Microbundles and differentiable structures

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A piecewise linear microbundle is an object something like a vector space brindle, but having only a "germ" of a piecewise linear n-cell as fibre. This paper develops a theory of such bundles, and uses it to study piecewise linear manifolds.

In § 1 the concepts are defined, and many standard constructions for vector bundles are modified so as to apply to microbundles. In particular every PL-manifold M has a tangent microbundle τ_M. Any microbundle over M determines a homotopy class of maps from M to a classifying space $B(PL_m)$.

§ 2 shows that microbundles have inverses with respect to the Whitney sum operation, and § 3 describes a theory of normal microbundles. In § 4 the problem of smoothing a PL-manifold (i.e. imposing a well behaved differentiable structure) is considered. It is shown M is smoothable if and only if there exists a map f from M to a certain "universal" manifold U so that the diagram

$$\begin{tikzcd} M \arrow[r, dashed] \arrow[d] & U \arrow[d] \\ B(PL_m) \arrow[r, phantom, "\subset"] & B(PL_n) \end{tikzcd}$$

is homotopy commutative, where the vertical arrows are classifying maps for the tangent microbundles. This manifold U is essentially a classifying space for the orthogonal group $O(n)$. It follows that one can set up an obstruction theory for the problem of smoothing a PL-manifold. (Compare Thom [18], Munkres [13].)

In a later paper these methods will be used to show that the tangent vector bundle of a certain differentiable manifold is not a topological invariant. (Compare § 4 Corollary 6.4.)

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1. Microbundles

This paper will work in the category of locally finite simplicial complex and piecewise linear maps. (Briefly: PL-maps.) However the definitions, and many of the theorems, would also make sense in the larger category of topological space and maps.

Definition: A function $f : K \longrightarrow L$ between locally finite simplicial complexes is piecewise linear if there exists a rectlinear subdivision K' of K so that f maps each simplex of K' linearly into a simplex of L. (Compare Lemma 9 on page 27.)

Note that any open subset of a locally finite simplicial complex can be triangulated so that the inclusion map is piecewise linear. (See Alexandroff and Hopf [1, pg. 143].) The resulting simplicial complex is unique up to PL-homeomorphism.

Definition: A PL-microbundle ξ of dimension n (or briefly a "bundle") is a diagram

$$B \xrightarrow{i} E \xrightarrow{j} B$$

where B, E are locally finite simplicial complexes and i, j are PL-maps; such that the following local triviality condition is satisfied. For each $b \in B$, there should exist neighborhoods B_0 of b, E_0 of $i(b)$ and a PL-homeomorphism $h: E_0 \longrightarrow B_0 \times R^n$ so that the diagram

$$\begin{tikzcd} & E_0 \arrow[dd, "h"] \arrow[dr, "j|E_0"] & \\ B_0 \arrow[ur, "i|B_0"] \arrow[dr, "\times 0"'] & & B_0 \\ & B_0 \times R^n \arrow[ur, "p_1"'] & \end{tikzcd}$$

is commutative. Here the notion $\times 0$ stands for the map $b \longrightarrow (b,0)$, p_1 denotes the projection into the first factor, and R^n denotes euclidean n-space.

B will be called the base space of ξ, E the total space, i the injection map, and j the projection map. Note that the composition $ji : B \longrightarrow B$ is the identity map of B.

<u>Definition</u>: A second PL-microbundle

$$\xi': \quad B \xrightarrow{i'} E' \xrightarrow{j'} B$$

over the same base space is <u>isomorphic</u> to ξ (written $\xi' \approx \xi$) if there exist neighborhoods E_1 of $i(B)$ and E_1' of $i'(B)$, and a PL-homeomorphism $E_1 \longrightarrow E_1'$ so that the diagram

$$\begin{tikzcd} & E_1 \arrow[dd] \arrow[dr, "j|E_1"] & \\ B \arrow[ur, "i"] \arrow[dr, "i'"'] & & B \\ & E_1' \arrow[ur, "j'|E_1'"'] & \end{tikzcd}$$

is commutative.

Example 1. For any B and any $n \geq 0$ the trivial bundle ε_B^n is defined to be the diagram

$$B \xrightarrow{\times 0} B \times R^n \xrightarrow{p_1} B.$$

Any bundle isomorphic to ε_B^n is also called a trivial bundle.

Example 2. A simplicial complex M will be called a PL-manifold[1] if each point has a neighborhood U which is PL-homeomorphic to R^n. The tangent microbundle τ_M is then defined to be the diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M \quad ,$$

where Δ denotes the diagonal map.

Proof that τ_M is a microbundle. First consider the special case $M = R^n$. The PL-homeomorphism $h(x,y) = (x,x-y)$ makes the diagram

$$\begin{tikzcd} & R^n \times R^n \arrow[dr, "p_1"] \arrow[dd, "h"] & \\ R^n \arrow[ur, "\Delta"] \arrow[dr, "\times 0"'] & & R^n \\ & R^n \times R^n \arrow[ur, "p_1"'] & \end{tikzcd}$$

commutative. Thus τ_{R^n} is a microbundle. (In fact τ_{R^n} is a trivial microbundle.) Since each point of M has a neighborhood which is PL-homeomorphic to R^n it follows that τ_M is also a microbundle.

Just as in the theory of vector space bundles, there are a number of ways of building new PL-microbundles out of old ones. Given two bundles

$$\xi_\alpha: \quad B \xrightarrow{i_\alpha} E_\alpha \xrightarrow{j_\alpha} B \quad , \alpha=1,2,$$

over the same base space, the Whitney sum

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[1] The terms "formal manifold" and "combinatorial manifold" have also been used for this concept.

$$\xi_1 \oplus \xi_2 : \quad B \xrightarrow{i} E \xrightarrow{j} B$$

is obtained as follows. Let $E \subset E_1 \times E_2$ be the set of (e_1, e_2) with $j_1 e_1 = j_2 e_2$; and let

$$ib = (i_1 b, i_2 b) \ , \quad j(e_1, e_2) = j_1 e_1 \quad .$$

It is easily verified that $\xi_1 \oplus \xi_2$ is a PL-microbundle.

