Lectures on Homology Operations

by

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Contents

Introduction and historical comments

These notes are chapter 2 of the lecture notes of a course I gave at Brown University in the spring of 1976. This chapter is the heart of the course--the geometry of maps of polyhedral cycles to Euclidean spaces. Chapter 1 will discuss maps of smooth surfaces to the plane and to 3-space, centering on the work of Whitney. Chapter 3 will show how the constructions of chapter 2 can be used to give geometric descriptions of the dual Steenrod operations in mod 2 homology.

My discussion of double points of cycles generalizes that of Hudson for manifolds [H]. The geometry of Steenrod homology operations was announced in [M1] and developed in [M2] and [HM].

I begin with a summary of definitions of polyhedral topology. The main idea is that of "localizing" a polyhedral map f at a point x to obtain a <u>link map</u> Lf_x . The global topology of f will be described using the topology of its links Lf_x . This is an old idea which seems very natural from Rourke and Sanderson's definition of a polyhedron as a locally conical space [RS].

The second section is about <u>spreading out</u>, usually called "general position". This is also an old technique, given its modern form by Zeeman [Z].

In the next section I give a geometric definition of singular homology. This idea is due to Sullivan. Similar interpretations of his idea occur in [A], [RS], [BRS], and [M2].

The fourth section contains a generalization of the classical simplicial definition of the degree of a map. The concept of an irreducible cycle is borrowed from algebraic geometry.

Next a definition of linking numbers is given using the degree (cf. [AH], [Mi]), and intersection numbers are defined using local linking numbers. This is a relatively undeveloped idea of Lefschetz [L].

After these preliminaries come the double point and branch point cycles. (The name "branch point" for a nonimmersion point was coined by Zeeman [Z].) Let $f : X^k \to R^n$ be a spread-out map of a k-cycle to Euclidean n-space with $k \leq n$. I show that the set of double points of f in X is a cycle, provided that double points are counted with multiplicities defined using local linking numbers. Any two such maps have homologous double point cycles, so the homology class of double point cycles of spread-out maps $X^k \to R^n$ is an intrinsic invariant of X . Next the analogous results are proved for branch points. Finally I show that the double point class and the branch point class are equal, using an idea of Tom Banchoff [B].

At the end of each section are some illustrations and exercises. Statements in these notes which are not proved are intended to be exercises for the reader.

In lieu of chapter 1, I recommend chapter 1 of Seifert and Threlfall [ST] and the papers of Whitney [W] and Banchoff [B] on characteristic cycles, as a geometric introduction to chapter 2.

In chapter 3 I will give an axiomatic characterization of the double point class [HM], give a combinatorial formula for it [BM], and discuss its relation with Smith operations [Wu].

Many thanks go to Dave Damiano and James Stormes, who did most of the work of writing up these notes.

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§1. The polyhedral category

This section is a brief introduction to the category of polyhedra and piecewise-linear maps. It contains all the background material necessary for understanding this chapter. For a more detailed and more general discussion of the PL category, see [RS].

By R^n we will mean Euclidean n-space. R^n embeds in R^{n+1} in the first n coordinates $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. A k-plane in R^n , $k \leq n$, is a translated k-dimensional vector subspace of R^n . The span of $\{x_0, \ldots, x_k\} \subset R^n$ is the smallest plane containing $\{x_0, \ldots, x_k\}$. In terms of coordinates, $\mathrm{span}\{x_0, \ldots, x_k\} = \{x \mid x = t_0 x_0 + \ldots + t_k x_k \text{ , } t_i \in R \text{ , } \sum_i t_i = 1\}$. We say that the points $x_0, \ldots, x_k$ are independent if their span has dimension k or, equivalently, if $x_1 - x_0, \ldots, x_k - x_0$ are linearly independent vectors.

A k-simplex σ is the convex hull of $k + 1$ independent points $x_0, \ldots, x_k$ in R^n . That is, $\sigma = \{x \mid x = \sum_i t_i x_i$, $\sum_i t_i = 1 \text{ , } t_i \geq 0\}$. The points $x_0, \ldots, x_k$ are the vertices of σ . They are uniquely determined by σ . To denote that σ is a k-simplex we will write σ^k . The simplex τ is a face of σ , written $\tau < \sigma$, if all the vertices of τ are vertices of σ . (Notice that $\sigma < \sigma$.) The interior of σ , σ° , consists of all the points of σ which are not in any proper face of σ . (Or, if the dimension of σ is greater than zero, σ° is the topological interior of σ in the plane spanned by the vertices of σ .)

A simplicial complex K is a collection of simplices in R^n which satisfies three conditions. First, if $\sigma, \tau \in K$ and $\sigma \neq \tau$, then $\sigma^\circ \cap \tau^\circ = \emptyset$. Second, if $\sigma \in K$ and $\tau < \sigma$ then $\tau \in K$. Third, if $\tau \in K$, there is a neighborhood U of τ in R^n such that $\{\sigma \in K | \tau < \sigma$ and $\sigma \cap U \neq \emptyset\}$ is finite. The realization of the simplicial complex K is the topological space $|K| = \bigcup_{\sigma \in K} \sigma$.

A polyhedron X is a topological subspace of R^n such that $X = |K|$ for some simplicial complex K. A subpolyhedron of X is a subspace of X which is a polyhedron. The complex K is called a triangulation of X. Notice that $|K|$ is compact if and only if K is finite. X has dimension $\leq k$ if X is the realization of a complex all of whose simplices have dimension $\leq k$.

The complex J in R^n is a subdivision of K if $|J| = |K|$ and each simplex of J is contained in a simplex of K. For example, given $x \in |K|$, there is a subdivision J of K which has x as a vertex.

Proposition 1.1 Let K and L be simplicial complexes in R^n. If $|K| = |L|$, then K and L have a common subdivision. ∎

If A and B are subsets of R^n, their join AB is the set $\{\lambda a + \mu b | a \in A, b \in B; \lambda, \mu \in R; \lambda, \mu \geq 0; \lambda + \mu = 1\}$. AB consists of all points on straight line segments with endpoints in both A and B. If $A = \emptyset$ then we define $AB = B$. The sets A and B are called joinable if for each x in AB there is only one representation of x as $x = \lambda a + \mu b$ with a, b, λ and μ as above.

We write aB for $\{a\}B$. If $\{a\}$ and B are joinable, aB is called the <u>cone</u> with <u>apex</u> a and <u>base</u> B . For example, if $B \subset R^{n-1}$, then using the embedding $R^{n-1} \to R^n$ we have the cones $C_+(B) = (0, \ldots ,0,1)B$ and $C_-(B) = (0, \ldots ,0,-1)B$.

A <u>suspension</u> of a subset B of R^n is the union of two cones aB and $a'B$ whose intersection is B . For example, if $B \subseteq R^{n-1}$, we have the suspension $S(B) = C_+(B) \cup C_-(B)$ (Figure 1.1). If X is a compact polyhedron, and $\{a\}$ and X are joinable in R^n , then the cone aX is a polyhedron. Thus if X is a compact polyhedron, so are $C_+(X)$, $C_-(X)$, and $S(X)$.

If X is a polyhedron and $x \in X$, then a <u>link</u> of x in X , denoted by LX_x , is any compact subpolyhedron of X such that $\{x\}$ and LX_x are joinable and the cone xLX_x is a neighborhood of x in X . More generally, if $\sigma \in K$, $|K| = X$, a link of σ in X is any compact subpolyhedron LX_σ of X such that σ and LX_σ are joinable and σLX_σ is a neighborhood of σ° in X . It follows from proposition 1.2 below that any two links of X in x are (piecewise linearly) homeomorphic.

The <u>simplicial link</u> of $\sigma \in K$ is defined to be the subcomplex $LK_\sigma = \{\tau \in K |\ \sigma \cap \tau = \emptyset$ and there exists $\omega \in K$ with $\sigma < \omega$ and $\tau < \omega\}$.

The realization of LK_σ is a suitable choice for LX_σ (Figure 1.2).

If X is a polyhedron in R^m and Y is a polyhedron in R^n, the map $f : X \to Y$ is <u>polyhedral</u> (or <u>piecewise-linear</u>, or PL) if the graph of f is a polyhedron in $R^{m+n} = R^m \times R^n$. If X is compact, then $f : X \to Y$ is polyhedral if and only if there exist triangulations K of X and L of Y such that for each $\sigma \in K$, $f(\sigma)$ is a simplex of L and $f|\sigma : \sigma \to f(\sigma)$ is the restriction of a linear map from the span of σ to the span of $f(\sigma)$. Equivalently, if $x_0, \ldots, x_k$ are the vertices of $\sigma \in K$, then $f(x_0), \ldots, f(x_k)$ are the vertices (possibly with repetitions) of a simplex of L, and $f(\sum_i t_i x_i) = \sum_i t_i f(x_i)$ for $\sum_i t_i = 1$ and $t_i \geq 0$. In this situation, we will say that f is <u>simplicial</u> with respect to the triangulations K and L.

Let A and B be subsets of R^{n-1}. If $f : A \to B$, define $C_+(f) : C_+(A) \to C_+(B)$ by

$$C_+(f)((0, \ldots, 0, \lambda) + \mu a) = (0, \ldots, 0, \lambda) + \mu f(a) .$$

$C_-(f) : C_-(A) \to C_-(B)$ is defined similarly, and $S(f) : S(A) \to S(B)$ is the union of $C_+(f)$ and $C_-(f)$. If A and B are compact polyhedra, and $f : A \to B$ is PL, then $C_+(f)$, $C_-(f)$, and $S(f)$ are PL.

Now let $f : X \to Y$ be a PL map, and $x \in X$. Suppose that there is a neighborhood U of x in X such that $U \cap f^{-1}(f(x)) = \{x\}$. (If X is compact, this is true for all $x \in X$ if and only if f is <u>finite</u>; i.e., $f^{-1}(y)$ is a finite set for all $y \in Y$.) Then we can choose a link LX_x and a link $LY_{f(x)}$ so that $f(LX_x) \subset LY_{f(x)}$. A <u>link</u> of f at x is then the restriction of f to LX_x ,

$$Lf_x : LX_x \to LY_{f(x)}$$

(Figure 1.3).

<u>Proposition</u> 1.2 Let $f : X \to Y$ be a PL map. If $\ell : L_1 \to L_2$ and $\ell' : L_1' \to L_2'$ are links of f at $x \in X$, there are PL homeomorphisms $h_1 : L_1 \to L_1'$ and $h_2 : L_2 \to L_2'$ such that $\ell' \circ h_1 = h_2 \circ \ell$. ∎

Thus any two choices for Lf_x are polyhedrally equivalent.

<u>Exercises</u>

1. If X and Y are compact polyhedra, then so are $X \cup Y$ and $X \cap Y$.

2. The composition of polyhedral maps is polyhedral.

3. The subspace X of R^n is <u>locally conical</u> if each point $x \in X$ has a cone neighborhood xL in X , where L is compact. Show that X is locally conical if and only if X is a polyhedron (cf. [RS]).

4. Define a locally conical map of locally conical spaces. Show that $f : X \to Y$ is locally conical if and only if f is polyhedral.

5. Describe the possible links of a finite polyhedral map from the plane to itself. Do the same for maps of R^2 to R^3 .

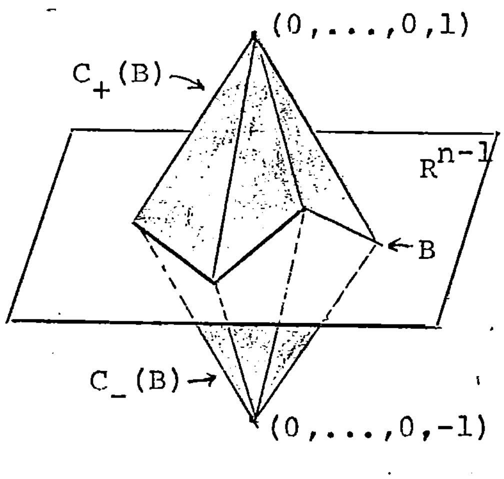


Figure 1.1. $S(B)=C_+(B) \cup C_-(B)$

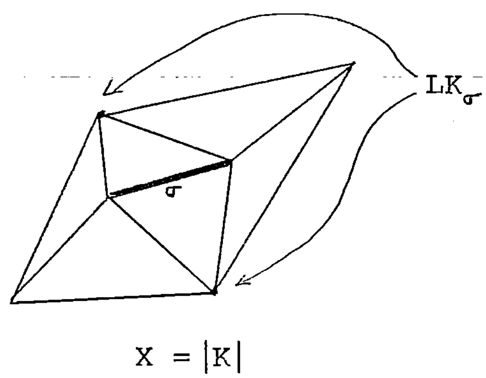


Figure 1.2. $LX_\sigma = |LK_\sigma|$

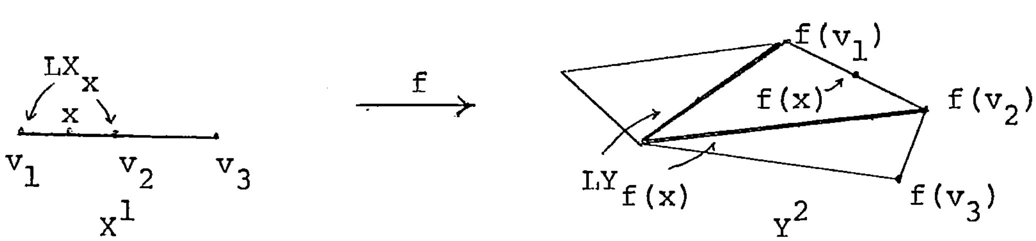


Figure 1.3. $f(LX_x) \subset LY_{f(x)}$

§2. The spreading out theorem

In this section we recall the double point set and the branch point set of a map, and prove the spreading out theorem, which plays an important role in the following theory.

Let $f : A \to B$ be a continuous map of topological spaces with A compact. Let $D^o(f) = \{x \in A \mid \text{there exists } y \neq x \text{ with } f(y) = f(x)\}$. The closure of $D^o(f)$, denoted $D(f)$, is the <u>double point set</u> of f. The closed set $B(f) = \{x \in A \mid \text{for every neighborhood } U \text{ of } x \text{, there exist } y,w \in U,\ y \neq w,\ f(y) = f(w)\}$ is the <u>branch point set.</u> Clearly $D(f) = D^o(f) \cup B(f)$, but this is not necessarily a disjoint union. Examples of branch point sets are the fold set of a stable map of a surface to R^2 and the pinch point set of a stable map of a surface to R^3 (chapter 1).

<u>Proposition</u> 2.1 If X and Y are polyhedra, X compact, and $f : X \to Y$ is a PL map, then $D(f)$ and $B(f)$ are polyhedra.

