb{Z}/2)$ consists of an even number of $\mathbb{Z}_2$ terms. $\square$

Remark. Since the lemma utilizes only integral Poincaré duality and the universal coefficient theorem, it follows that, in fact, $w([X]) = 0$ for any IP space[9]; these are spaces that satisfy local conditions guaranteeing that intersection homology Poincaré duality holds over the integers and that a universal coefficient theorem holds (see [3, 10]).

A slightly more elaborate argument demonstrates that it is also not possible to have $w([X]) \neq 0$ if X is a $\mathbb{Z}$-oriented $\mathbb{Z}_2$-Witt space with at worst isolated singularities:

Proposition. *Let X be a closed $\mathbb{Z}$-oriented $4k+2$-dimensional $\mathbb{Z}_2$-Witt space with at worst isolated singularities. Then $w([X]) = 0$.*

Proof. Since X has at worst point singularities, it follows from basic intersection homology calculations (see [2, Section 6.1]) that $I^{\bar{m}}H_{2k+1}(X;\mathbb{Z}_2) \cong \mathrm{im}(H_{2k+1}(M;\mathbb{Z}_2) \to H_{2k+1}(M, \partial M;\mathbb{Z}_2))$, where M is the compact $\mathbb{Z}$-oriented PL ∂-manifold obtained by removing an open regular neighborhood of the singular set of X. We will show that if $[z] \in \mathrm{im}(H_{2k+1}(M;\mathbb{Z}_2) \to H_{2k+1}(M, \partial M;\mathbb{Z}_2))$, then the intersection product $[z] \cdot [z] = 0$. It follows that the intersection pairing on $I^{\bar{m}}H_{2k+1}(X;\mathbb{Z}_2)$ is split by [7, Lemma III.1.1], since then there can be no non-trivial anisotropic subspace. This implies that $w([X]) = 0$ by the definition of the Witt group.

The following argument that $[z] \cdot [z] = 0$ was suggested by "Martin O" on the web site MathOverflow [9]. By Poincaré duality, it suffices to show that $\alpha \cup \alpha = 0$, where α is the Poincaré dual of $[z]$ in $H^{2k+1}(M, \partial M;\mathbb{Z}_2)$. But now $\alpha \cup \alpha = Sq^{2k+1}\alpha = Sq^1 Sq^{2k}\alpha = \beta^* Sq^{2k}\alpha$, where β^* is the Bockstein associated with the sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ (see [5, Section 4.L]). In the case at hand, this is the Bockstein $\beta^* : H^{4k+1}(M, \partial M;\mathbb{Z}_2) \to H^{4k+2}(M, \partial M;\mathbb{Z}_2)$. But this map is trivial. To see this, observe that there is a commutative

[9] Also called "intersection homology Poincaré spaces," though this is perhaps a misnomer as "Poincaré spaces" are generally not required to be manifolds while IP spaces are still expected to be pseudomanifolds.

diagram

$$\begin{array}{ccc} H^{4k+1}(M,\partial M;\mathbb{Z}_2) & \xrightarrow{\beta^*} & H^{4k+2}(M,\partial M;\mathbb{Z}_2) \\ \downarrow \cong & & \downarrow \cong \\ H_1(M;\mathbb{Z}_2) & \xrightarrow{\beta_*} & H_0(M;\mathbb{Z}_2), \end{array}$$

where β_* is the homology Bockstein and the vertical maps are Poincaré duality. The existence of this diagram follows as in [8, Lemma 69.2]. But now $\beta_* : H_1(M;\mathbb{Z}_2) \to H_0(M;\mathbb{Z}_2)$ is trivial, as the standard map $\times 2 : H_0(M;\mathbb{Z}_2) \to H_0(M;\mathbb{Z}_4)$ is injective. □

Hence any candidate to have $w([X]) = 1$ must have singular set of dimension > 0 and must not be an IP space. Given that all K-Witt spaces for $\mathrm{char}(K) \neq 2$ are K-Witt bordant to spaces with at worst isolated singularities [11, 1], it is unclear how to proceed to determine whether $\mathbb{Z}_2$-Witt spaces with $w([X]) = 1$ exist. One method to prove that they do not would be to try to show "by hand" that every $\mathbb{Z}_2$-Witt space is $\mathbb{Z}_2$-Witt bordant to a space with at most isolated singularities, but the only proof currently known to the author of this fact for fields of other characteristics utilizes the bordism computations of [11, 1].

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