Given a microbundle

$$\xi : \qquad B \xrightarrow{i} E \xrightarrow{j} B$$

and given a subcomplex $B_o \subset B$ the <u>restricted bundle</u> $\xi | B_o$ is defined to be the diagram

$$B_o \xrightarrow{i | B_o} E_o \xrightarrow{j | E_o} B_o$$

where $E_o = j^{-1} B_o$. More generally given a complex B_1 and a PL-map $f \colon B_1 \longrightarrow B$ the <u>induced bundle</u>

$$f^* \xi : \qquad B_1 \xrightarrow{i_1} E_1 \xrightarrow{j_1} B_1$$

is obtained as follows. Let $E_1 \subset B_1 \times E$ be the set of (b_1, e) with $f(b_1) = j(e)$; and let

$$i_1(b_1) = (b_1, if(b_1)) \ , \ j_1(b_1, e) = b_1 .$$

The verification that $f^*\xi$ is a microbundle is straightforward.

Theorem 1 (Covering homotopy theorem) <u>Let</u> f <u>and</u> g <u>be</u> <u>two homotopic</u> PL-<u>maps from</u> B_1 <u>to</u> B. <u>Then the induced bundles</u> $f^*\xi$ <u>and</u> $g^*\xi$ <u>over</u> B_1 <u>are isomorphic</u>.

The proof given in Steenrod [15, § 11] applies without essential change. It is only necessary to be sure that all maps occuring in the the proof can be made piecewise linear. In particular, one must show that there exists a homotopy

$$h : B_1 \times I \longrightarrow B$$

between f and g which is a PL-map. [Proof. Starting with any homotopy $B_1 \times I \longrightarrow B$ and applying the simplicial approximation theorem one concludes that there exist simplicial approximations f' to f and g' to g so that f' is PL-homotopic to g'. But it is easily seen that any PL-map is PL-homotopic to its simplicial approximations.] Further details will be left to the reader.

Theorem 2. (Universal bundle theorem). <u>There exists a PL-microbundle</u>

$$\gamma^n: \qquad B(PL_n) \longrightarrow E(PL_n) \longrightarrow B(PL_n)$$

<u>which is "universal" in the following sense. For any locally finite complex</u> B <u>and any bundle</u> ξ <u>over</u> B <u>of dimension</u> n <u>there is a unique homotopy class of maps</u> $f: B \longrightarrow B(PL_n)$ <u>such that</u> $f^*\gamma^n \approx \xi$.

A proof based on the theory of semi-semplicial complexes will be given in § 5 at the end of this paper. (A. Shapiro has pointed out that an easier proof could be given using the methods of E. Brown [2].

A theory of characteristic classes can also be developed for microbundles. For example the Thom definition of Stiefel-Whitney classes (see [16]) applies easily to microbundles. The combinatorial definition of Pontrjagin classes (see Thom [17], Rohlin and Švarc [14]) can be used to define Pontrjagin classes

$$p_i(\xi) \in H^{4i}(B \,;\, Q)$$

for a PL-microbundle. No details will be given, since these characteristic classes will not be used in the present paper.

2. Inverse bundles

Definition Two bundles ξ, η over B are inverse to each other if the Whitney sum $\xi \oplus \eta$ is trivial.

[Note: Inverses are not unique. For example the tangent microvundle τ_{S^2} of the 2-sphere can be considered as an inverse to a trivial bundle over S^2. Yet τ_{S^2} is not trivial.]

Theorem 3. Every PL-microbundle ξ over a finite dimensional complex B has an inverse.

It is sufficient to consider the case of an orientable microbundle. For even if ξ is not orientable, the sum $\xi \oplus \xi$ clearly is orientable.

First suppose that B is a suspension. It is necessary to be careful here since the usual double cone construction destroys local finiteness. However if one imbeds a given complex B' in a contractible locally finite complex C(B') and thm takes two copies of C(B') matched along B', one obtains an acceptible substitute for the suspension of B':

Let B have this form, and let $r: B \longrightarrow B$ be the "reflection" which interchanges the two copies of C(B').

Lemma 1. If ξ is an orientable microbundle over such a complex B, then $\xi \oplus r^*\xi$ is trivial.

<u>Proof</u>. Let $B_v B$ denote the union with a simple point b_o in common. Given oriented microbundles ξ and η over B with the same fibre dimension q , let $\xi \vee \eta$ denote an oriented bundle over $B_v B$ whose restriction to the first summand is ξ and whose restriction to the second summand is η. This construction is well defined up to orientation preserving isomorphism: the only choice involved in forming $\xi \vee \eta$ is the choice of how to identify the fibre of ξ over b_o with the fibre of η over b_o. But according to Gugenheim [6, Theorem 3], this identification is unique up to piecewise linear isotopy.

Note that

(1) $(\xi \vee \eta) \oplus (\xi' \vee \eta') \approx (\xi \oplus \xi') \vee (\eta \oplus \eta')$.

If these four bundles all have the same fibre dimension q then, since $\eta \oplus \eta' \approx \eta' \oplus \eta$, this implies:

(2) $(\xi \vee \eta) \oplus (\xi' \vee \eta') \approx (\xi \vee \eta') \oplus (\xi' \vee \eta)$.

Let $c : B \longrightarrow BvB$ be such that the composition with each of the "projection maps"

$$BvB \longrightarrow Bvb_o = B \quad \text{or} \quad BvB \longrightarrow b_ovB = B$$

is homotopic to the identity. Then clearly

(3) $c^*(\xi \vee \varepsilon^q) \approx c^*(\varepsilon^q \vee \xi) \approx \xi$

Furthermore

(4) $c^*(\xi \vee r^*\xi) \approx \varepsilon^q$;

since $\xi \vee r^*\xi$ is isomorphic to $f^*\xi$ for a suitible map $f: BvB \longrightarrow B$, with fc homotopic to zero.