<u>proof</u>. Let K and L be triangulations of X and Y with respect to which f is simplicial. Then $D^o(f)$ is the union of the interiors of all the simplices σ of K such that there exists $\tau \in K$, $\tau \neq \sigma$, with $f(\tau) = f(\sigma)$. Thus $D(f)$ is a union of simplices of K. Similarly, $B(f)$ is the union of all the simplices $\sigma \in K$ such that there exist $\tau, \omega \in K$, $\sigma < \tau$, $\sigma < \omega$, $\tau \neq \omega$, $f(\tau) = f(\omega)$. ∎

Suppose that $f : X \to Y$ is a PL map from the compact polyhedron X to the polyhedron Y, where $\dim X \leq \dim Y$. The map f is spread-out if

(1) f is finite (i.e. for all $y \in Y$, $f^{-1}(y)$ is finite)

(2) $\dim D(f) \leq 2\dim X - \dim Y$. In other words, the codimension of $D(f)$ $(\dim X - \dim D(f))$ is greater than or equal to the codimension of f $(\dim Y - \dim X)$.

If $\dim X = \dim Y$, condition (2) is vacuous. If $\dim Y > 2\dim X$, then f is spread-out means f is an embedding. If $\dim Y = 2\dim X$, then f is spread-out implies f is an immersion. (Figure 2.1)

There are three versions of the spreading out theorem. We prove the first using the simplicial approximation theorem and a general position argument. The proofs of the second and third versions are refinements of the proof of the first version. (The second will be used to prove 7.3 and 8.3, and the third to prove 9.2.) Let $R^n_+ = \{x_1, \ldots, x_n) \in R^n | x_n \geq 0\}$.

Theorem 2.2 (Spreading-out theorem) Let X be a compact polyhedron of dimension k, and let $g : X \to R^n$ be a continuous map, with $n \geq k$. Then

(a) Given $\epsilon > 0$ there is a spread-out map $f : X \to R^n$ with $||f(x) - g(x)|| < \epsilon$ for every $x \in X$.

(b) If $A \subset X$ is a compact subpolyhedron and $g|A$ is finite, there exists a PL map $f : X \to R^n$ such that $\|f(x) - g(x)\| < \epsilon$ for all $x \in X$, $f|A = g|A$, and $\dim(D(f)\setminus A) \leq 2\dim(X\setminus A) - n$. If $g(X) \subset R^n_+$ and $g(A) \subset R^{n-1}$, then f can be chosen so that $f(X) \subset R^n_+$ and $f^{-1}(R^{n-1}) = A$.

(c) Let $p : R^n_+ \to R^{n-1}_+$ by $p(x_1, \ldots, x_n) = (x_2, \ldots, x_n)$. Again let A be a compact subpolyhedron of X. Suppose that $g(X) \subset R^n_+$, $g(A) \subset R^{n-1}$, $g|A$ is finite, and $p \circ g|A$ is finite. Then there exists a PL map $f : X \to R^n$ such that $\|f(x) - g(x)\| < \epsilon$ for all $x \in X$, $f(X) \subset R^n_+$, $f^{-1}(R^{n-1}) = A$, $f|A = g|A$, $\dim(D(f)\setminus A) \leq 2\dim(X\setminus A) - n$, and $\dim(D(p\circ f)\setminus A) \leq 2\dim(X\setminus A) - n + 1$.

The following is an immediate corollary.

Corollary 2.3 Any compact k-dimensional polyhedron embeds in R^{2k+1} and immerses in R^{2k}.

Remark. The theorem is true with R^n replaced by an arbitrary PL n-manifold, i.e. a polyhedron locally PL homeomorphic with R^n.

proof of (a). By the simplicial approximation theorem there is a PL map $g' : X \to R^n$ such that $\|g'(x) - g(x)\| < \epsilon/2$ for all $x \in X$. Let K be a triangulation of X with respect to which g' is simplicial.

Let $v_o, \ldots, v_m$ be an ordering of the vertices of K. The new map f will be linear on the simplices of K, so f will be determined by its values on the vertices of K. Let

$f(v_o) = g'(v_o)$. Now suppose that $f(v_o), \ldots, f(v_i)$ have already been defined.

Let L_i be the union of all the proper planes in R^n spanned by the subsets of $\{f(v_o), \ldots, f(v_i)\}$. Since $\dim L_i < n$, it follows that $R^n \setminus L_i$ is dense in R^n . Let $f(v_{i+1})$ be any point in $R^n \setminus L_i$ such that $\|f(v_{i+1}) - g'(v_{i+1})\| < \epsilon/2$. Proceeding in this manner we define f on all the vertices of K .

Now we verify that f has the desired properties. Suppose $x \in \sigma°$. If $v_{i0}, \ldots, v_{is}$ are the vertices of σ , then $x = \sum_j t_j v_{ij}$, $\sum_j t_j = 1$, $t_j \geq 0$. Thus

$$
\begin{aligned}
\|f(x) - g'(x)\| &= \|\textstyle\sum_j t_j f(v_{ij}) - \sum_j t_j g'(v_{ij})\| \\
&= \|\textstyle\sum_j t_j (f(v_{ij}) - g'(v_{ij}))\| \\
&< \textstyle\sum_j t_j \epsilon/2 = \epsilon/2 \quad .
\end{aligned}
$$

Thus $\|f(x) - g(x)\| < \epsilon$ for all $x \in X$.

In order to show f is finite we must show that $f|\sigma$ is injective for each $\sigma \in K$. But if $f|\sigma$ were not injective, some vertex of σ would have been mapped to a point in the span of the preceding vertices of σ .

We prove that $\dim D(f) \leq 2\dim X - n$ by showing that if σ is a k-simplex and τ is an ℓ-simplex of K , then $\dim(f(\sigma°) \cap f(\tau°)) \leq k + \ell - n$. This is sufficient since

$f(D(f)) \subset C\ell(\bigcup_{\sigma,\tau}(f(\sigma^\circ) \cap f(\tau^\circ)))$, and f has already been proven finite, so f preserves dimension. Let $v_{i0}, \ldots, v_{is}$ be the vertices of σ and τ with no repetitions. Then since the dimension of the intersection of two linear spaces is the sum of the dimensions of the spaces minus the dimension of their span,

$$\begin{aligned}\dim(f(\sigma^\circ) \cap f(\tau^\circ)) &\leq k + \ell - \dim(\mathrm{span}\{f(v_{i0}), \ldots, f(v_{is})\}) \\ &= k + \ell - \min\{s,n\}\end{aligned}$$

by the construction of f . If $s \leq n$ then the images of the open simplices are disjoint (the <u>closed</u> simplices intersect on a $(k+\ell-n)$-dimensional face) so in general we can say the dimension of intersection is $\leq k + \ell - n$.

<u>Exercises</u>

1. Extend the proof of 2.2(a) to the case when the target is an arbitrary polyhedral manifold.

2. For each $n > 0$ find an example of an n-dimensional polyhedron which does not embed in R^{2n} .

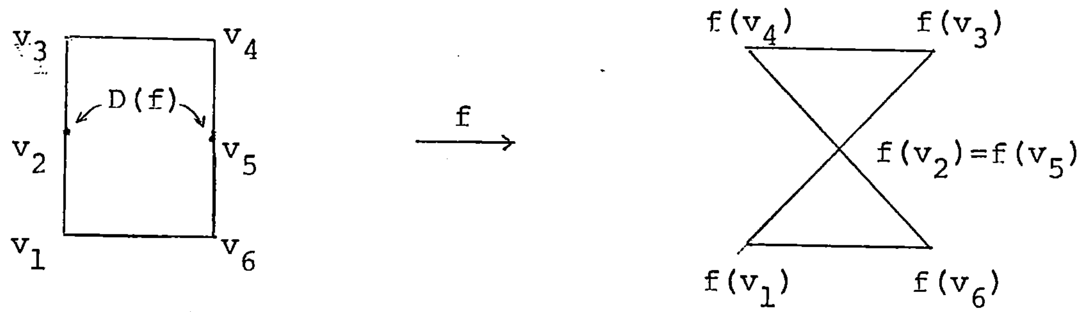


Figure 2.1a. f: $X^1 \longrightarrow R^2$ Spread-out

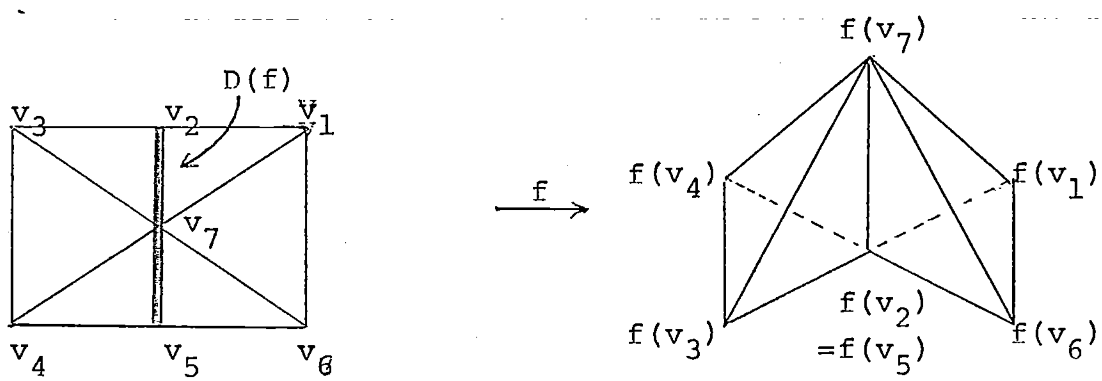


Figure 2.1b. f: $X^2 \longrightarrow R^3$ Spread-out

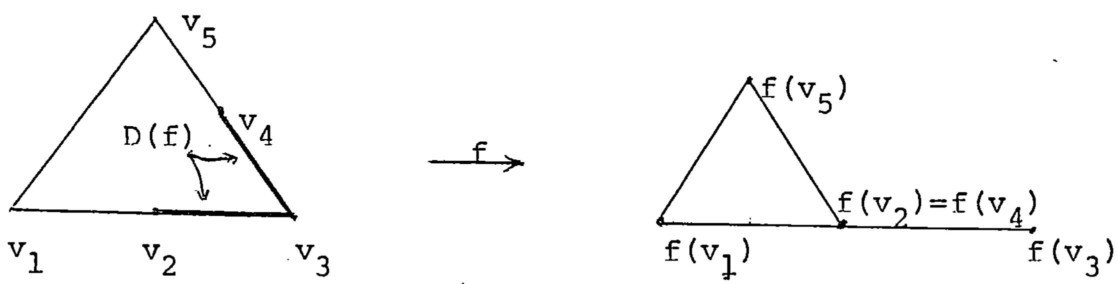


Figure 2.1c. f: $X^1 \longrightarrow R^2$ not Spread-out

§3. Geometric homology

In this section we show how to define the homology of a topological space using maps of polyhedra into the space.

The simplicial complex K is <u>purely</u> k-<u>dimensional</u> if each simplex of K is a face of a k-simplex. If K is purely k-dimensional, the <u>boundary</u> (mod 2) of K is the subcomplex $\partial K = \{\tau \mid \tau < \sigma^{k-1}$, σ^{k-1} is a face of an odd number of k-simplices of $K\}$.

<u>Proposition</u> 3.1 If $|K| = |L|$ and K is purely k-dimensional, then L is purely k-dimensional, and $|\partial K| = |\partial L|$. ∎

Therefore we say that a polyhedron X is purely k-dimensional if $X = |K|$, where K is purely k-dimensional. A polyhedral k-<u>chain</u> (mod 2) is a compact, purely k-dimensional polyhedron. If $X = |K|$ is purely k-dimensional, its <u>boundary</u> (mod 2) is $\partial X = |\partial K|$. A polyhedral k-<u>cycle</u> (mod 2) is a k-chain with empty boundary.

If A is a topological space, we are going to define the k^{th} mod 2 homology of A to be the set of maps $f : X \to A$, where X is a k-cycle, modulo those maps f such that $f = F|X$, where $F : W \to A$, and W is a (k+1)-chain with boundary X .

Suppose that the k-chain X is contained in the k-chain Y . Recall that the <u>frontier</u> of X in Y , or $\mathrm{Fr}(Y,X)$, is the set of points $y \in Y$ such that every neighborhood of y in Y contains points of both X and $Y \setminus X$. Clearly $Fr(Y,X) \subset \partial X$ if and only if $\partial X = Fr(Y,X) \cup (X \cap \partial Y)$ (Figure 3.1).

Now let (A,B) be a pair of spaces. We define $Z_k(A,B)$ to be the set of all continuous maps $f : (X,\partial X) \to (A,B)$, where X is a k-chain, and $B_k(A,B)$ to be the subset of $Z_k(A,B)$ consisting of those maps which are boundaries in the following sense. The map $f : (X,\partial X) \to (A,B)$ is in $B_k(A,B)$ if there is a (k+1)-chain W together with

(1) a polyhedral embedding $e : X \to \partial W$ such that $Fr(\partial W,e(X)) \subset e(\partial X)$, and

(2) a map $F : (W,C\ell(\partial W \setminus e(X))) \to (A,B)$ such that $F \circ e = f$. (Figure 3.2)

$Z_k(A,B)$ is an abelian semigroup under disjoint union, and $B_k(A,B)$ is a subsemigroup.

Definition. $H_k(A,B) = Z_k(A,B)/B_k(A,B)$, the k^{th} mod 2 geometric homology group of (A,B) . The group $H_k(A)$ equals $H_k(A,\emptyset)$.

Note that $H_k(A,B)$ is indeed a group, since if $f : (X,\partial X) \to (A,B)$ is in $Z_k(A,B)$, then $f + f$ is in $B_k(A,B)$, as can be seen by taking $W = X \times I$. If $f \in Z_k(A,B)$, the element of $H_k(A,B)$ determined by f is called the homology class of f . It is denoted by [f].

The <u>boundary morphism</u> $\partial : H_k(A,B) \to H_{k-1}(B)$ is defined by $\partial[f] = [f|\partial X]$.

If $g : (A,B) \to (C,D)$ is a map, then $g_*[f] = H_k(g)[f] = [g \circ f]$. If X is a k-chain, the <u>fundamental class</u> $[X] \in H_k(X,\partial X)$ is the homology class of the identity map $(X,\partial X) \to (X,\partial X)$

<u>Theorem</u> 3.2 The pair (H,∂) satisfies the axioms of Eilenberg and Steenrod [ES]. ∎

Therefore, geometric mod 2 homology is naturally isomorphic with ordinary mod 2 homology on the category of pairs of triangulable spaces. If $X = |K|$ is a k-chain, let $\langle X \rangle$ be the fundamental class of X in the k^{th} simplicial homology group of $(X,\partial X)$. The class $\langle X \rangle$ is represented by the sum of all the k-simplices of K . The transformation from geometric homology to simplicial homology is defined by sending the class of $f : (X,\partial X) \to (A,B)$ to $f_\# \langle X \rangle$, where $f_\#$ is the induced map in simplicial homology.

