Some free actions of cyclic groups on spheres

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Let $p \geq 5$ be prime and let $n \geq 5$ be odd. <u>This note will show that the cyclic group Π of order p can act differentiably on the n-sphere, without fixed points, in infinitely many different ways.</u> These actions are "different" in the sense that the corresponding quotient manifolds $M = S^n/\Pi$ can be distinguished by their Reidemeister-Franz-de Rham torsion invariants. Hence two such "different" manifolds M, M' cannot have the same simple homotopy type, cannot be piecewise-linearly homeomorphic, and cannot be diffeomorphic. (It is not known whether or not M and M' can be homeomorphic.)

First let me review the basic properties of the torsion invariant, following [3], [4]. Let K be a finite, connected CW-complex and let Π denote the fundamental group of K. Let

$$f: Z[\Pi] \longrightarrow \mathbb{C}$$

be a ring homomorphism from the integral group ring to the complex numbers. If the homology groups $H_i(K; \mathbb{C}_f)$ are all zero (homology with local coefficients twisted by f) then the torsion invariant $\Delta_f \tilde{K} \in \mathbb{C}_0/\pm f\Pi$ is defined. (Here $\tilde{K}$ denotes the universal covering complex, $\mathbb{C}_0$ the multiplicative group of non-zero complex numbers, and $\pm f\,\Pi$ the subgroup generated by $f(\Pi)$ and ± 1.) To simplify the notation we will henceforth leave off the tilde, and write simply $\Delta_f K$.

Similarly, given a pair K, L with $H_*(K, L; C_f) = 0$ the torsion $\Delta_f(K, L)$ is defined. This satisfies the identity

$$(1) \qquad \Delta_f(K, L) = \Delta_f K/\Delta_f L ,$$

providing that the three terms are defined. (If two out of three are defined, then the third is automatically defined.)

If W is a triangulated manifold of dimension n with boundary bW, then the following duality theorem holds. We must assume that $|f(t)| = 1$ for $t \in \Pi = \pi_1(W)$. Then

$$(2) \qquad \Delta_f(bW) = (\Delta_f W)(\overline{\Delta}_f W)^{\varepsilon(n)}$$

where $\overline{\Delta}$ denotes the complex conjugate and $\varepsilon(n) = (-1)^n$. We will also need the following variant form. If M is a triangulated manifold without boundary of dimension $n - 1$ then

$$(3) \qquad \Delta_f M = (\Delta_f M)^{\varepsilon(n)} .$$

Now consider an h-cobordism $(W; M, M')$. That is, assume that W is a smooth manifold with boundary $M + M'$, and that both M and M' are deformation retracts of W. Choosing a C^1-triangulation of $(W; M, M')$ we will assume that the torsion

$$\Delta_f M \in C_0/\pm f\,\Pi$$

is defined.

Lemma 1. _Then_ $\Delta_f M'$ _is defined, and equal to_ $(\Delta_f M)\,\Delta_f(W, M)(\bar{\Delta}_f(W, M))^{\varepsilon(n)}$.

Proof. Since M is a deformation retract of W it is clear that $\Delta_f(W, M)$ is defined. Thus $\Delta_f W$ is defined, and similarly $\Delta_f M'$ is defined. Consider the duality statement

$$\Delta_f(bW) = (\Delta_f W)(\bar{\Delta}_f W)^{\varepsilon(n)} .$$

Since $\Delta_f(bW) = (\Delta_f M)(\Delta_f M')$ and since $\Delta_f W = (\Delta_f M)\,\Delta_f(W, M)$, this can be rewritten as

$$(\Delta_f M)(\Delta_f M') = (\Delta_f M)\,\Delta_f(W, M)(\bar{\Delta}_f M)^{\varepsilon(n)}(\bar{\Delta}_f(W, M))^{\varepsilon(n)} .$$

Now dividing through by

$$\Delta_f M = (\bar{\Delta}_f M)^{\varepsilon(n)}$$

we obtain the required formula

$$\Delta_f M' = (\Delta_f M)\,\Delta_f(W, M)(\bar{\Delta}_f(W, M))^{\varepsilon(n)} .$$

Henceforth we will assume that the dimension n of W is even. Thus Lemma 1 can be rewritten in the form

(4) $$\Delta_f M' = (\Delta_f M)|\,\Delta_f(W, M)|^2 .$$

Suppose that we are given the manifold M with fundamental group Π , and wish to construct the h-cobordism $(W; M, M')$.

Lemma 2 (Stallings). *If* $\dim(M) \geq 5$ *then the h-cobordism* $(W; M, M')$ *can be constructed so that* $\Delta_f(W, M)$ *is equal to the image, in* $C_0/\pm f\,\Pi$, *of any unit of the ring* $Z[\Pi]$.