Now combining (2) and (4) one has

$$\varepsilon^q \oplus \varepsilon^q \approx c^*((\xi \vee r^*\xi) \oplus (\varepsilon^q \vee \varepsilon^q)) \approx c^*(\xi \vee \varepsilon^q) \oplus c^*(\varepsilon^q \vee r^*\xi) \quad .$$

Together with (3) this implies that

$$\varepsilon^q \oplus \varepsilon^q \approx \xi \oplus r^*\xi \quad ,$$

which proves Lemma 1.

Proof of Theorem 3 by induction on the dimension of B. If B has dimension 1 then it has the homotopy type of a suspension and the conclusion follows. Suppose that B has dimension $n + 1$, and that ξ restricted to the skeleton B^n has an inverse η. First we will show that $\eta \oplus \varepsilon^q$ can be extended in some way over B ; where q is the fiber dimension of ξ. Clearly a bundle over B^n can be extended over a given $(n + 1)$-simplex if and only if its restriction to the boundary n-sphere Σ^n is trivial. Thus $\xi|\Sigma^n$ is trivial. Hence $(\eta \oplus \varepsilon^q)|\Sigma^n$ is isomorphic to $(\eta \oplus \xi)|\Sigma^n$ which is known to be trivial. This proves that $\eta \oplus \varepsilon^q$ extends to some bundle η' over B.

Consider the complex $B \cup C'(B^n)$ obtained from B by adjoining a contractible complex over the n-skeleton. Since $\xi \oplus \eta'$ restricted to B^n is trivial, it follows that $\xi \oplus \eta'$ extends to some bundle ζ over $B \cup C'(B^n)$. But $B \cup C(B^n)$ has the homotopy type of a suspension: namely of a bouquet of $(n + 1)$-spheres. Hence ζ has an inverse ζ^* ; and $\xi \oplus \eta' \oplus (\zeta^*|B)$ is trivial. This completes the proof of Theorem 3.

Definition. Two PL-microbundles ξ and ξ' over B belong to the same s-class if $\xi \oplus \varepsilon_B^q$ is isomorphic to $\xi' \oplus \varepsilon_B^r$ for some q, r. We will also say that ξ is s-isomorphic to ξ'. The s-class of ξ will be denoted by (ξ).

As an immediate consequence of Theorem 3 we have:

Corollary 3.1. The s-classes of PL-microbundles over a finite dimensional complex B form an abelian group under the composition operation:

$$(\xi) + (\eta) = (\xi \oplus \eta) \quad .$$

The proof is striaghtforward.

Definition. This group will be denoted by $k_{PL}(B)$.

Clearly the correspondnece

$$B \dashrightarrow k_{PL}(B)$$

defines a contravariant functor from complexes to abelian groups.

The analogues of Theorem 3 and of Corollary 3.1 for vector bundles are well known. We will use the notation $k_O(B)$ for the group of s-classes of vector bundles over a finite dimension complex B. Individual vector bundles will be denoted by lower case Latin letters.

It will be seen later (§ 4, Lemma 6) that there is a natural transformation

$$T : k_O(B) \longrightarrow k_{PL}(B)$$

between these two functors. Intuitively, $T(v)$ is obtained by triangulating the vector bundle v so as to make a PL-bundle out of it.

3. Normal bundles

Consider PL-manifolds $M \subset N$ with inclusion map $i: M \longrightarrow N$.

Definition. M has a normal microbundle ν in N if there exists a neighborhood U of M in N and a retraction $j: U \longrightarrow M$ so that the diagram

$$\nu: \qquad M \xrightarrow{i} U \xrightarrow{j} M$$

is a PL-microbundle over M. In particular M has a trivial normal bundle in N if U and j can be chosen so that ν is a trivial bundle

It is not known that M has a normal bundle in N even if the imbedding $M \subset N$ is locally flat. Furthermore, even if the normal bundle does exist, it is not known to be unique up to isomorphism. However the following two results will be proved.

Theorem 4. Given PL-manifolds $M \subset N$ there exists an integer q so that the submanifold $M \times 0 \subset N \times R^q$ has a normal microbundle.

Theorem 5. If $M \subset N$ has a normal microbundle ν then the Whitney sum $\tau_M \oplus \nu$ is isomorphic to the restriction $\tau_N | M$.

This result implies that ν is at least unique up to s-isomorphism:

Corollary 5.1. If $M \subset N$ has two distinct normal bundles ν, ν' (corresponding to two distinct choices of j : (neighborhood) $\longrightarrow M$) then ν and ν' belong to the same s-class.

The proofs will depend on the concept of the composition of two microbundles

$$\xi: \qquad B \xrightarrow{i} E \xrightarrow{j} B \quad , \text{ and}$$

$$\eta: \qquad E \xrightarrow{I} F \xrightarrow{J} E$$

where the total space of ξ coincides with the base space of η. This composition is defined to be the microbundle

$$B \xrightarrow{Ii} F \xrightarrow{jJ} B.$$

Example. Given bundles ξ and ξ' over B, consider the induced bundle $j^*\xi'$ over the total space E of B. Then it is easily verified that the composition of ξ and $j^*\xi^*$ is exactly the Whitney sum $\xi \oplus \xi'$. In fact this example is the most general one:

Lemma 2. Given ξ and η as above, the composition of ξ and η is isomorphic to the Whitney sum $\xi \oplus i^*\eta$.

Proof. Let E_0 be a neighborhood of $i(B)$ in E which is sufficiently small so that the map

$$ij_0 \colon E_0 \longrightarrow E$$

is homotopic to the inclusion map (where $j_0 = j|E_0$). Thus the bundle $j_0{}^*i^*\eta$ over E_0 will be isomorphic to $\eta|E_0$. But the composition of ξ and $j_0^*(i^*\eta)$ is isomorphic to the Whitney sum $\xi \oplus i^*\eta$. (Compare the example above.) This proves Lemma 2.