In fact, H is naturally isomorphic with mod 2 singular homology for all pairs of spaces (D. Sullivan). Integral geometric homology can be defined similarly; it is isomorphic with integral singular homology.

<u>Exercises</u>

1. If X is a k-chain, then $\partial(\partial X) = \emptyset$.

2. How is the definition of $H_k(A,B)$ affected by the removal of condition (1) in the definition of $B_k(A,B)$?

3. Define the suspension homomorphism $\sigma_k : H_k(A,B) \to H_{k+1}(SA,SB)$. Prove that it is an isomorphism. (This can be done directly or by using theorem 3.2.)

4. Show that if X is a polyhedron then every class in $H_k(X)$ is represented by an embedding of a k-cycle in X .

5. Give a similar definition of integral homology.

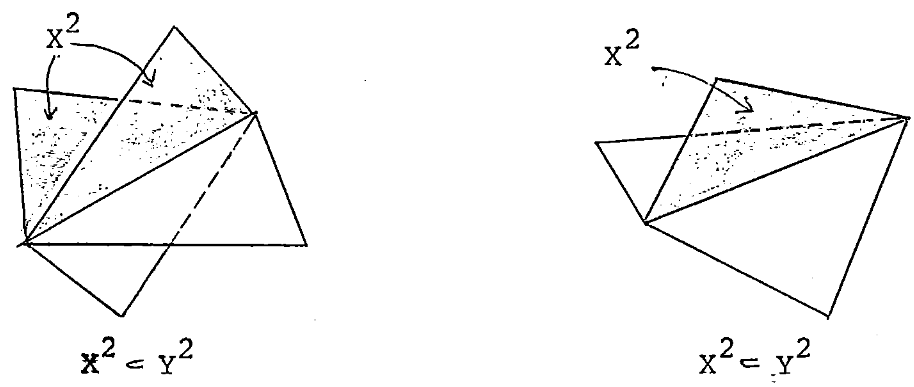


Figure 3.1. $Fr(Y,X) \not\subset \partial X$ and $Fr(Y,X) \subset \partial X$

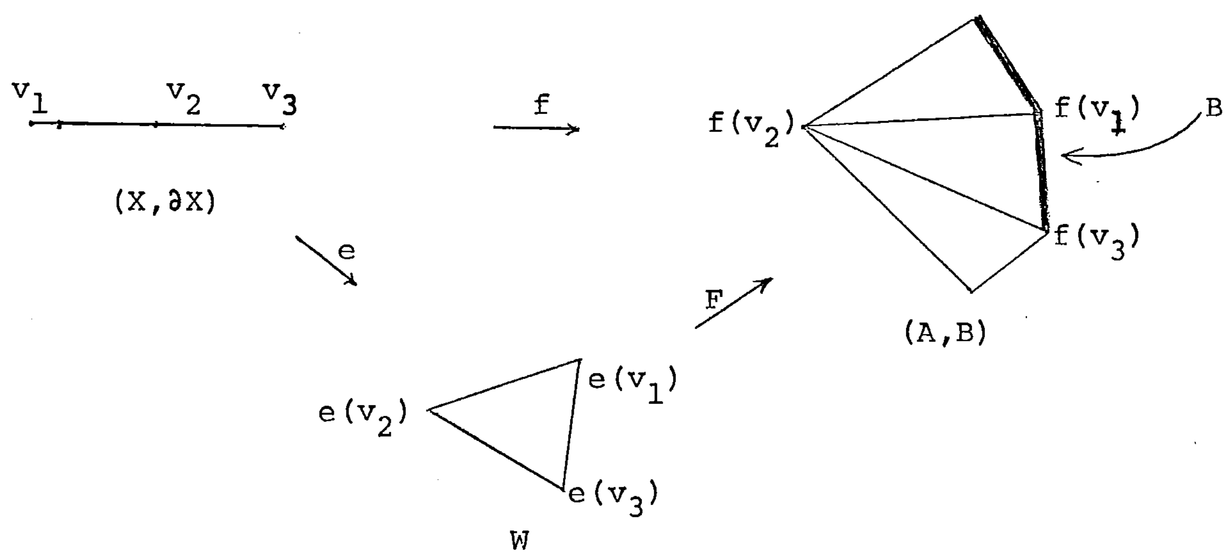


Figure 3.2. A map $f\colon (X,\partial X) \longrightarrow (A,B)$ in $B_1(A,B)$

§4. Degree

The theory of linking and intersection will allow us to distinguish between qualitatively different types of double points. In particular, we will find that we do not want to count certain double points, because of their local linking behavior. If X is a k-cycle, and $f : X \to R^n$ is spread out, $n > k$, then the double points of f which we do count will form a cycle in X (Figure 4.1).

We will carry out part of the program of Lefschetz [L, ch. IV, §6] to define intersection in the homology of a manifold in terms of linking numbers. We adopt the convention that a "number" means an integer mod 2. Much of this section and the next could be extended to the integers by taking orientations into account.

Let Y^k be a k-cycle. Y is irreducible if Y is not the union of two k-cycles properly contained in Y.

Let $f : X^k \to Y^k$ be a PL map of k-cycles, with Y^k irreducible. The degree of f may be defined as follows. Choose triangulations of X and Y with respect to which f is simplicial. Let σ be a k-simplex of Y and $y \in \sigma^\circ$. The degree of f is then the number (mod 2) of points of $f^{-1}(y)$. This number is well defined, for if we let

$$Y_0 = C\ell\{y \in Y | f^{-1}(y) \text{ has an even number of points}\} \quad ,$$
$$Y_1 = C\ell\{y \in Y | f^{-1}(y) \text{ has an odd number of points}\} \quad ,$$

then Y_0 and Y_1 are k-cycles and $Y = Y_0 \cup Y_1$, so either $Y = Y_0$ or $Y = Y_1$.

Proposition 4.1 Let $f : X^k \to Y^k$ be a PL map of k-cycles, with Y irreducible. If $f \in B_k(Y)$, then the degree of f is zero.

proof. Let W be a (k+1)-chain with $\partial W = X$, and $F : W \to Y$ a PL map with $F|X = f$. Fix triangulations of W and Y for which F is simplicial. If y is a point in the interior of a k-simplex of Y, then $F^{-1}(y)$ is a 1-chain with $\partial F^{-1}(y) = f^{-1}(y)$. As the boundary of a 1-chain must have an even number of points, the degree of f is zero (Figure 4.2). ∎

Now suppose that $g : X^k \to Y^k$ is any continuous map of k-cycles, with Y irreducible. Define the degree of g to be the degree of a PL map homotopic to g. The degree of g is well defined, since if f_0 and f_1 are homotopic PL maps, the relative simplicial approximation theorem implies that f_0 and f_1 are PL homotopic. In other words, there is a PL map $H : X \times I \to Y$ such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for all $x \in X$. Thus proposition 4.1 implies that $\deg(f_0) = \deg(f_1)$.

The degree of a continuous map has several basic properties.

Proposition 4.2

(a) If $f \in B_k(Y^k)$, $\deg(f) = 0$.

(b) If f is homotopic to g , $\deg(f) = \deg(g)$.

(c) $\deg(f + g) = \deg(f) + \deg(g)$.

(d) $\deg(g \circ f) = \deg(g)\deg(f)$.

(e) $\deg(S(f)) = \deg(f)$.

proof. Property (a) follows from proposition 4.1 and the relative simplicial approximation theorem. Properties (c) through (e) are immediate consequences of the definition of degree. (Note that for (d), the target of g = the source of f must be irreducible. Implicit in (e) is the assertion that if Y is irreducible, then SY is irreducible.) Property (b) follows from (a) and (c). ∎

Exercises

1. The k-cycle Y is irreducible if and only if the $\mathbb{Z}/2$ vector space $H_k(Y)$ has dimension one. If $f : X \to Y$ is a continuous map of k-cycles with Y irreducible, then $f_*[X] = \deg(f)[Y]$.

2. Reprove proposition 4.2 using exercise 1.

3. Prove that an n-ball does not retract to its boundary. Deduce Brouwer's theorem that any map from an n-ball to itself must have a fixed point.

4. Define the degree in integral homology. Let X and Y be closed orientable polyhedral surfaces. Show that there exists a continuous map $f : X \to Y$ with nonzero integral degree if and only if genus $(X) \geq$ genus (Y) .

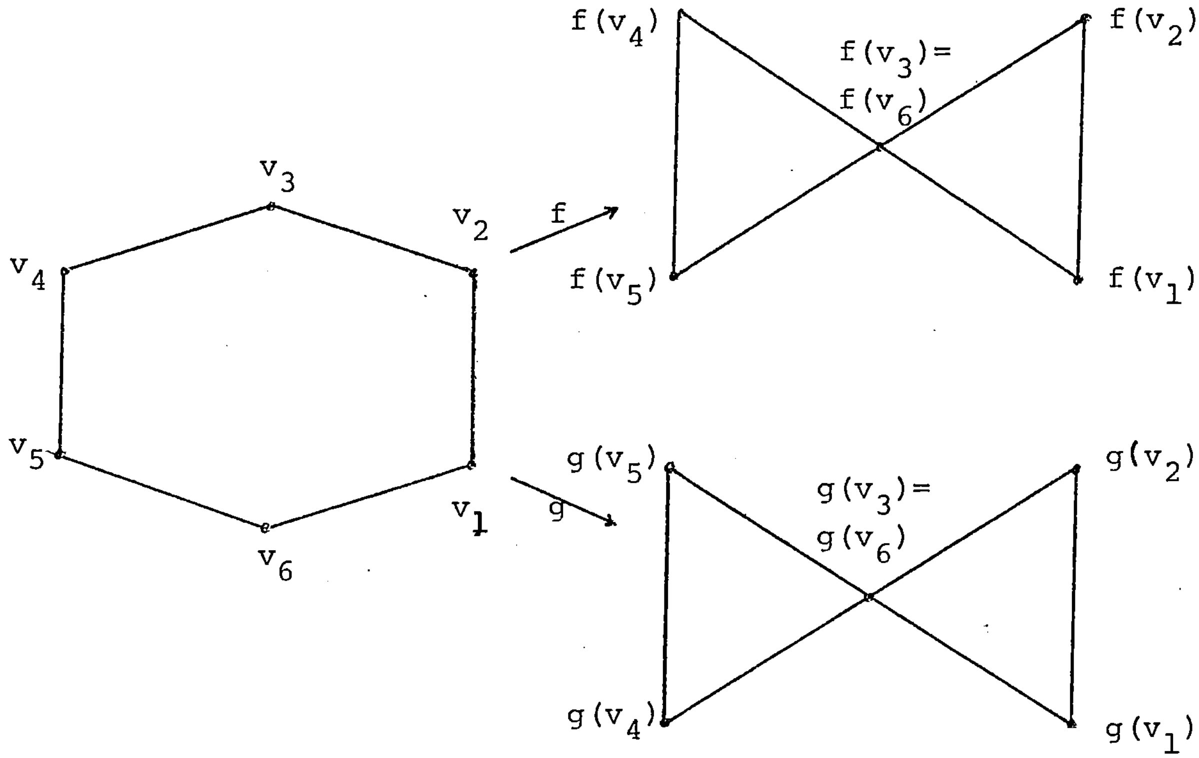


Figure 4.1. The double points of f and g are qualitatively different. This difference is detected by linking theory.

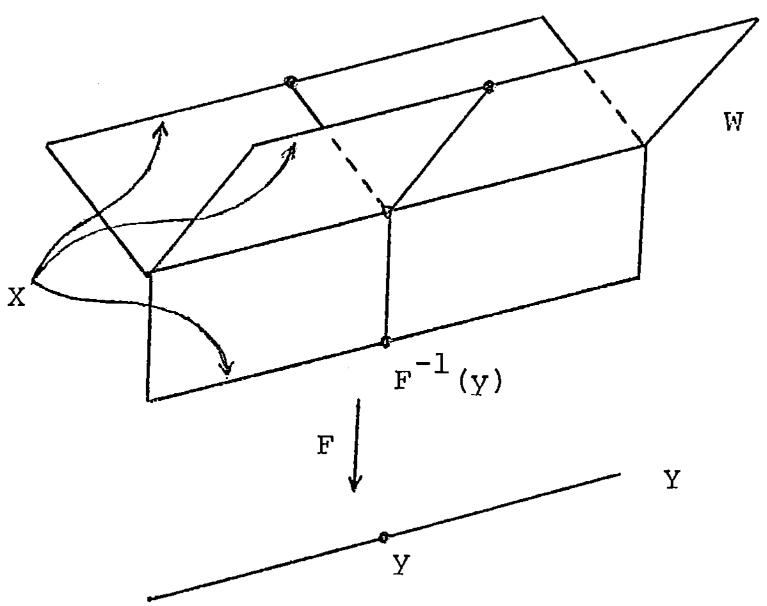


Figure 4.2. $\partial F^{-1}(y) = f^{-1}(y)$.

§5. Linking

Let X^k and Y^ℓ be cycles, and $f : X^k \to R^n$, $g : Y^\ell \to R^n$ be continuous maps, where $n = k + \ell + 1 > 1$. Assuming $f(X) \cap g(Y) = \emptyset$, we may define

$$D(f,g) : X \times Y \to S^{n-1}$$

$$D(f,g)(x,y) = \frac{f(x) - g(y)}{||f(x) - g(y)||}$$

where $||(x_1, \ldots, x_n)|| = \sum_{i=1}^{n} |x_i|$, and $S^{n-1} = \{(x_1, \ldots, x_n) \in R^n \,|\, ||(x_1, \ldots, x_n)|| = 1\}$, the standard polyhedral $(n-1)$-sphere. (Even if f and g are PL, $D(f,g)$ will not in general be PL.) The <u>linking number</u> of f and g is defined by

$$\mathcal{L}(f,g) = \deg(D(f,g))$$

(Figure 5.1).

We can also define the linking number of two maps to a sphere. Let X^k and Y^ℓ be cycles, $k > 0$, $\ell > 0$, and let $f : X^k \to S^n$, $g : Y^\ell \to S^n$, where $n = k + \ell + 1$, with $f(X) \cap g(Y) = \emptyset$. Choose an embedding e of $B^n = \{(x_1, \ldots, x_n) \in R^n \,|\, ||(x_1, \ldots, x_n)|| \leq 1\}$ in S^n such that $f(X) \cup g(Y) \subset e(B^n)$. Let $h : e(B^n) \to R^n$ be defined by $h(e(b)) = b$, and define

$$\mathcal{L}(f,g) = \mathcal{L}(h \circ f, h \circ g) .$$

Of course it must be checked that this definition is independent of the choice of embedding e .