Proof. Stallings actually observes that the h-cobordism can be constructed so that the Whitehead torsion invariant $\tau(W, M)$ is any desired element of the Whitehead group

$$Wh(\Pi) = GL(\infty, Z[\Pi])/(\text{Commutators}, \pm \Pi).$$

(See Stallings [6,§2].) In particular if u is a unit of $Z[\Pi]$ then W can be chosen so that $\tau(W, M)$ is the element of $Wh(\Pi)$ corresponding to the matrix

$$\begin{pmatrix} u & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \in GL(\infty, Z[\Pi]) .$$

It is then clear that $\Delta_f(W, M)$ is equal to the image of u in $C_0/\pm f\,\Pi$. (Compare Cockcroft [1], or [3, pg. 589].) This completes the proof.

Thus in order to construct examples of h-cobordisms, we need only look for units in $Z[\Pi]$. To be more specific, let us now assume that Π is cyclic of order p with generator t. Define $f\colon Z[\Pi] \longrightarrow C$ by $f(t) = \exp(2\pi i/p)$.

Lemma 3 (Higman). *If* $p \geq 5$ *is an integer of the form* $6k \pm 1$ *then* $Z[\Pi]$ *contains a unit* u *with* $|f(u)| \neq 1$.

<u>Proof.</u> This follows easily from Higman [2]. Alternatively, here is a direct proof. Let

$$u = t + t^{-1} - 1$$

so that $f(u) = 2\cos(2\pi/p) - 1 \neq \pm 1$. To see that u is a unit it is only necessary for the reader to verify the identity

$$u(1 + t - t^3 - t^4 + t^6 + t^7 - - + + \ldots - - + t^{p-1}) = 1$$

for $p \equiv 1 \pmod 6$; or

$$u(-1 + t^2 + t^3 - t^5 - t^6 + + - - \ldots - - + t^{p-3} + t^{p-2}) = 1$$

for $p \equiv -1 \pmod 6$. This completes the proof.

Now combining the three lemmas we have the following.

Theorem. <u>Let M be a smooth manifold of odd dimension ≥ 5 whose fundamental group is cyclic of order $p = 6k \pm 1$, $p \geq 5$. Then there exist infinitely many manifolds $M_1, M_2, M_3, \ldots$ which are h-cobordant to M, but such that no two have the same simple homotopy type.</u>

<u>Proof.</u> For each integer m we can choose the h-cobordism $(W_m; M, M_m)$ so that

$$|\Delta_f(W_m, M)| = |f(u^m)| .$$

Then
$$\Delta_f M_m = (\Delta_f M)|f(u)|^{2m}.$$

Since $|f(u)| \neq 0,1$ the real numbers $|\Delta_f M_m|$ are all distinct. This does not yet prove that the M_m all have distinct simple homotopy types, since the invariant $|\Delta_f M_m|$ depends on the choice of f. But there are only finitely many homomorphisms from $Z[\Pi]$ to $\mathbb{C}$, so out of the infinite sequence $M_1, M_2, \ldots$ one can certainly extract an infinite subsequence consisting of pairwise distinct manifolds. This completes the proof.

In particular let us apply this theorem to a Lens space

$$L = S^{2k-1}/\Pi .$$

The resulting h-cobordant manifolds $L_1, L_2, \ldots$ will all have universal covering spaces diffeomorphic to the sphere. (See Smale [5].) Thus we have infinitely many distinct free actions of the cyclic group Π on S^{2k-1}. But there are only finitely many orthogonal actions of Π on S^{2k-1}. Thus we have:

Corollary. *For $2k - 1 \geq 5$ and p prime ≥ 5 there exist infinitely many smooth fixed point free actions of the cyclic group of order p on S^{2k-1} which are not smoothly equivalent to orthogonal actions.*

It would be interesting to know whether any corresponding phenomenon occurs in dimension 3.

References

1. W. H. Cockcroft, Simple homotopy type torsion and the Reidemeister-Franz torsion, Topology, 1 (1962), 143-150.

2. G. Higman, The units of group rings, Proc. Lond. Math. Soc., 46 (1940), 231-248.

3. J. Milnor, Two complexes which are homeomorphic but combinatorially distinct, Annals of Math., 74 (1961), 575-590.

4. ____________, A duality theorem for Reidemeister torsion, Annals of Math., 76 (1962), 137-147.

5. S. Smale, On the structure of manifolds, Amer. J. Math., 84 (1962), 387-399.

6. J. Stallings, On infinite processes leading to differentiability in the complement of a point, to appear.