Proof of Theorem 5. The bundle

$$\tau_N|M\colon \qquad M \xrightarrow{\Delta} M \times U \xrightarrow{p_1} M$$

can be considered as the composition of the tangent bundle

$$\tau_M\colon \qquad M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M$$

and the induced bundle

$$p_2^*\nu : \quad M \times M \xrightarrow{1\times i} M \times U \xrightarrow{1\times j} M \times M.$$

Hence by Lemma 2 we have

$$\tau_N|M \approx \tau_M \oplus \Delta^* p_2^* \nu = \tau_M \oplus \nu \quad ;$$

which completes the proof.

The corollary 5.1 follows immediately. For if ν' is a second normal bundle, then choosing η so that $\tau_M \oplus \eta \approx \varepsilon^q$, we have $\nu' \oplus \varepsilon^q \approx (\nu' \oplus \tau_M) \oplus \eta \approx (\nu \oplus \tau_M) \oplus \eta \approx \nu \oplus \varepsilon^q$.

<u>Proof</u> of Theorem 4. Replacing N by a neighborhood of M if necessary, we may assume that some PL-retraction $r\colon N \longrightarrow M$ exists.

Consider the induced bundle

$$r^*\tau_M\colon \qquad N \xrightarrow{(I,r)} N \times M \xrightarrow{p_1} N \quad ;$$

where (I,r) denotes the map $x \longrightarrow (x, r(x))$. Choose a bundle η over N so that $r^*\tau_M \oplus \eta$ is trivial;

$$r^*\tau_M \oplus \eta \approx \varepsilon_N^q .$$

This implies that the bundle ε_N^q is isomorphic to the composition of $r^*\tau_M$ and the bundle $p_1{}^*\eta$ over $N \times M$.

If V is a sufficiently small neighborhood of $(I,r)N$ in $N \times M$, it follows that we have a commutative diagram

$$\begin{tikzcd} N \arrow[rr, "\times 0"] \arrow[dr, "(I,r)"'] & & N \times R^q \\ & V \arrow[ur, "i'"'] & \end{tikzcd}$$

where i' is the inclusion map of the PL-microbundle $p_1{}^*\eta$. On the other hand the composition

$$M \xrightarrow{i} N \xrightarrow{(I,r)} V$$

is also the inclusion map of a microbundle. In fact this composition is the diagonal map $\Delta: M \longrightarrow V \subset N \times M$, which is the inclusion map for $\tau_N|M$. Now the map $M \times 0 \xrightarrow{i\times 0} N \times R^q$ is equal to the composition $i'\Delta$ of the inclusion maps of two microbundles. Hence $M \times 0$ has a normal bundle ν in $N \times R^q$ which is isomorphic to the composition of these bundles. This completes the proof of Theorem 4.

(Remark: By Lemma 2, ν is isomorphic to

$$\tau_N|M \oplus \eta|M\text{:})$$

Combining Theorems 4 and 5 we have:

Corollary 5.2 $M \times 0$ *has a trivial normal bundle in* $N \times R^q$ *for large* q *if and only if* τ_M *and* $\tau_N|M$ *belong to the same* s-*class*.

Definition (J. H. C. Whitehead [20]) A PL-manifold M is *of class* Π if for any imbedding of M in a high dimensional Euclidean space, the regular neighborhood of M is PL-homeomorphic with the product $M \times$ (simplex).

Corollary 5.3 (Theorem of M. Curtis and R. Lashof.) M *is of class* Π *if and only if its tangent bundle* τ_M *is* s-*trivial*.

Proof It is only necessary to observe that every imbedding of M in a high dimensional euclidean space can be deformed so that it lies in a $(2m + 1)$-dimensional hyperplane. Thus Corollary 5.2 applies.

4. Differentiable structures

The notation M_σ will be used for a PL-manifold M together with a differentiable structure σ of class C^r on M which satisfies the following:

Compatibility condition. For some rectilinear subdivision M' of M, the identity map

$$M' \longrightarrow M_\sigma$$

should be a C^r-triangulation of the smooth manifold M_σ. (See J. H. C. Whitehead [19]. σ must induce the usual differentiable structure on each simplex of M'.) Here r denotes some fixed integer, $1 \leq r \leq \infty$.

Definition. If such a σ exists then M will be called a smoothable PL-manifold; and σ will be called a smoothing of M.

According to Cairns [3] as corrected by Whitehead [22], every PL-manifold of dimension ≤ 4 is smoothable. On the other hand according to Thom [17] and Rohlin-Švarč [14] there exists a PL-manifold of dimension 8 which is not smoothable.

The following basic result is due to S. Cairns [4] and M. Hirsch [8].

Theorem C-H. *A PL-manifold M is smoothable if and only if the product $M \times R$ is smoothable.*

It follows by induction that M is smoothable if any product $M \times R^q$ is smoothable. Since an open subset of R^{m+q} is certainly smoothable, this implies the following.

Corollary C-H.1 <u>Every manifold of class</u> Π <u>is smoothable.</u>

If the manifold M is contractible then its tangent micro-bundle τ_M is trivial (Theorem 1), hence M is of class Π. This gives a new proof of the following result.

Corollary C-H.2 (Theorem of A. Gleason [5].) <u>Every contractible PL-manifold is smoothable.</u>

The theorem of Cairns and Hirsch can also be stated in the following slightly sharper form. Let e_M^q denote the trivial vector space bundle over M.

Theorem C-H$^{\#}$. <u>If</u> $M \times R^q$ <u>has a smoothing</u> σ_1 <u>with tangent vector bundle</u> t_1 <u>then a smoothing</u> σ <u>of</u> M <u>can be chosen so that the tangent vector bundle</u> $t(M_\sigma)$ <u>is</u> s-<u>isomorphic</u> to $t_1|M$.

<u>Proof</u> This follows immediately from the argument in [4] or [8].

In order to study the smoothing problem for PL-manifolds, the following concept will be useful.

<u>Definition</u>. A PL-manifold U of dimension n with a smoothing μ will be called m-<u>universal</u> if the tangent vector bundle $t(U_\mu)$ is an m-universal bundle for the orthogonal group $O(n)$, in the sense of Steenrod [15, § 19].