Our definitions have omitted certain special cases. The linking number in Euclidean space has been defined only for $k + \ell > 0$. If $k = \ell = 0$, then $\mathcal{L}(f,g)$ can be defined only if either X^0 or Y^0 is an <u>even</u> 0-cycle (that is, consists of an even number of points). We then define $\mathcal{L}(f,g)$ to be the "degree" of $D(f,g) : X \times Y \to S^0$; that is, the number (mod 2) of points of $D(f,g)^{-1}(z)$, where z is either of the two points of S^0 .

The linking number in a sphere has been defined only for $k,\ell > 0$. To extend the definition, say for $k = 0$, we must require that X be an even 0-cycle. We then choose $x_0 \in X$ and $e : B^n \to S^n$ such that $f(X \setminus \{x_0\}) \cup g(Y) \subset e(B^n)$ and $f(x_0) \in S^n \setminus e(B^n)$. Defining $\mathcal{L}(f,g)$ as before, one easily verifies that $\mathcal{L}(f,g)$ is independent of these choices. Notice that if $k = \ell = 0$, then the linking number in S^1 is defined only if both X and Y are even 0-cycles.

Linking numbers have several basic properties. Let f and g be maps of cycles to R^n or S^n such that $\mathcal{L}(f,g)$ is defined. Let $\mathrm{Im}(g)$ denote the image of the map g .

<u>Proposition</u> 5.1

(a) If $f \in B_k(R^n \setminus \mathrm{Im}(g))$ [or $B_k(S^n \setminus \mathrm{Im}(g))$], then $\mathcal{L}(f,g) = 0$.

(b) If f_1 is homotopic to f_2 by a homotopy which is disjoint from $\mathrm{Im}(g)$, then $\mathcal{L}(f_1,g) = \mathcal{L}(f_2,g)$.

(c) $\mathcal{L}(f,g) = \mathcal{L}(g,f)$.

(d) $\mathcal{L}(f_1 + f_2\ , g) = \mathcal{L}(f_1,g) + \mathcal{L}(f_2,g)$.

(e) $\mathcal{L}(f \circ h\ , g) = (\deg h)\mathcal{L}(f,g)$.

(f) (for linking in a sphere only) $\mathcal{L}(Sf\ , i \circ g) = \mathcal{L}(f,g)$, where $i : S^n \to S^{n+1}$ is the usual inclusion (Figure 5.2).

<u>proof</u>. Properties (a)-(e) are easy consequences of properties (a)-(e) of degree. To prove (f), let $m : (SX) \times Y \to S(X \times Y)$ be the obvious map. $S(X \times Y)$ will not in general be irreducible, but if $z \in \sigma^\circ$, where σ is a $(k+\ell+1)$-simplex of a triangulation of $S(X \times Y)$ with respect to which m is simplicial, then there will be precisely one point in $m^{-1}(z)$. As in property (e), we may conclude that

$$\deg(SD(f,g) \circ m) = \deg(SD(f,g)) .$$

Furthermore, $SD(f,g) \circ m$ is homotopic to $D(Sf\ , i \circ g)$. It follows that

$$\begin{aligned} \mathcal{L}(Sf\ , i \circ g) &= \deg(D(Sf\ , i \circ g)) \\ &= \deg(SD(f,g) \circ m) \\ &= \deg(SD(f,g)) \\ &= \deg(D(f,g)) \\ &= \mathcal{L}(f,g) \quad . \quad \blacksquare \end{aligned}$$

Exercises

1. Give a definition of integral linking numbers (cf. [AH]).

2. Give an example of a pair of disjoint simple closed curves in R^3 such that their integral linking number is zero, but neither curve is contractible in the complement of the other.

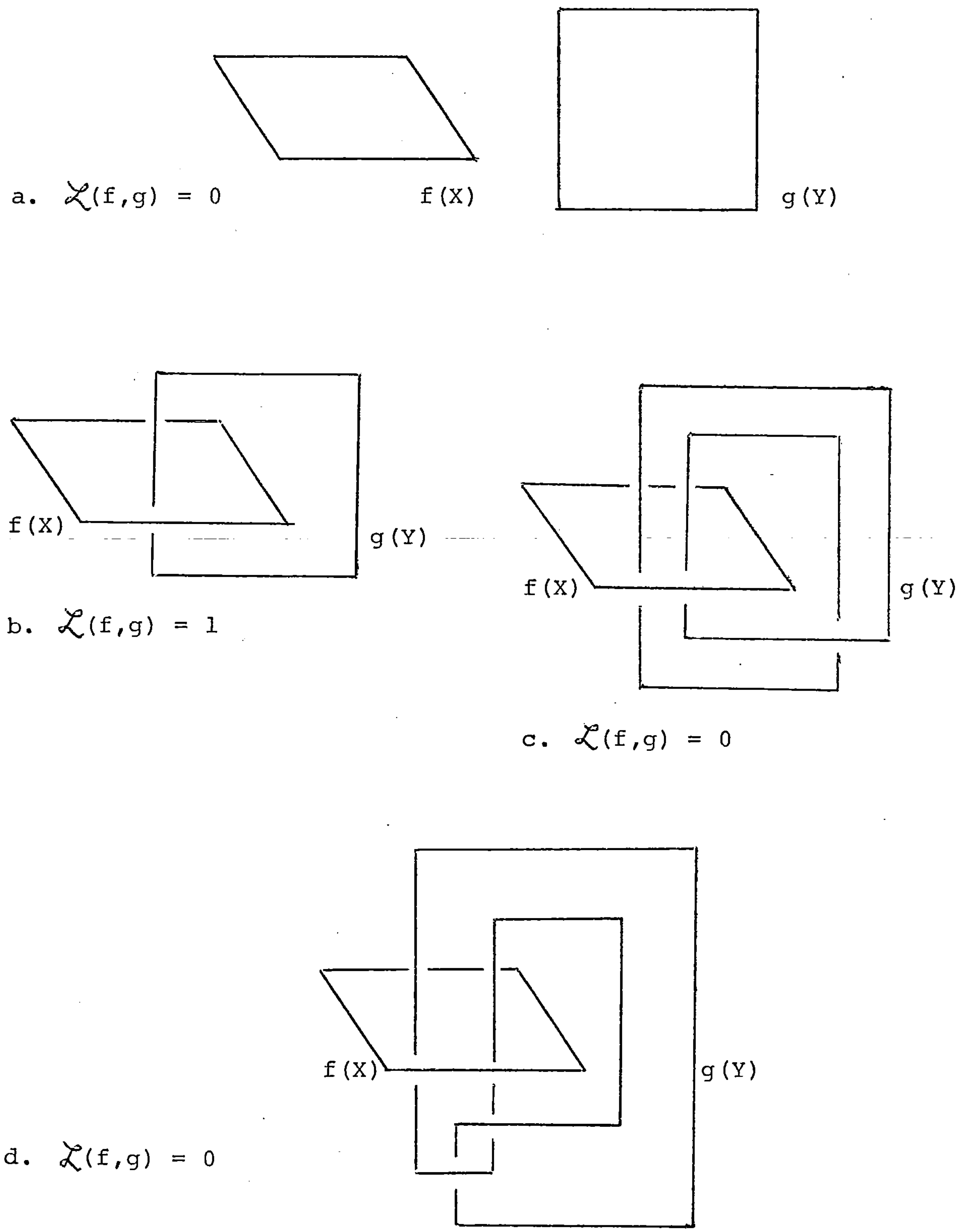


Figure 5.1. $\mathcal{L}(f,g) = \deg D(f,g)$.

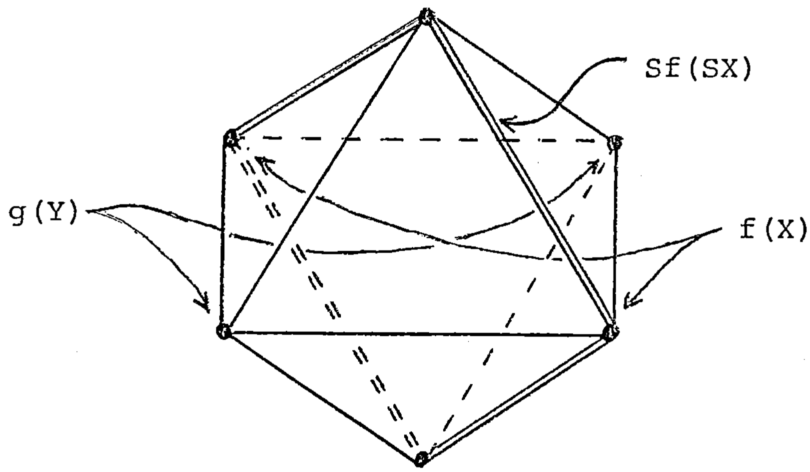


Figure 5.2. $\mathcal{L}(Sf, i \circ g) = \mathcal{L}(f,g)$.

§6. Intersection numbers

Let $f : X^k \to R^n$ and $g : Y^\ell \to R^n$ be PL maps, where X^k is a k-chain, Y^ℓ is an ℓ-chain, $k + \ell = n$, $k > 0$, and $\ell > 0$. Suppose that $f^{-1}(g(Y))$ and $g^{-1}(f(X))$ are finite sets, and that $f^{-1}(g(\partial Y))$ and $g^{-1}(f(\partial X))$ are empty. If $x \in X \setminus \partial X$, then LX_x is a (k-1)-cycle. If $y \in Y \setminus \partial Y$, LY_y is an (ℓ-1)-cycle. If $z \in R^n$, LR^n_z is a polyhedral (n-1)-sphere. The <u>intersection number</u> of f and g is defined to be

$$\mathcal{I}(f,g) = \sum_{x,y} \mathcal{L}(Lf_x , Lg_y) \quad , \quad f(x) = g(y) \quad .$$

(This means $\sum_I \mathcal{L}(Lf_x , Lg_y)$, $I = \{(x,y) \mid f(x) = g(y)\}$. We will always use such an abbreviation in writing sums.) Notice that the dimensions are correct:

$$\dim(LX_x) + \dim(LY_y) + 1 = \dim\left(LR^n_z\right) \quad , \quad z = f(x) = g(y) \quad .$$

If $k = 0$ and $\ell > 0$ then we define

$$\mathcal{I}(f,g) = \sum_{x,y} \deg(Lg_y) \quad , \quad f(x) = g(y) \quad .$$

If $k = \ell = 0$ we define $\mathcal{I}(f,g)$ to be the number (mod 2) of ordered pairs (x,y) such that $f(x) = g(y)$. (Figure 6.1)

This definition easily extends to intersection numbers in any PL manifold of dimension n . At this point we depart from the program of Lefschetz. We will return to it in chapter 3.

Theorem 6.1 (The linking theorem) Let $f : X^k \to R^n$ and $g : Y^\ell \to R^n$ be PL maps of chains for which $\mathcal{L}(f,g)$ is defined. If $\partial Y = \emptyset$, then

$$\mathcal{L}(f,g) = \mathcal{L}(f|\partial X , g) .$$

(Figure 6.2)

Corollary 6.2 Let $f : X^k \to R^n$ and $g : Y^\ell \to R^n$ be PL maps of chains for which $\mathcal{L}(f,g)$ is defined. If $\partial X = \emptyset$ and $\partial Y = \emptyset$, then $\mathcal{L}(f,g) = 0$.

If $f : (X,\partial X) \to (Z,\partial Z)$ is a map of pairs, let $\partial f : \partial X \to \partial Z$ be the restriction of f.

Corollary 6.3 Let $f : (X^k , \partial X) \to (B^n , S^{n-1})$ and $g : (Y^\ell , \partial Y) \to (B^n , S^{n-1})$ be PL maps, where X^k is a k-chain, Y^ℓ is an ℓ-chain, $k + \ell = n$, $f^{-1}(g(Y))$ and $g^{-1}(f(X))$ are finite, $f^{-1}(g(\partial Y)) = \emptyset$, $g^{-1}(f(\partial X)) = \emptyset$, $f^{-1}(S^{n-1}) = \partial X$, and $g^{-1}(S^{n-1}) = \partial Y$. Then

$$\mathcal{L}(f,g) = \mathcal{L}(\partial f , \partial g) .$$

(Figure 6.3)

Corollary 6.2 is an immediate consequence of the theorem.

Corollary 6.3 is proved as follows. Let $\tilde{X} = X \cup C_+(\partial X)$, $\tilde{Y} = Y \cup C_+(\partial Y)$, and $\tilde{B}^n = B^n \cup C_+(S^{n-1})$. Let $h : \tilde{B}^n \to S^n$ be the obvious PL homeomorphism. Define $\tilde{f} : \tilde{X} \to S^n$ and $\tilde{g} : \tilde{Y} \to S^n$ by $\tilde{f} = h\circ(f \cup C_+(\partial f))$, $\tilde{g} = h\ (g \cup C_+(\partial g))$. Since $\tilde{X}$ and $\tilde{Y}$ are cycles, corollary 6.2 implies that $\mathcal{I}(\tilde{f},\tilde{g}) = 0$. The observation that

$$\mathcal{I}(\tilde{f},\tilde{g}) = \mathcal{I}(f,g) + \mathcal{L}(\partial f\ ,\ \partial g)$$

proves corollary 6.3. ∎

<u>proof of theorem</u> 6.1 (for $k,\ell > 0$). Let $P = \{p_1, \ldots ,p_s\} = f(X) \cap g(Y)$. Let $f_i = \sum_{f(x)=p_i} Lf_x$ and $g_i = \sum_{g(x)=p_i} Lg_x$.

The theorem is a consequence of the following three statements:

(a) $\mathcal{I}(f,g) = \sum_i \mathcal{L}(f_i\ ,\ g_i)$

(b) $\mathcal{L}(f_i\ ,\ g_i) = \mathcal{L}(j_i\circ f_i\ ,\ g)$, where $j_i : LR^n_{p_i} \to R^n$ is the inclusion.

(c) $\sum_i \mathcal{L}(j_i\circ f_i\ ,\ g) = \mathcal{L}(\partial f\ ,\ g)$.

Statement (a) follows directly from the definition of $\mathcal{I}$.

Statement (c) is a consequence of the additivity of $\mathcal{L}$ (proposition 5.1(d)), which implies that

$$\sum_i \mathcal{L}(j_i\circ f_i\ ,\ g) + \mathcal{L}(\partial f\ ,\ g) = \mathcal{L}(\sum_i j_i\circ f_i + \partial f\ ,\ g) .$$

Let $W = C\ell(X \setminus \bigcup_{f(x)\in P} xLX_x)$. Since $\partial W = \partial X \cup \bigcup_{f(x)\in P} LX_x$, and $f(W) \subset R^n \setminus g(Y)$, proposition 5.1(a) implies that $0 = \mathcal{L}(f|\partial W , g) = \mathcal{L}(\partial f + \sum_i j_i \circ f_i , g)$.

Finally, statement (b) will be proved by constructing a homology between g and the "suspension" of g_i , in the complement of the image of f_i .

More precisely, let B be a compact polyhedron in R^n containing $f(X) \cup g(Y)$ and let $e : B \to S^n$ be a PL embedding with the following properties:

(i) $e(p_i) = (0, \ldots ,0,-1)$,

(ii) $e(LR^n_{p_i}) = S^{n-1}$,

(iii) if $g(y) = p_i$ and $\ell \in LY_y$, then $e(g(\lambda y + \mu\ell)) = (0, \ldots ,0,-\lambda) + \mu e(g(\ell))$, where $\lambda,\mu \geq 0$, $\lambda + \mu = 1$.

Now let $f' = e\circ f$, $g' = e\circ g$, $f_i' = e\circ f_i$, $g_i' = e\circ g_i$, and $L = \bigcup_{g(y)=p_i} LY_y$. Thus $\mathrm{Im}(f_i') \subset S^{n-1}$ and $g_i' : L \to S^{n-1}$ with $\mathrm{Im}(g_i') \cap \mathrm{Im}(f_i') = \emptyset$. We shall construct an $(\ell+1)$-chain W with $\partial W = Y \cup S(L)$ and a map $H : W \to S^n \setminus \mathrm{Im}(f_i')$ such that $H|Y = g'$ and $H|S(L) = S(g_i')$ (Figure 6.4). It then follows that, if $j : S^{n-1} \to S^n$ is the inclusion,

$$
\begin{aligned}
\mathcal{L}(j_i \circ f_i\ , g) &= \mathcal{L}(j \circ f_i^!\ , g') && \text{by } 5.1(e) \\
&= \mathcal{L}(j \circ f_i^!\ , S(g_i^!)) && \text{by } 5.1(a) \\
&= \mathcal{L}(f_i^!\ , g_i^!) && \text{by } 5.1(f) \\
&= \mathcal{L}(f_i\ , g_i) && \text{by } 5.1(e).
\end{aligned}
$$

Construction of W and H : Let $Y' = C\ell(Y \setminus \bigcup_{g(y)=p_i} yLY_y)$, and let $A_+ = Y' \cup (L \times I) \cup C_+(L)$, with the identifications of $L \times \{0\}$ with $L = \partial Y'$ and of $L \times \{1\}$ with $L = \partial C_+(L)$. Thus A_+ is an $(\ell+1)$-cycle. Let $B_+ = C_+(S^{n-1})$, and let $\alpha_+ : A_+ \to B_+$ be defined by

$$
\begin{aligned}
&\alpha_+ | Y' = g' | Y' \ , \\
&\alpha_+ | C_+(L) = S(g_i^!) | C_+(L) \ , \\
&\alpha_+(\ell, t) = g'(\ell) \quad \text{for} \quad (\ell, t) \in L \times I \ .
\end{aligned}
$$

Thus $\mathrm{Im}(\alpha_+) \cap \mathrm{Im}(f_i^!) = \emptyset$. Let V_+ be a cone with base A_+ . Since B_+ is contractible, there is a map $\beta_+ : V_+ \to B_+$ with $\beta_+ | A_+ = \alpha_+$. And β_+ can be chosen so that $\beta_+^{-1}(S^{n-1}) = \alpha_+^{-1}(S^{n-1})$, so $\mathrm{Im}(\beta_+) \cap \mathrm{Im}(f_i^!) = \emptyset$.

Now let $A_- = \bigcup_{g(y)=p_i} yLY_y \cup (L \times I) \cup C_-(L)$, with identifications along $L \times \{0\}$ and $L \times \{1\}$ as above. Let $B_- = C_-(S^{n-1})$. Define $\alpha_- : A_- \to B_-$ by

$$
\alpha_- \Big|_{\bigcup_{g(y)=p_i} yLY_y} = g' \Big|_{\bigcup_{g(y)=p_i} yLY_y} \ ,
$$

$$\alpha_-|C_-(L) = S(g_1^!)|C_-(L) \quad ,$$
$$\alpha_-(\ell,t) = g(\ell) \quad \text{for} \quad (\ell,t) \in L \times I \quad .$$

Let V_- be a cone with base A_- , and define $\beta_- : V_- \to B_-$ by $\beta_-(\lambda c + \mu a) = (0, \ldots ,0,-\lambda) + \mu\alpha_-(a)$, where $a \in A_-$ and c is the apex of V_- .

Let W be the union of V_+ and V_- identified along $L \times I$ (Figure 6.5). Define $H : W \to S^n$ by $H|V_+ = \beta_+$ and $H|V_- = \beta_-$. Then W and H have the desired properties. ∎

Exercises

1. What does theorem 6.1 say about winding numbers ($k = 1$, $\ell = 0$, and $n = 2$)?

2. Prove that a nonorientable polyhedral surface can't be embedded in R^3 (cf. chapter 1). (Hint: If the surface were embedded in R^3 , there would exist a curve in R^3 crossing the surface once.)

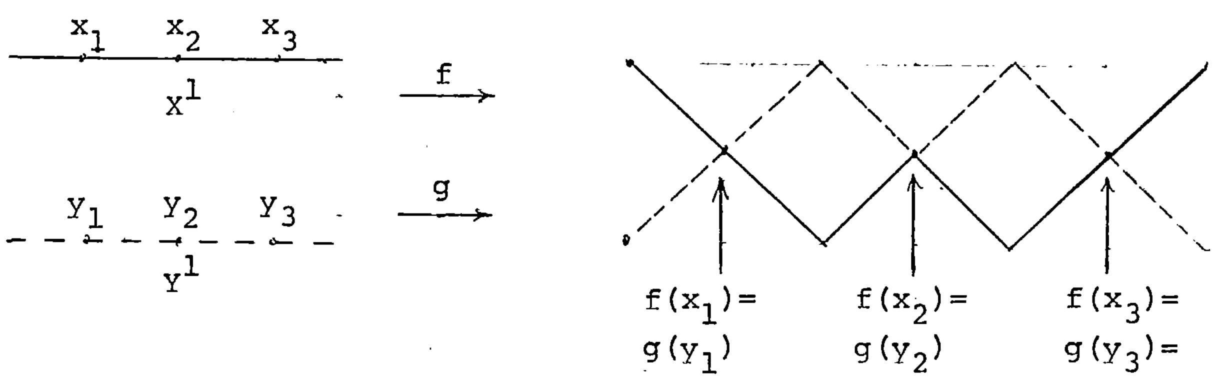
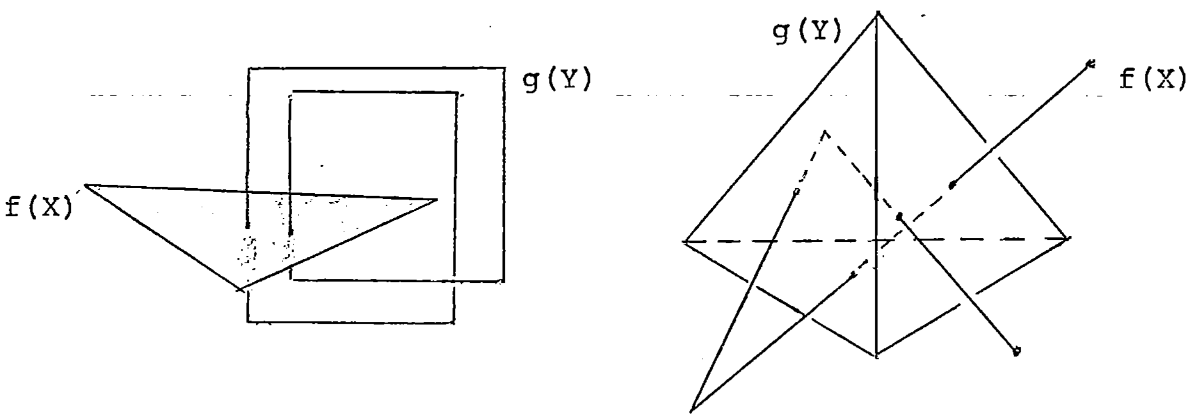
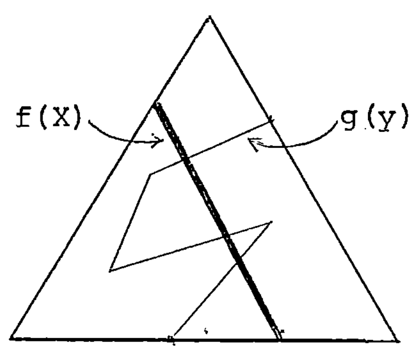


Figure 6.1. $f\colon X^1 \longrightarrow R^2$ and $g\colon Y^1 \longrightarrow R^2$ with $\mathcal{A}(f,g) = 0$.



a. $X=D^2$ $Y=S^1$ b. $X=[0,1]$ $Y=S^2$

Figure 6.2. $\mathcal{A}(f,g) = \mathcal{L}(f|\partial X,g)$



$X=Y=[0,1]$

Figure 6.3. $\mathcal{A}(f,g)= \mathcal{L}(\partial f,\partial g)$

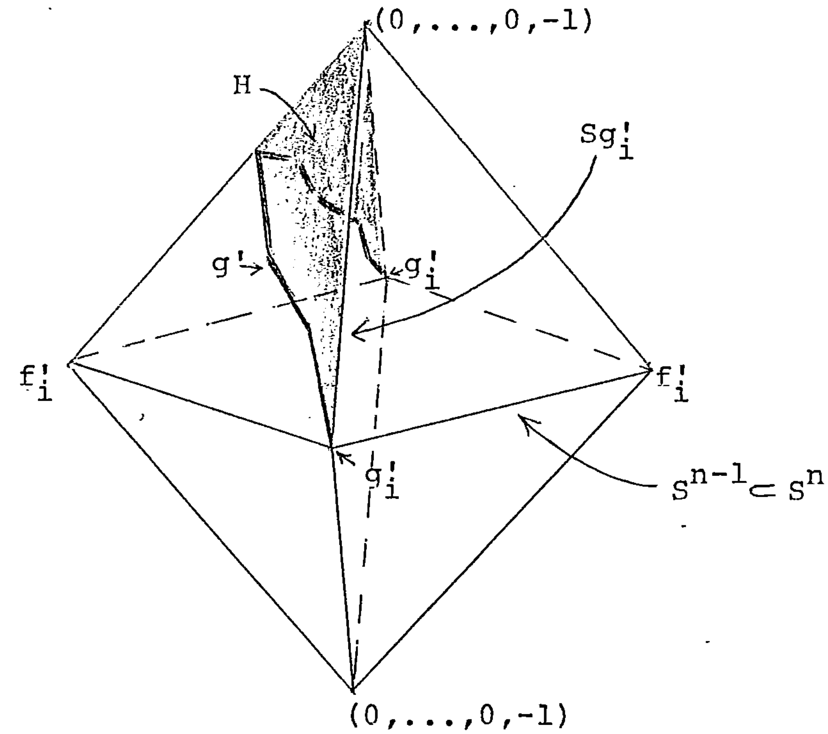


Figure 6.4. A homology between g and the "suspension" of g_i.

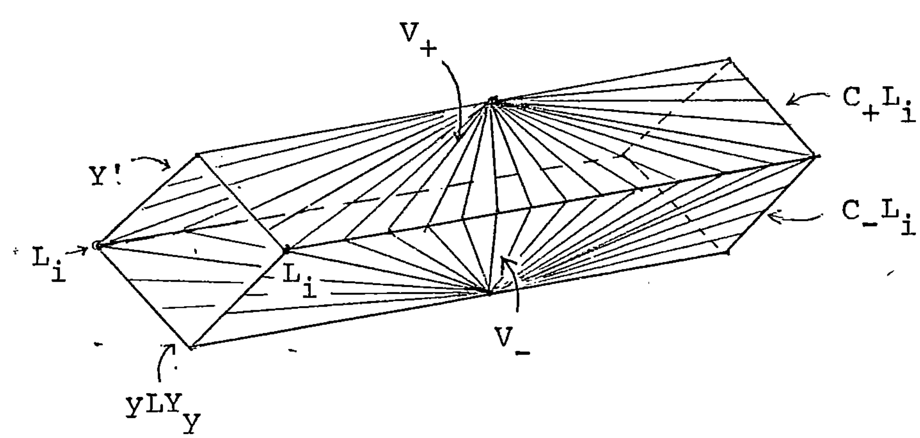


Figure 6.5. W = the union of V_+ and V_- along L x I.

§7. The double point cycle

Let X^k be a k-cycle, and let $f : X^k \to R^n$, $n > k$, be a spread-out map. Define a (2k-n)-chain $\mathbb{D}(f) \subset D(f)$ as follows. Let K be a triangulation of X with respect to which f is simplicial, and let σ be a (2k-n)-simplex in $D(f)$. Since $(f,\partial f)$ is spread-out, $\sigma \not\subset \partial X$. Define

$$d_f(\sigma) = \sum_{\sigma'} \mathcal{L}(Lf_{\sigma'}, Lf_{\sigma}) \quad , \quad \begin{array}{l} \dim(\sigma') = 2k-n \quad , \quad \sigma' \neq \sigma \quad , \\ f(\sigma') = f(\sigma) \quad , \end{array}$$

and let $\mathbb{D}(f)$ be the union of all (2k-n)-simplexes σ of K such that $d_f(\sigma) \neq 0$ (Figure 7.1). Note that $\dim(LX_{\sigma}) = k - (2k-n) - 1 = n - k - 1$, $LR^n_{f(\sigma)}$ is a sphere of dimension $n - (2k-n) - 1 = 2n - 2k - 1$, and $\mathrm{Im}(Lf_{\sigma'}) \cap \mathrm{Im}(Lf_{\sigma}) = \emptyset$, since $\dim(D(f)) \leq 2k - n$. Thus $\mathcal{L}(Lf_{\sigma'}, Lf_{\sigma})$ is defined.