(Note: the dimension n is necessarily $\geq 2m$.)

Lemma 3. <u>For every</u> m <u>there exists an</u> m-<u>universal manifold</u> U_μ.

<u>Proof</u> Start with the Grassmann manifold $G(m,m)$ of m-planes in 2m-space. This has the right homotopy type to be an m-universal manifold: that is there exists a smooth m-universal vector bundle u

over $G(m,m)$. However the tangent bundle t of $G(m,m)$ is not the right bundle. To correct this, choose a smooth vector bundle v over $G(m,m)$ so that the Whitney sum $t \oplus v$ is trivial. Now consider the total space $E(v \oplus u)$ of the smooth vector bundle $v \oplus u$. It is easily verified that the tangent bundle of this smooth manifold $E(v \oplus u)$ is an m-universal vector bundle. Choosing a C^r-triangulation $f: U \longrightarrow E(v \oplus u)$, and letting μ denote the induced smoothing of U, this proves Lemma 3.

Let M be a PL-manifold of dimension m, and let U_μ be an m-universal manifold of dimension $n \geq 2m + 1$.

Theorem 6. M *is smoothable if and only if the Whitney sum* $\tau_M \oplus \varepsilon_M^{n-m}$ *is isomorphic to* $f^*\tau_U$ *for some PL-map* $f: M \longrightarrow U$.

The proof will be based on two lemmas.

Let M and M' be PL-manifolds of dimensions m, $m + k$, and suppose that M' is smoothable, with smoothing μ.

Lemma 4 *If there exists a PL-map* $f: M \longrightarrow M'$ *such that* $f^*\tau_{M'}$ *is s-isomorphic to* $\tau_{M'}$, *then* M *is also smoothable. In fact there exists a smoothing* σ *of* M *so that the tangent vector bundle* $t(M_\sigma)$ *is s-isomorphic to* $f^*t(M'_\mu)$.

Proof. Replacing M' by some product $M' \times R^q$, the map f is homotopic to a PL-imbedding $f_1: M \longrightarrow M' \times R^q$. Since the restricted tangent bundle $\tau_{M' \times R^q}|f_1(M)$ is s-isomorphic to $\tau_{f_1(M)}$, it follows from Corollary 5.2 that the submanifold $f_1M \times 0 \subset M' \times R^q \times R^r$ has a trivial normal bundle, providing that r is sufficiently large. In other words the product $M \times R^{k+q+r}$ can be PL-imbedded as an open subset of the smoothable manifold $M' \times R^{q+r}$. Thus $M \times R^{k+q+r}$ is smoothable. Using the Cairns-Hirsch theorem it follows that M is smoothable.

The smoothing of $M \times R^{k+q+r}$ obtained in this way will evidently have a tangent bundle t_1 such that

$$t_1 | M \times 0 \approx f^*{}_1 t(M'_\mu \times R^{q+r}_\theta);$$

where θ denotes the standard smoothing of Euclidean space. Therefore $t_1 | M \times 0$ is s-isomorphic to $f^* t(M'_\mu)$. According to the sharpened form of the Cairns-Hirsch theorem, it follows that M has a smoothing σ with $t(M_\sigma)$ s-isomorphic to $f^* t(M'_\mu)$. This proves Lemma 4.

Conversely suppose that M and M' are both smoothable, with smoothings σ, μ respectively. Let $f: M \longrightarrow M'$ be such that the induced vector bundle $f^* t(M'_\mu)$ is s-isomorphic to $t(M_\sigma)$.

Lemma 5. Then the induced microbundle $f^* \tau_{M'}$ is s-isomorphic to τ_M. In fact if the dimensions $m + k$ and m satisfy $m + k \geq 2m + 1$, then $f^* \tau_{M'} \approx \tau_M \oplus \varepsilon^k_M$.

Proof. Replacing M'_μ by some product $M'_\mu \times R^q_\theta$ the map f is homotopic to a smooth imbedding

$$f_1 : M_\sigma \longrightarrow M'_\mu \times R^q_\theta.$$

Since $f_1^* t(M'_\mu \times R^q_\theta)$ is s-isomorphic to $t(M_\sigma)$ it follows that the normal vector bundle of $f_1 M_\sigma$ in $M'_\mu \times R^q_\theta$ is s-trivial; and hence is trivial. (Compare [10, Lemma 4].) Therefore a tubular neighborhood N_λ of $f_1 M_\sigma$ in $M'_\mu \times R^q_\theta$ is diffeomorphic to $M_\sigma \times R^{k+q}_\theta$.

According to J. H. C. Whitehead [19, Theorem 8] any diffeomorphism $g: M_\sigma \times R^{k+q}_\theta \longrightarrow N_\lambda$ can be approximated by a PL-homeomorphism

$$g_1 : M \times R^{k+q} \longrightarrow N \quad .$$

Clearly the induced microbundle $g_1^* \tau_N$ is isomorphic to $\tau_{M \times R^{k+q}}$. Since the diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & M' \\ \downarrow{\times 0} & & \uparrow{p_1} \\ M \times R^{k+q} & \xrightarrow{g \text{ or } g_1} & N \subset M' \times R^q \end{array}$$

is homotopy commutative, this implies that

$$f^* \tau_{M'} \oplus \varepsilon^q \approx \tau_M \oplus \varepsilon^{k+q} \quad ,$$

which proves the first part of Lemma 5.

If the dimension $m + k$ of M' is $\geq 2m + 1$, then f can actually be approximated by a smooth imbedding of M_σ into M'_μ itself (rather than $M'_\mu \times R^q$).

<u>Proof</u>. Let $f_0 : M \longrightarrow M'$ be a simplicial approximation to f ; and let f_1 be a differentiable approximation to f_0 so that

a) f_1 is a one-one immersion of M_σ in M'_μ and

b) the image $f_1(M)$ is disjoint from the m-skeleton of M'.