We will prove that $\mathbb{D}(f)$ is a <u>cycle</u>. This can be seen intuitively as follows. If τ is a (2k-n-1)-simplex of $\mathbb{D}(f)$, then the number of (2k-n)-simplices σ of $\mathbb{D}(f)$ such that $\tau < \sigma$ is equal to the sum of the intersection numbers $\mathcal{L}(Lf_{\tau'}, Lf_{\tau})$, where $\tau' \neq \tau$ and $f(\tau') = f(\tau)$, plus the number of points of $\mathbb{D}(Lf_{\tau})$. Each intersection number $\mathcal{L}(Lf_{\tau'}, Lf_{\tau})$ is zero because $LX_{\tau'}$ and LX_{τ} are cycles (corollary 6.2). The link LX_{τ} is an (n-k)-cycle, and $Lf_{\tau} : LX_{\tau} \to S^{2n-2k}$ is spread-out. Therefore Lf_{τ} has an even number of double points $\mathbb{D}(Lf_{\tau})$, since double points - in the source! - occur in pairs (Figure 7.2).

More generally, suppose that X is a k-chain, and that

$(f, \partial f) : (X, \partial X) \to (R^n_+ , R^{n-1})$ is spread-out, $n > k$. This means that f and ∂f are spread-out and $f^{-1}(R^{n-1}) = \partial X$. (Recall that $R^n_+ = \{(x_1, \ldots, x_n) \in R^n | x_n \geq 0\}$.) Defining $\mathbb{D}(f)$ just as above, we have the following basic result.

Theorem 7.1 Let X^k be a k-chain. If $(f, \partial f) : (X^k, \partial X) \to (R^n_+ , R^{n-1})$, $n > k$, is spread-out, then

$$\partial\mathbb{D}(f) = \mathbb{D}(\partial f) .$$

(Figure 7.3)

Corollary 7.2 If X^k is a k-cycle and $f : X^k \to R^n$, $n > k$, is spread-out, then $\mathbb{D}(f)$ is a $(2k-n)$-cycle.

Corollary 7.3 Let X^k be a k-chain and let $(f, \partial f)$ and $(g, \partial g)$ be spread-out maps from $(X, \partial X)$ to (R^n_+ , R^{n-1}), $n > k$. Let $i : (\mathbb{D}(f), \partial\mathbb{D}(f)) \to (X, \partial X)$ and $j : (\mathbb{D}(g), \partial\mathbb{D}(g)) \to (X, \partial X)$ be the inclusions. Then $i + j \in B_{2n-k}(X, \partial X)$.

Definition. Let X be a k-chain. For each integer $i > 0$, the double point class $\mathbb{D}^i(X) \in H_{k-i}(X, \partial X)$ is the homology class of the inclusion map $(\mathbb{D}(f), \partial\mathbb{D}(f)) \to (X, \partial X)$, where $(f, \partial f) : (X, \partial X) \to (R^{k+i}_+ , R^{k+i-1})$ is a spread-out map. ($\mathbb{D}^i(X)$ is well-defined by corollary 7.3.) $\mathbb{D}^0(X) = [X] \in H_k(X, \partial X)$, the fundamental class of X.

Corollary 7.2 follows immediately from the theorem.

<u>proof</u> of 7.3. Let $h : \partial X \times I \to \mathbb{R}^{n-1}$ be a homotopy from ∂f to ∂g such that $h(x,t) = \partial f(x)$ for all $t \in [0,\varepsilon]$ and $h(x,t) = \partial g(x)$ for all $t \in [1-\varepsilon,1]$ for some $\varepsilon > 0$. Let $(e,\partial e)$: $(\mathbb{R}^n_+ \times I , \partial(\mathbb{R}^n_+ \times I)) \to (\mathbb{R}^{n+1}_+ , \mathbb{R}^n)$ be a PL homeomorphism. Define $h' : \partial(X \times I) \to \mathbb{R}^n$ by

$$h'(x,t) = \begin{cases} \partial e(h(x,t),t) & \text{if } x \in \partial X \\ \partial e(f(x),t) & \text{if } t \in [0,\varepsilon] \\ \partial e(g(x),t) & \text{if } t \in [1-\varepsilon,1] \end{cases}$$

By theorem 2.2(b), there is a spread out map $\tilde{h}' : \partial(X \times I) \to \mathbb{R}^n$ such that $h'(x,t) = \tilde{h}'(x,t)$ if $t \in [0,\varepsilon] \cup [1-\varepsilon,1]$.

Let $H : (X \times I , \partial(X \times I) \to (\mathbb{R}^{n+1}_+ , \mathbb{R}^n)$ be an extension of $\tilde{h}'$, such that $H(x,t) = e(f(x),t)$ for all $t \in [0,\varepsilon]$ and $H(x,t) = e(g(x,t),t)$ for all $t \in [1-\varepsilon,1]$. By theorem 2.2(b), there is a spread out map $\tilde{H} : X \times I \to \mathbb{R}^{n+1}_+$, such that $\tilde{H}(x,t) = H(x,t)$ if $t \in [0,\varepsilon] \cup [1-\varepsilon,1]$ or $x \in \partial X$, and $\tilde{H}^{-1}(R^n) = \partial(X \times I)$.

Now let $W = \mathbb{D}(\tilde{H})$, and let $F : W \to X$ be the restriction of the projection $X \times I \to X$. Theorem 7.1 implies that W is a $(2k-n+1)$-chain with $\partial W = \partial\mathbb{D}(\tilde{H}) = \mathbb{D}(\partial\tilde{H})$. Thus $\mathbb{D}(f)$ and $\mathbb{D}(g)$ are contained in ∂W . Furthermore, $Fr(\partial W , \mathbb{D}(f)) \subset \partial\mathbb{D}(f)$ and $Fr(\partial W , \mathbb{D}(g)) \subset \partial\mathbb{D}(g)$, since $\tilde{H}$ agrees with H near $X \times \{0\}$

and $X \times \{1\}$. Finally, $F(\partial W \setminus (\mathbb{D}(f) \cup \mathbb{D}(g))) \subset \partial X$ since $H(\partial X \times I) \subset R^n$. Thus $F : W \to X$ is a homology between $\mathbb{D}(f) \subset X$ and $\mathbb{D}(g) \subset X$ (Figure 7.4). ∎

<u>proof of</u> 7.1 Let K be a triangulation of X with respect to which f is simplicial, and let τ be a $(2k-n-1)$-simplex of K . Let $\partial d_f(\tau)$ be the number (mod 2) of $(2k-n)$-simplices σ such that $d_f(\tau) \neq 0$ and $\tau < \sigma$. We must show that $\partial d_f(\tau) = 0$ if $\tau \not\subset \partial X$ and $\partial d_f(\tau) = d_{\partial f}(\tau)$ if $\tau \subset \partial X$.

We claim the following:

(a) $\partial d_f(\tau) = \sum_{\tau'} \mathcal{L}(Lf_{\tau'}, Lf_\tau)$, $\tau' \neq \tau$, $f(\tau') = f(\tau)$.

(b) $\mathcal{L}(Lf_{\tau'}, Lf_\tau) = \mathcal{L}(\partial Lf_{\tau'}, \partial Lf_\tau)$.

(c) If $\tau \not\subset \partial X$, then $\partial(LX_\tau) = \emptyset$. If $\tau \subset \partial X$, then $\partial(LX_\tau) = L(\partial X)_\tau$.

From (a) and (b) we conclude that

$$\partial d_f(\tau) = \sum_{\tau'} \mathcal{L}(\partial L f_{\tau'} , \partial L f_\tau) \quad , \quad \tau' \neq \tau \quad , \quad f(\tau') = f(\tau) \quad .$$

Thus (c) implies that if $\tau \not\subset \partial X$, then $\partial d_f(\tau) = 0$, and that if $\tau \subset \partial X$, then

$$\partial d_f(\tau) = \sum_{\tau'} \mathcal{L}(L\partial f_{\tau'} , L\partial f_\tau) \quad , \quad \tau' \neq \tau \quad , \quad \partial f(\tau') = \partial f(\tau) \quad ,$$

which equals $d_{\partial f}(\tau)$, as desired.

Proof of (c): If τ is an i-simplex of K , $\dim(LK_\tau) = k - i - 1$. The ℓ-simplices of τLK_τ are all the joins $\tau\alpha$, where α is an $(\ell-i-1)$-simplex of LK_τ . If $\alpha, \beta \in LK_\tau$, then $\alpha < \beta$ if and only if $\tau\alpha < \tau\beta$. Thus $\alpha \subset \partial LX_\tau$ if and only if $\tau\alpha \subset \partial X$ if and only if $\alpha \subset L(\partial X)_\tau$ (Figure 7.5).

Proof of (a): If $\tau \in K$ and α is a simplex of LK_τ , then $L(LK_\tau)_\alpha = LK_{\tau\alpha}$ (Figure 7.6). If τ is an (i-1)-face of an i-simplex $\sigma \in K$, and σ' is an i-simplex of K such that $f(\sigma') = f(\sigma)$, then there exists a unique (i-1)-face τ' of σ' such that $f(\tau') = f(\tau)$.

Now let τ be a (2k-n-1)-simplex of K . Then

$$\partial d_f(\tau) = \sum_\sigma d_f(\sigma) \quad , \quad \dim(\sigma) = 2k - n \quad , \quad \tau < \sigma$$

$$= \sum_{\sigma',\sigma} \mathscr{L}(Lf_{\sigma'}\, , Lf_{\sigma}) \quad , \quad \dim(\sigma') = \dim(\sigma) = 2k - n \quad , \quad \sigma' \neq \sigma \quad , \quad f(\sigma') = f(\sigma) \quad , \quad \tau < \sigma$$

$$= \sum_{\substack{\tau' \\ \alpha,\beta}} \mathscr{L}(L(Lf_{\tau'})_{\alpha}\, , L(L(f_{\tau})_{\beta}) \quad , \quad \begin{array}{l} \dim(\tau') = \dim(\tau) = 2k - n - 1 \quad , \\ \dim(\alpha) = \dim(\beta) = 0 \quad , \\ \tau'\alpha \neq \tau\beta \quad , \quad f(\tau') = f(\tau) \quad , \\ f(\alpha) = f(\beta) \end{array}$$

$$= \sum_{\alpha,\beta} \mathscr{L}(L(Lf_{\tau})_{\alpha}\, , L(Lf_{\tau})_{\beta}) \quad , \quad \alpha \neq \beta \quad , \quad f(\alpha) = f(\beta)$$

$$+ \sum_{\substack{\tau' \\ \alpha,\beta}} \mathscr{L}(L(Lf_{\tau'})_{\alpha}\, , L(Lf_{\tau})_{\beta}) \quad , \quad \tau' \neq \tau \quad , \quad f(\tau') = f(\tau) \quad , \quad f(\alpha) = f(\beta) \quad .$$

In this last expression, every term in the first summand appears twice, so it equals zero (mod 2). the second summand equals

$$\sum_{\tau'} \mathscr{L}(Lf_{\tau'}\, , Lf_{\tau}) \quad , \quad \tau' \neq \tau \quad , \quad f(\tau') = f(\tau) \quad .$$

Proof of (b): If $f(\tau') = f(\tau) \not\subset R^{n-1}$, then $\tau',\tau \not\subset \partial X$, $LX_{\tau'}$ and LX_{τ} are $(n-k)$-cycles, and $L(R^n_+)_{f(\tau)}$ is a $(2n-2k)$-sphere. Thus corollary 6.2 implies that $\mathscr{L}(Lf_{\tau'}\, , Lf_{\tau}) = 0$. On the other hand, if $f(\tau') = f(\tau) \subset R^{n-1}$, then $\tau,\tau' \subset \partial X$, $LX_{\tau'}$ and LX_{τ} are $(n-k)$-chains, and $L(R^n_+)_{f(\tau)}$ is a $(2n-2k)$-ball. Thus corollary 6.3 implies that $\mathscr{L}(Lf_{\tau'}\, , Lf_{\tau}) = \mathscr{L}(\partial Lf_{\tau'}\, , \partial Lf_{\tau})$. ∎

Let $Y^k \subset X^k$ be k-chains such that $Fr(X,Y) \subset \partial Y$ or equivalently, $\partial Y = Fr(X,Y) \cup (Y \cap \partial X)$. The <u>restriction morphism</u>

$$r(X,Y) : H_i(X,\partial X) \to H_i(Y,\partial Y)$$

is defined by the diagram

$$\begin{array}{ccc} H_i(X,\partial X) & \xrightarrow{\quad r(X,Y) \quad} & H_i(Y,\partial Y) \\ \downarrow & & \| \\ H_i(X,\partial X \cup C\ell(X\setminus Y)) & \underset{\text{excision}}{\xleftarrow{\ \approx\ }} & H_i(Y,Fr(X,Y) \cup (Y \cap \partial X)) \end{array}$$

<u>Proposition</u> 7.4 If $Y^k \subset X^k$ are k-chains with $Fr(X,Y) \subset \partial Y$, then $r(X,Y)\mathbb{D}^i(X) = \mathbb{D}^i(Y)$ for all $i \geq 0$.

<u>proof.</u> If k or i equals zero, this is clear. If k and i are positive, let $n = k + i$, and let

$$Q^n = \{(x_1, \ldots ,x_n) \in R^n \mid x_n \geq 0 \text{ and } x_{n-1} \geq 0\} ,$$

$$\partial Q = \{(x_1, \ldots ,x_n) \in Q^n \mid x_n = 0 \text{ or } x_{n-1} = 0\} .$$

There exists a spread-out map $f : (X,\partial X) \to (R^n_+ , R^{n-1})$ such that $f^{-1}(Q) = Y$ and $f^{-1}(\partial Q) = \partial Y$. Let $h : (Q,\partial Q) \to (R^n_+ , R^{n-1})$ be a PL homeomorphism. Then

$$\mathbb{D}^i(Y) = [\mathbb{D}(h\circ f|Y)] = r(X,Y)[\mathbb{D}(f)] = r(X,Y)\mathbb{D}^i(X) . \quad ∎$$

<u>Exercises</u>

1. If X is an orientable k-cycle, and $f : X \to R^n$, $n > k$, is spread out, is $\mathbb{D}(f)$ orientable?

2. Let X be the Moebius strip. Construct an embedding of X in

R^3 so that $\partial X \subset R^2$. Prove that there exists no embedding of X in R^3_+ with $\partial X \subset R^2$.

3. If $f : X^k \to M^n$ is a spread-out map of the k-cycle X to the polyhedral n-manifold M , $k < n$, then $\mathbb{D}(f)$ is a cycle whose homology class in X depends only on the homotopy class of f .

4. $\mathbb{D}^i(SX) = \sigma\mathbb{D}^i(X)$ (cf. exercise 3 of §3).

5. If X and Y are k-cycles and $f : X \to Y$ is a map, then $f_*\mathbb{D}^i(X) = \deg(f)\mathbb{D}^i(Y)$.

6. Illustrate proposition 7.4 with X the Klein bottle and Y the Moebius strip.

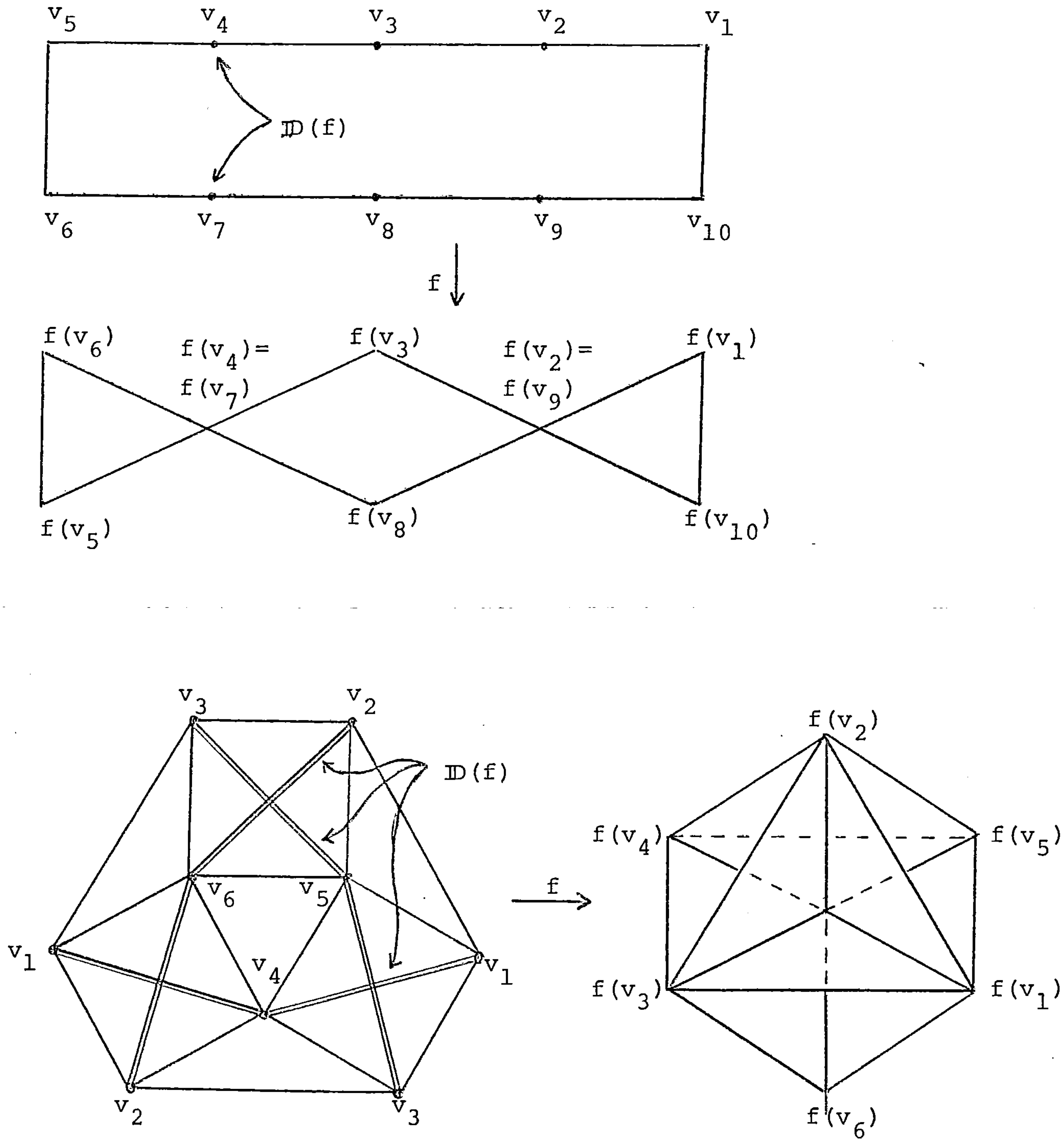


Figure 7.1. ID(f).

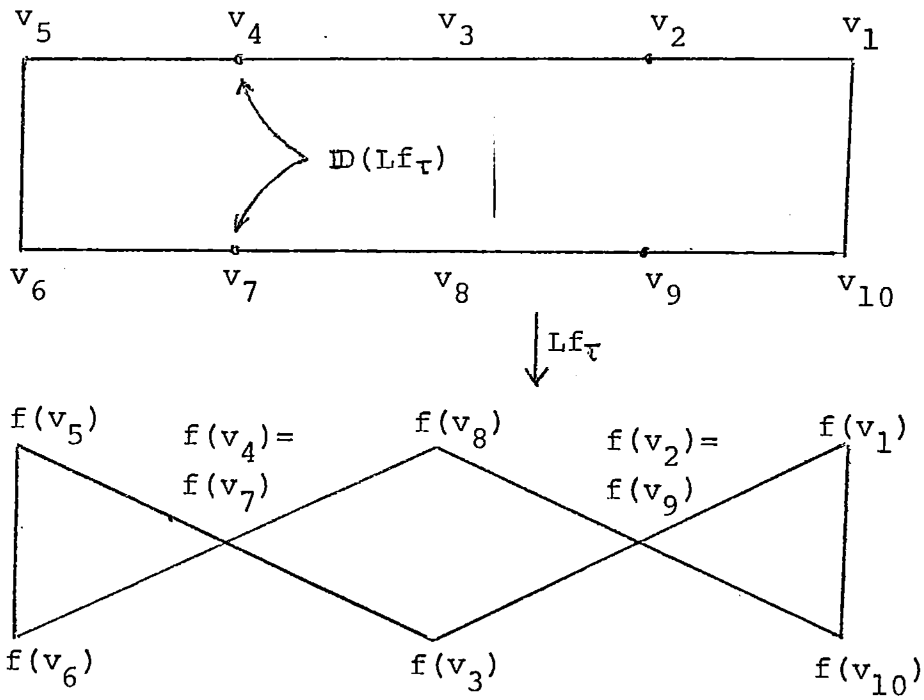


Figure 7.2. $\mathbb{D}(Lf_\tau)$ must consist of an even number of points.

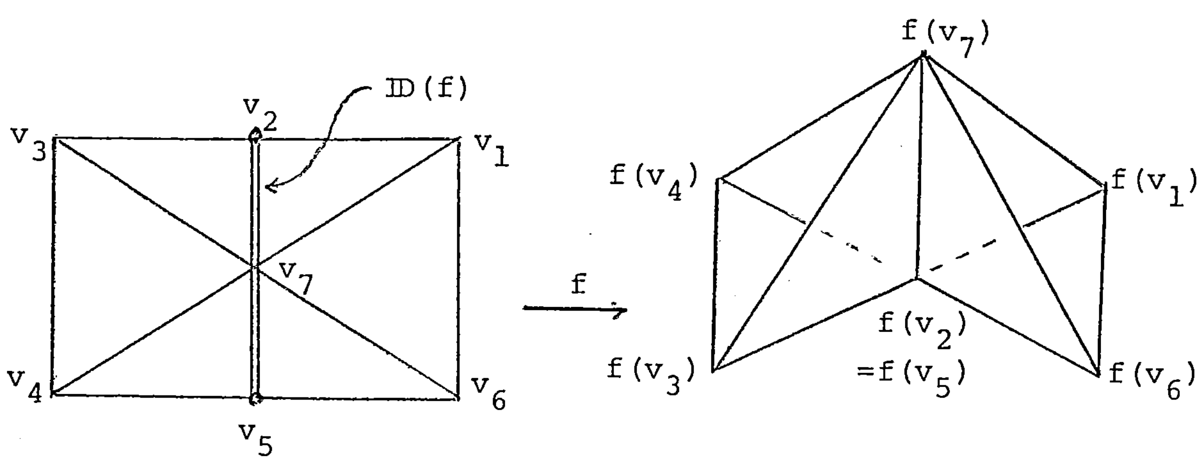


Figure 7.3. $\partial\mathbb{D}(f) = \mathbb{D}(\partial f)$.

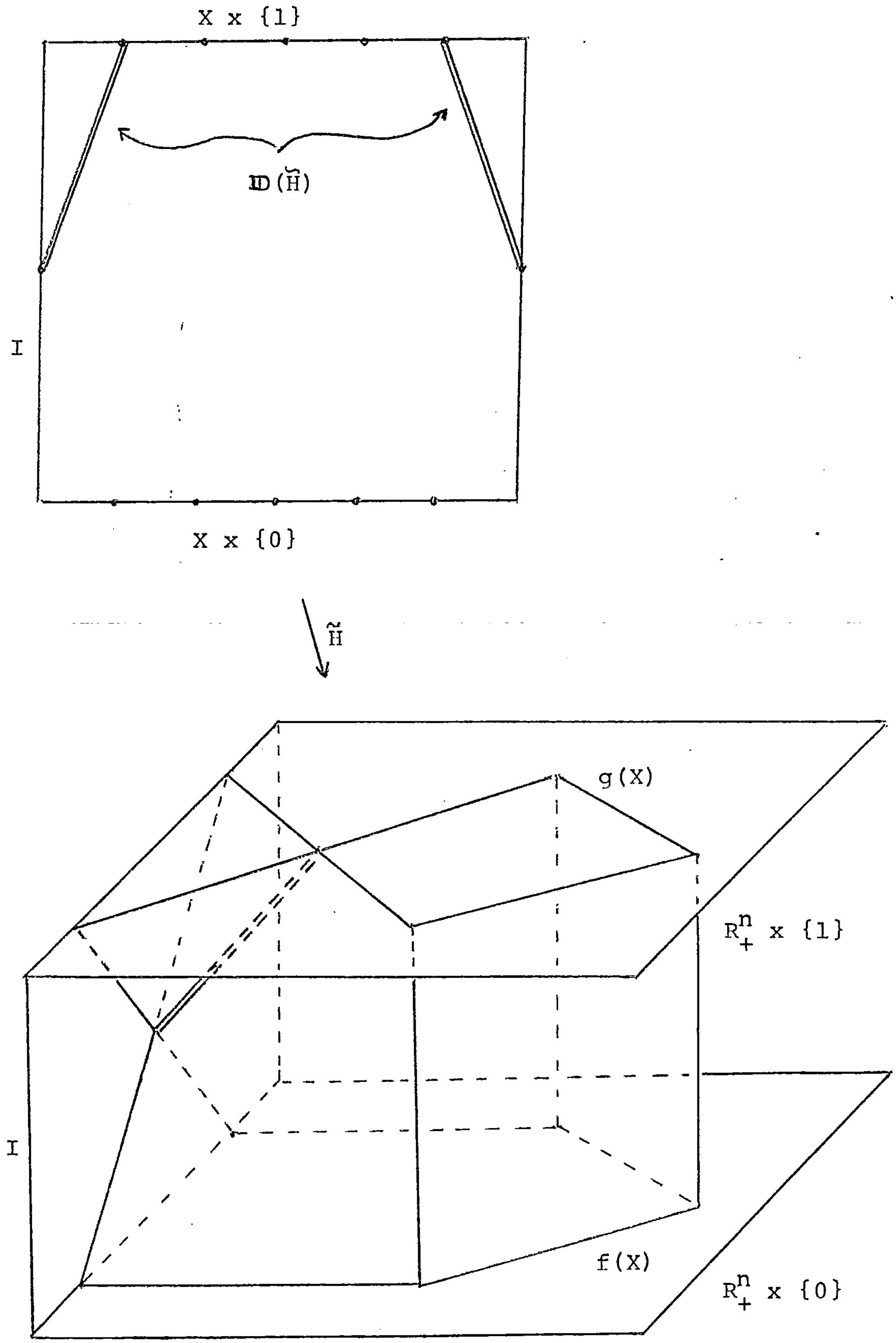


Figure 7.4. ID(**f**) is homologous to ID(g).

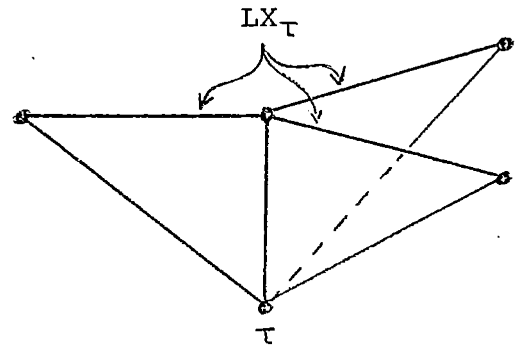


Figure 7.5. $\partial LX_\tau = L(\partial X)_\tau$

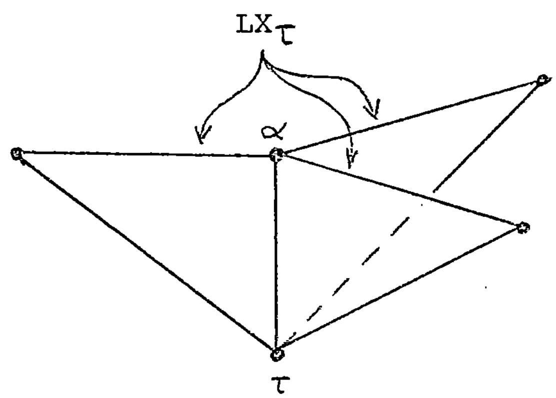


Figure. 7.6. $L(LX_\tau)_\alpha = LX_{\tau\alpha}$

§8. The branch point cycle

Let X^k be a k-chain, and let $(f,\partial f) : (X,\partial X) \to (R^n_+ , R^{n-1})$ be spread-out, $n > k$. Define a (2k-n-1)-chain $\mathbb{B}(f) \subset B(f)$ as follows. Let K be a triangulation of X with respect to which f is simplicial, and let τ be a (2k-n-1)-simplex in $B(f)$, Since $(f,\partial f)$ is spread-out, $\tau \not\subset \partial X$. Define

$$b_f(\tau) = \sum_{\{\sigma,\sigma'\}} \mathcal{L}(Lf_{\sigma'}, Lf_{\sigma}) \quad , \quad \begin{array}{l} \dim(\sigma) = \dim(\sigma') = 2k - n \ , \\ \sigma \neq \sigma' \ , \quad f(\sigma) = f(\sigma') \ , \\ \tau < \sigma \ , \quad \tau < \sigma' \ . \end{array}$$

This sum is taken over all <u>unordered</u> pairs $\{\sigma,\sigma'\}$. Let $\mathbb{B}(f)$ be the union of all (2k-n-1)-simplices τ of K such that $b_f(\tau) \neq 0$. (Figure 8.1).