This is certainly possible if $m + k \geq 2m + 1$. (Compare Whitney [23].) Now if the approximation is sufficiently close, then the limit set of f_1 will equal the limit set of f_0. Since this is contained in the m-skeleton it will be disjoint from $f_1(M)$. Therefore f_1 will be an imbedding.] The argument above now shows that $f^*\tau_{M'} \approx \tau_M \oplus \varepsilon^k$; which proves Lemma 5.

Proof of Theorem 6. If M possesses a smoothing σ, then since $t(U_\mu)$ is an m-universal bundle, the sum $t(M_\sigma) \oplus \varepsilon_M^{n-m}$ is isomorphic to $f^*t(U_\mu)$ for some $f\colon M \longrightarrow U$. Hence by Lemma 5.

$$f^*\tau_U \approx \tau_M \oplus \varepsilon_M^{n-m}.$$

Conversely if $f^*\tau_U \approx \tau_M \oplus \varepsilon_M^{n-m}$ then Lemma 4 asserts that M is smoothable. This completes the proof of Theorem 6.

Using the universal bundle theorem (§ 5), Theorem 6 can be reformulated as follows. Let

$$c\colon U \longrightarrow B(PL_n)\ , \quad c'\colon M \longrightarrow B(PL_n)$$

be classifying maps for the microbundles τ_U and $\tau_M \oplus \varepsilon_M^{n-m}$ respectively.

Corollary 6.1. M *is smoothable if and only if there exists a map* $f\colon M \longrightarrow U$ *so that the diagram*

$$\begin{tikzcd} & U \arrow[dr, "c"] & \\ M \arrow[ur, dashed, "f"] \arrow[rr, "c'"'] & & B(PL_n) \end{tikzcd}$$

is homotopy commutative.

With this formulation it is natural to make use of obstruction theory. Using a mapping cylinder construction, one can assume that c imbeds U as a subcomplex of $B(PL_n)$.

Corollary 6.2. M <u>is smoothable if and only if a sequence of obstructions</u>

$$o_i \in H^i(M; \pi_i(B(PL_n),U))$$

<u>are all equal to zero</u>.

The proof is standard. (As is usual in obstruction theory, the higher obstructions need not be well defined.)

On the other hand Munkres [13] has defined a sequence of obstructions

$$\alpha_*[\lambda \underline{f}] \in \mathcal{H}_{m-i}(M ; \Gamma_{i-1}) \approx H^i(M ; \Gamma_{i-1})$$

whose vanishing implies that M can be given a (not necessarily compatible) differentiable structure. A similar theory has been outlined by Thom [18]. <u>This suggests the conjecture that the relative</u> homotopy group $\pi_i(B(PL_n),U)$, $n >> i$, <u>is isomorphic to the group</u> Γ_{i-1} <u>considered by Thom and Munkres</u>. Since the Γ_{i-1} are now known to be finite groups, this conjecture would yield quite a bit of information about the $B(PL_n)$.

Still another formulation can be given as follows. Recall that the s-classes of PL-microbundles over a finite dimensional complex B form an abelian group $k_{PL}(B)$. Similarly the s-classes of vector bundles over B form an abelian group $k_O(B)$. Define a natural transformation T from the functor k_O to the functor k_{PL} as follows.

Given a vector bundle v of dimension q over B, choose a PL-manifold M with smoothing σ, and a PL-map $f: B \longrightarrow M$, so that

$$v \oplus e_B^{m-q} \approx f^*t(M_\sigma).$$

(For example one could choose M_σ to be a universal manifold.) Now define

$$T(v) = (f^*\tau_M).$$

This s-class $T(v)$ does not depend on the choice of M_σ and f. For if U_μ is a k-universal manifold, where $k > \mathrm{Max}(m, \dim B)$, then there are maps

$$c : M \longrightarrow U \;, \quad c_1 : B \longrightarrow U \;,$$

unique up to homotopy, so that

$$c^*t(U_\mu) \approx t(M_\sigma) \oplus e^{n-m} \;, \quad c_1^*t(U_\mu) \approx v \oplus e^{n-q}.$$

Thus cf is homotopic to c_1 ; and $(c^*\tau_U) = (\tau_M)$ by Lemma 5. Hence

$$(c_1^*\tau_U) = (f^*c^*\tau_U) = (f^*\tau_M).$$

Therefore $T(v)$ is well defined.

For a Whitney sum $v \oplus v'$ one can use the product $M_\sigma \times M'_{\sigma'}$ of two suitible manifolds to show that

$$T(v \oplus v') = T(v) + T(v').$$

Since T clearly commutes with mappings, and carries trivial bundles into zero, this proves the following.

Lemma 6. <u>The homomorphisms</u>

$$T : k_O(B) \longrightarrow k_{PL}(B)$$

constitute a natural transformation from the functor k_O to the functor k_{PL}.

Now Theorem 6 can be reformulated in a new way:

Corollary 6.3. The PL-manifold M is smoothable if and only if the s-class τ_M lies in the image of the homomorphism

$$T : k_O(M) \longrightarrow k_{PL}(M).$$

In fact a given s-class $(v) \in k_O(M)$ contains the tangent vector bundle $t(M_\sigma)$ of some smoothing of M if and only if $(v) \in T^{-1}(\tau_M)$.

The proof is straightforward, making use of Lemma 4.

As a final consequence of Theorem 6:

Corollary 6.4. Suppose that for some finite complex B the homomorphism

$$T : k_O(B) \longrightarrow k_{PL}(B)$$

has a non-trivial kernel. Then the tangent vector bundle of any manifold having the homotopy type of B is not a topological invariant. For example a suitable open subset M of euclidean space can be given a new smoothing σ so that M_σ is not parallelizable.

The proof is immediate.

In a subsequent paper it will be shown that this phenominon actually occurs. For example if B consists of a 7-sphere with an 8-cell attached by a map of degree 7, then the homomorphism

$$T : k_O(B) \longrightarrow k_{PL}(B)$$

is zero. However the group $k_O(B)$ is cyclic of order 7 generated by any s-class (v) with Pontrjagin class $p_2(v) \neq 0$.

5. The universal bundle theorem.

This section will construct a classifying space $B(PL_n)$ for PL-microbundles of dimension n.

It is first necessary to define the concept of an "isomorphism-germ" between microbundles. Let

$$\xi_\alpha : \qquad B \xrightarrow{i_\alpha} E_\alpha \xrightarrow{j_\alpha} B \quad , \qquad \alpha = 1, 2$$

be two PL-microbundles over B. Recall that ξ_1 and ξ_2 are isomorphic if there exist neighborhoods U_α of $i_\alpha(B)$ in E_α for $\alpha = 1, 2$, and a PL-homeomorphism $f\colon U_1 \to U_2$ so that the diagram

$$\begin{tikzcd} & U_1 \arrow[dd, "f"] \arrow[dr] & \\ B \arrow[ur] \arrow[dr] & & B \\ & U_2 \arrow[ur] & \end{tikzcd}$$

is commutative.

Definition. Two such PL-homeomorphisms, f and

$$f'\colon U_1' \to U_2'$$

define the same <u>isomorphism-germ</u> F from ξ_1 to ξ_2, if the two maps coincide on some sufficiently small neighborhood of $i_1(B)$. (Thus an isomorphism-germ

$$F\colon \xi_1 \to \xi_2$$

is an equivalence class of such PL-homeomorphisms.)

Now consider the bundles $g^*\xi_1$ and $g^*\xi_2$ induced by some PL-map $g\colon B' \to B$. Any isomorphism-germ $F\colon \xi_1 \to \xi_2$ clearly gives rise to an isomorphism-germ $g^*\xi_1 \to g^*\xi_2$. This induced isomorphism-germ will be denoted by g^*F.

$g^*\xi_1 \longrightarrow g^*\xi_2$. This induced isomorphism-germ will be denoted by g^*F.

For each integer n, construct a c.s.s. group complex[2] PL_n as follows. Let Δ_k denote the standard ordered k-simplex. As usual let $\varepsilon^n_{\Delta_k}$ denote the trivial microbundle

$$\Delta_k \xrightarrow{\times 0} \Delta_k \times R^n \xrightarrow{p_1} \Delta_k .$$

<u>Definition</u>. A k-simplex F of the c.s.s. complex PL_n is an isomorphism-germ $F: \varepsilon^n_{\Delta_k} \longrightarrow \varepsilon^n_{\Delta_k}$.

The operation of composing isomorphism-germs makes the set $PL_n^{(k)}$ of k-simplexes into a group. For each monotone simplicial map $\lambda: \Delta_\ell \longrightarrow \Delta_k$ define a homomorphism

$$\lambda^{\#}: PL_n^{(k)} \longrightarrow PL_n^{(\ell)}$$

as follows. Let $\lambda^{\#}$ carry each isomorphism-germ F to the induced isomorphism-germ λ^*F. Thus $PL_n = (PL_n^{(k)}, \lambda^{\#})$ is a c.s.s. group complex.

<u>Note</u>: PL_n seems to play a role for PL-manifolds which is analogous to the role of the orthogonal group $O(n)$ in the theory of differentiable manifolds. Roughly speaking PL_n may be thought of as the singular complex of the group of germs of PL-automorphisms of the pair $(R^n,0)$.

Now consider a PL-microbundle ξ of dimension n over a simplicial complex B. Choose some ordering for the vertices of B.

---

[2] For the theory of c.s.s. (complete semi simplicial) complexes, see for example Moore [11], Heller [7].

Definition. The associated principal bundle $\tilde{\xi}$ is the c.s.s. principal bundle with group PL_n which is constructed as follows. The base space $\tilde{B}$ is the c.s.s. complex consisting of all monotone simplicial maps $f: \Delta_k \longrightarrow B$; with $\lambda^{\#}: \tilde{B}^{(k)} \longrightarrow \tilde{B}^{(\ell)}$ defined by $\lambda^{\#}f = f \circ \lambda$. A k-simplex of the total space $\tilde{E}$ consists of

1) a k-simplex $f \in \tilde{B}^{(k)}$, together with

2) an isomorphism-germ $F: \varepsilon^n_{\Delta_k} \longrightarrow f^*\xi$.

The functions $\lambda^{\#}: \tilde{E}^{(k)} \longrightarrow \tilde{E}^{(\ell)}$ are defined by the formula $\lambda^{\#}(f,F) = (f \circ \lambda, \lambda^*F)$. The right translation function

$$\tilde{E} \times PL_n \longrightarrow \tilde{E}$$

is just the operation of composing isomorphism germs. Since each group $PL_n^{(k)}$ permutes the set $\tilde{E}^{(k)}$ freely, with orbit set $\tilde{B}^{(k)}$; it follows that $\tilde{\xi}$ is a principal PL_n-bundle.

Lemma 7. Let B be locally finite. Then two PL-microbundles ξ, η over B are isomorphic if and only if the associated c.s.s. principal bundles $\tilde{\xi}$, $\tilde{\eta}$ are isomorphic.

Proof. Suppose that an isomorphism $\iota: \tilde{\xi} \longrightarrow \tilde{\eta}$ of c.s.s. bundles is given. In other words, to each monotone simplicial map $f: \Delta_k \longrightarrow B$ and each isomorphism-germ $F: \varepsilon^n_{\Delta_k} \longrightarrow f^*\xi$ there is assigned an isomorphism-germ $\iota(F): \varepsilon^n_{\Delta_k} \longrightarrow f^*\eta$. Note that the composition

$$\iota(F)F^{-1} : f^*\xi \longrightarrow f^*\eta$$

does not depend on the choice of F.

For each k-simplex Σ of B let f be the unique element of $\tilde{B}^{(k)}$ which maps Δ_k onto Σ. then there exists a unique isomorphism-germ

$$I_\Sigma : \xi|\Sigma \longrightarrow \eta|\Sigma$$

so that f^*I_Σ: $f^*\xi \longrightarrow f^*\eta$ is equal to $\iota(F)F^{-1}$. If Σ' is a face of Σ, then it is easily checked that $I_\Sigma|\Sigma' = I_{\Sigma'}$. Now, using the fact that B is locally finite, it follows that these isomorphism-germs I_Σ piece together to yeild the required isomorphism-germ

$$I : \xi \longrightarrow \eta \quad .$$

Lemma 8. <u>Again let</u> B <u>be locally finite.</u> <u>Then any principal</u> PL_n-<u>bundle</u> π <u>over</u> $\tilde{B}$ <u>is isomorphic to</u> $\tilde{\xi}$ <u>for some microbundle</u> ξ <u>over</u> B.

<u>Proof</u>. Construct ξ as follows. For each k-simplex Σ of B choose a k-simplex $[\Sigma]$ in the total space of π which lies over the corresponding simplex f: $\Delta_k \longrightarrow \Sigma \subset B$ of $\tilde{B}^{(k)}$. Passing to the i-th face $\partial_i\Sigma$ note that the two (k - 1)-simplexes $[\partial_i\Sigma]$ and $\partial_i[\Sigma]$ both lie over the same simplex $\partial_i f$ of $\tilde{B}^{(k-1)}$. Therefore

$$\partial_i[\Sigma] = [\partial_i\Sigma]\cdot F$$

for some uniquely defined

$$F = F(i,\Sigma) \in PL_n^{(k-1)} \quad .$$

Now form the topological sum of all of the simplexes Σ of B, and take the trivial microbundle ε^n_Σ over each Σ. Paste these all together, identifying each $\varepsilon^n_{\partial_i\Sigma}$ with $\varepsilon^n_\Sigma|\partial_i\Sigma$ using the isomorphism-germ $F'(i,\Sigma)$: $\varepsilon^n_{\partial_i\Sigma} \longrightarrow \varepsilon^n_\Sigma|\partial_i\Sigma$ which corresponds to

$$F(i,\Sigma) : \varepsilon^n_{\Delta_{k-1}} \longrightarrow \varepsilon^n_{\Delta_{k-1}}$$

under the PL-homeomorphism $\partial_i f\colon \Delta_{k-1} \longrightarrow \partial_i\Sigma$. It is not difficult to verify that these identifications are compatible, when one passes to a face of a face of Σ. Therefore, using the fact that B is locally finite, we see that the identification space yields a PL-microbundle ξ over B. Furthermore $\tilde{\xi}$ is isomorphic to π. This proves Lemma 8.

According to A. Heller [7] (see also MacLane [9], Moore [12]) for any c.s.s. group complex G there exists a "classifying complex", say $\overline{W}(G)$, with the following property. Any principal G-bundle over any c.s.s. complex K is induced by a unique homotopy class of maps $K \longrightarrow \overline{W}(G)$. The next two lemmas will be used to show that the c.s.s. complex $\overline{W}(PL_n)$ has the homotopy type of some locally finite simplicial complex.

Lemma 9. _Given a PL-map_ $f\colon X \longrightarrow Y$ _between finite simplicial complexes, there exist rectilinear subdivisions_ X' _of_ X _and_ Y' _of_ Y _so that the induced map_ $X' \longrightarrow Y'$ _is simplicial_.

Proof. First choose a subdivision X_1 of X so that each simplex of X_1 maps linearly into a simplex of Y. Next choose a subdivision Y_1 of Y so that, for each simplex Σ of X_1, the image $f(\Sigma)$ is a subcomplex of Y_1. (The finiteness of X is used for this step.)

For each simplex Σ of X_1 and each simplex Δ of Y_1 consider the convex cell $\Sigma \cap f^{-1}(\Delta)$. These cells form a cell-subdivision X_2 of X_1. Let X_3 be the first barycentric subdivision of X_2.

The new vertex v which is selected in each cell $\Sigma \cap f^{-1}(\Delta)$ must be chosen so that $f(v)$ is the barycenter of Δ. Let Y_2 be the first barycentric subdivision of Y_1. Then $f: X_3 \longrightarrow Y_2$ is clearly a simplicial map.

Lemma 10. <u>There are at most a countable number of non-isomorphic PL-microbundles over a finite complex</u> B. <u>If</u> B <u>is a circle then there are exactly two isomorphism classes of</u> n-<u>dimensional bundles over</u> B.

<u>Proof</u>. For any microbundle

$$\xi: \qquad B \xrightarrow{i} E \xrightarrow{j} B$$

it may be assumed that the total space E is also a finite complex, and that iB is a subcomplex of E. Choose subdivisions E', B' so that j is a simplicial map. Then i will automatically be simplicial. Hence the subdivided microbundle ξ' can be completely described by a finite scheme of incidence relations and mappings. Thus there are only a countable number of such bundles ξ'. Since $\xi \approx f^*\xi'$ where $f: B \longrightarrow B'$ is the identity map, and since there are also only a countable number of homotopy classes of maps $B \longrightarrow B'$. It follows that there are only countable many ξ, up to isomorphism.

Any bundle over a circle can be obtained from the trivial bundle over a line segment by matching the end fibres. It follows from Gugenheim [6, Theorem 3] that there only two essentially different ways of doing this. Thus there are only two bundles over a circle (namely the trivial bundle and the non-orientable bundle.) This pfoves Lemma 10.

It follows that the homotopy groups $\pi_i \overline{W}(PL_n)$ are all countable. Now using [21, Theorem 13] it is not hard to show that $\overline{W}(PL_n)$ has the homotopy type of a locally finite simplicial complex $B(PL_n)$.

A homotopy equivalence $\tilde{B}(PL_n) \longrightarrow \overline{W}(PL_n)$ induces a principal PL_n-bundle over $\tilde{B}(PL_n)$; and therefore gives rise to a PL-microbundle γ^n over $B(PL_n)$. Clearly γ^n is a universal n-dimensional microbundle. This completes the proof of the universal bundle theorem (Theorem 2).

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