If $n = k$, $\mathbb{B}(f) \subset B(f)$ is defined as follows. Let K and L be triangulations of X^k and R^k_+ with respect to which f is simplicial. Let $\tau \in K$ be a (k-1)-simplex of $B(f)$, and let $t = f(\tau) \in L$. Since $(f,\partial f)$ is spread-out, $t \not\subset R^{n-1}$. Let s and s' be the two k-simplices of L which have t as a face. Let a be the (integral) number of k-simplices $\sigma \in K$ such that $f(\sigma) = s$ and $\tau < \sigma$, and let b be the number of k-simplices $\sigma' \in K$ such that $f(\sigma') = s'$ and $\tau < \sigma'$. Since $\tau \not\subset \partial X$, $a + b$ is even. Let $b_f(\tau) = \frac{1}{2}(a - b)$ (mod 2), and let $\mathbb{B}(f)$ be the union of all (k-1)-simplices τ of K such that $b_f(\tau) \neq 0$.

We will prove that if X is a cycle, then $\mathbb{B}(f)$ is a cycle. This can be seen intuitively (for $n > k$) as follows. If ω is a (2k-n-2)-simplex of $\mathbb{B}(f)$, the number of (2k-n-1)-simplices τ

of $\mathbb{B}(f)$ such that $\omega < \tau$ equals the number of points in $\mathbb{B}(Lf_\omega)$. The link LX_ω is an $(n-k+1)$-cycle, and $Lf_\omega : LX_\omega \to S^{2n-2k+1}$ is a spread-out map. Such a map always has an even number of branch points, since the image of the branch points in $S^{2n-2k+1}$ is the boundary of the image of the double points (Figure 8.2).

Theorem 8.1 Let X^k be a k-chain. If $(f,\partial f) : (X^k,\partial X) \to (R^n_+ , R^{n-1})$, $n \geq k$, is a spread-out map, then

$$\partial \mathbb{B}(f) = \mathbb{B}(\partial f) \quad .$$

Corollary 8.2 If X^k is a k-cycle and $f : X^k \to R^n$, $n \geq k$, spread-out, then $\mathbb{B}(f)$ is a $(2k-n-1)$-cycle.

Corollary 8.3 Let X^k be a k-chain, and let $(f,\partial f)$ and $(g,\partial g)$ be spread-out maps from $(X,\partial X)$ to (R^n_+ , R^{n-1}) , $n \geq k$. Let $i : (\mathbb{B}(f) , \partial\mathbb{B}(f)) \to (X,\partial X)$ and $j : (\mathbb{B}(g) , \partial \mathbb{B}(g)) \to (X,\partial X)$ be the inclusions. Then $i + j \in B_{2n-k}(X,\partial X)$.

Corollary 8.2 follows immediately from the theorem, and the proof of 8.3 is just like the proof of 7.3.

Definition. Let X be a k-chain. For each integer $i > 0$, the branch point class $\mathbb{B}^i(X) \in H_{k-i}(X,\partial X)$ is the homology class of the inclusion map $(\mathbb{B}(f) , \partial\mathbb{B}(f)) \to (X,\partial X)$, where $(f,\partial f) : (X,\partial X) \to (R^{k+i-1}_+ , R^{k+i-2})$ is a spread-out map. $(\mathbb{B}^i(X)$ is well-defined by corollary 8.3.) $\mathbb{B}^0(X) = [X] \in H_k(X,\partial X)$.

<u>proof of</u> 8.1 $(n > k)$. Let K be a triangulation of X with respect to which f is simplicial, and let ω be a $(2k-n-2)$-simplex of X. Let $\partial b_f(\omega)$ be the number (mod 2) of $(2k-n-1)$-simplices τ such that $b_f(\tau) \neq 0$ and $\omega < \tau$. We must show that $\partial b_f(\omega) = 0$ if $\omega \not\subset \partial X$ and $\partial b_f(\omega) = b_{\partial f}(\omega)$ if $\omega \subset \partial X$.

If $v \in LX_\omega$ and $\tau = \omega v \subset \omega LX_\omega$, then $v \in \mathbb{B}(Lf_\omega)$ if and only if $\tau \subset \mathbb{B}(f)$, so $\partial b_f(\omega)$ is the number of points in $\mathbb{B}(Lf_\omega)$. On the other hand, if $\omega \subset \partial X$, then

$$b_{\partial f}(\omega) = \sum_{\{\tau,\tau'\}} \mathscr{L}(L\partial f_\tau, L\partial f_{\tau'}) , \quad \begin{array}{l} \dim(\tau) = \dim(\tau') = 2k-n-1 , \\ \tau \neq \tau' , \quad f(\tau) = f(\tau') , \\ \omega < \tau , \quad \omega < \tau' \end{array}$$

$$= \sum_{\{v,v'\}} \mathscr{L}(L(L\partial f_\omega)_v , L(L\partial f_\omega)_{v'}) , \quad \begin{array}{l} \dim(v) = \dim(v') = 0 , \\ v \neq v' , \quad f(v) = f(v') . \end{array}$$

Therefore, theorem 8.1 is a consequence of the following lemma applied to $(g,\partial g) = (Lf_\omega , L\partial f_\omega)$.

<u>Lemma</u> 8.4 If Y^ℓ is an ℓ-chain and $(g,\partial g) : (Y,\partial Y) \to (B^{2\ell-1}, S^{2\ell-2})$ is a spread-out map, the number of points in $\mathbb{B}(g)$ equals

$$\sum_{\{v,v'\}} \mathscr{L}(L\partial g_v , L\partial g_{v'}) , \quad \begin{array}{l} \dim(v) = \dim(v') = 0 , \quad v \neq v' , \\ g(v) = g(v') . \end{array}$$

(Figure 8.3).

proof. Let K and L be triangulations of Y^{ℓ} and $B^{2\ell-1}$ with respect to which g is simplicial. For each 1-simplex σ of L, let

$$i(\sigma) = \sum_{\{\tau,\tau'\}} \mathcal{L}(Lg_{\tau}\ ,\ Lg_{\tau'}) \quad , \quad \begin{array}{l} \dim(\tau) = \dim(\tau') = 1 \ , \\ \tau \neq \tau' \ , \quad g(\tau) = g(\tau') = \sigma \ . \end{array}$$

Let u be a vertex of $B^{2\ell-1}$. If $u \notin S^{2\ell-2}$, then

$$\sum_{u<\sigma} i(\sigma) = \sum_{u=g(v)} b_g(v) = \text{number of points in } \mathbb{B}(g) \text{ with } g(v) = u \ .$$

If $u \in S^{2\ell-2}$, then

$$\begin{aligned} \sum_{u<\sigma} i(\sigma) &= \sum_{\sigma} \sum_{\{\tau,\tau'\}} \mathcal{L}(Lg_{\tau}\ ,\ Lg_{\tau'}) \quad , \quad u<\sigma \ , \quad \tau \neq \tau' \ , \quad g(\tau) = g(\tau') = \sigma \\ &= \sum_{\{v,v'\}} \sum_{\{w,w'\}} \mathcal{L}(L(Lg_v)_w\ ,\ L(Lg_{v'})_{w'}) \quad , \quad v \neq v' \ , \quad g(v) = g(v') = u \ , \quad g(w) = g(w') \\ &= \sum_{\{v,v'\}} \mathcal{L}(Lg_v\ ,\ Lg_{v'}) \quad , \quad v \neq v' \ , \quad g(v) = g(v') = u \\ &= \sum_{\{v,v'\}} \mathcal{L}(L\partial g_v\ ,\ L\partial g_{v'}) \quad , \quad v \neq v' \ , \quad g(v) = g(v') = u \ . \end{aligned}$$

(The last step is justified by corollary 6.3.)

Therefore, $\sum_u \sum_{u<\sigma} i(\sigma)$ equals this last expression plus the number of points in $\mathbb{B}(g)$. But $\sum_u \sum_{u<\sigma} i(\sigma)$ is zero (mod 2), since each 1-simplex σ appears twice in the sum, once for each endpoint. ∎

Proposition 8.5 If $Y^k \subset X^k$ are k-chains with $Fr(X,Y) \subset \partial Y$, then $r(X,Y)\mathbb{B}^i(X) = \mathbb{B}^i(Y)$ for all $i \geq 0$.

This can be proved just as proposition 7.4, or it can be derived from the following result.

If X^k is a k-chain and $f : X^k \to R^n$ is a spread-out map, $n \geq k$, the number $b_f(\tau)$ can be defined as above for any $(2k-n-1)$-simplex $\tau \not\subset \partial X$. Let $\mathbb{B}(f)$ be the union of all such simplices τ for which $b_f(\tau) \neq 0$.

Proposition 8.6 If X^k is a k-chain, and $f : X^k \to R^n$, $n \geq k$, is spread-out, then $\partial\mathbb{B}(f) \subset \partial X$, and the homology class of the inclusion $(\mathbb{B}(f), \partial\mathbb{B}(f)) \to (X,\partial X)$ is $\mathbb{B}^{n-k+1}(X)$.

Remark. The corresponding statement about $\mathbb{D}$ is false (Figure 8.4).

proof. The statement that $\partial\mathbb{B}(f) \subset \partial X$ follows from the proof of theorem 8.1. That $(\mathbb{B}(f), \partial\mathbb{B}(f)) \to (X,\partial X)$ represents $\mathbb{B}^{n-k+1}(X)$ follows from the fact that f is homotopic to a map $g : X \to R^n$ with $g(X) \subset R^n_+$ and $g(\partial X) \subset R^{n-1}$, and an argument similar to the proof of corollary 7.3. ∎

Exercises

1. Prove theorem 8.1 for $n = k$.

2. Let Y be an ℓ-cycle and let $g : Y \to R^{2\ell}$ be a spread-out map. The crossing number of g is

$$c(g) = \sum_{\{p,q\}} \mathscr{L}(Lg_p , Lg_q) \quad , \quad g(p) = g(q) \quad ,$$

summed over all unordered pairs $\{p,q\}$. Show that if X is a k-cycle and $f : X \to R^n$ is spread-out, $n \geq k$, then $\mathbb{B}(f)$ is the union of all $(2n-k-1)$-simplices τ of a triangulation of f with $c(Lf_\tau) \neq 0$.

3. Illustrate theorem 8.1 with X the Moebius strip and $n = 2$.

4. Illustrate lemma 8.4 with Y the Moebius strip.

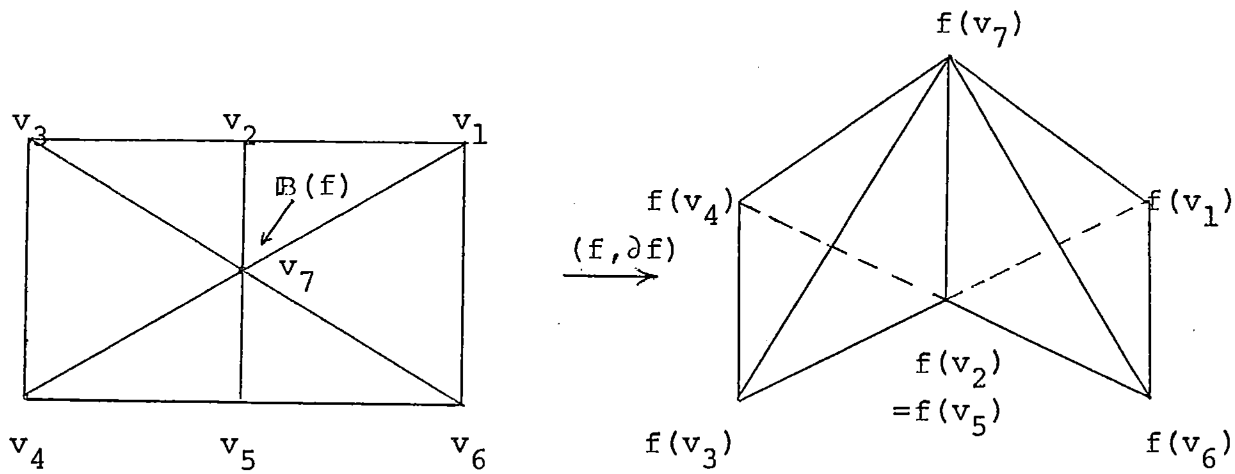


Figure 8.1. $(f,\partial f)$: $(X^2,\partial X) \longrightarrow (R^3_+,R^2)$

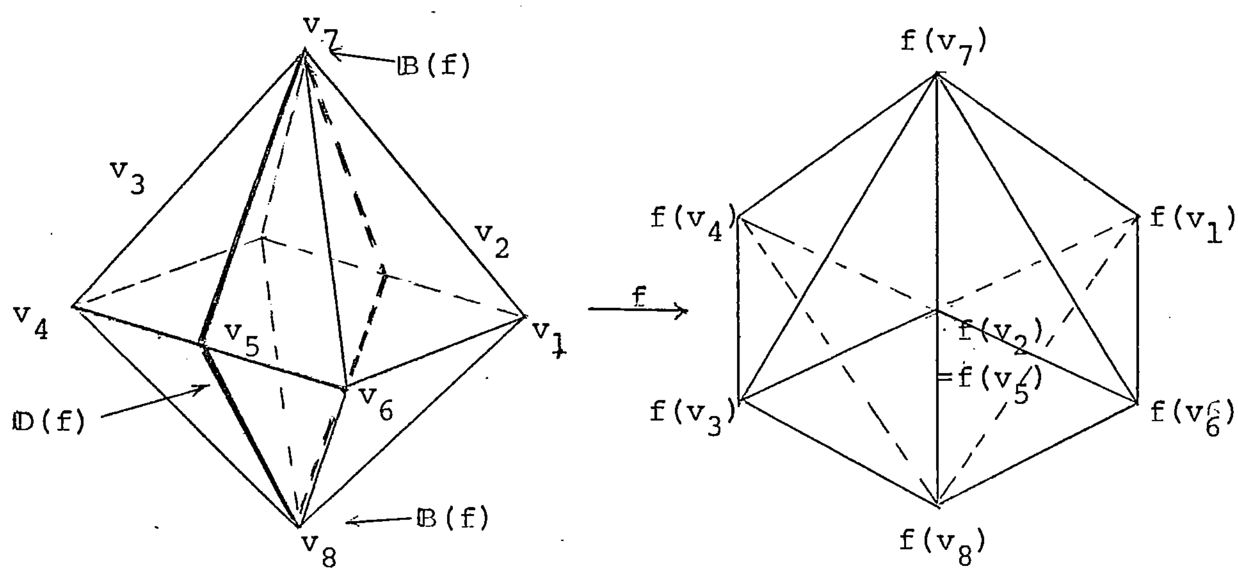


Figure 8.2. f: $S^2 \longrightarrow R^3$ spread-out. The image of the branch points is the boundary of the image of the double points.

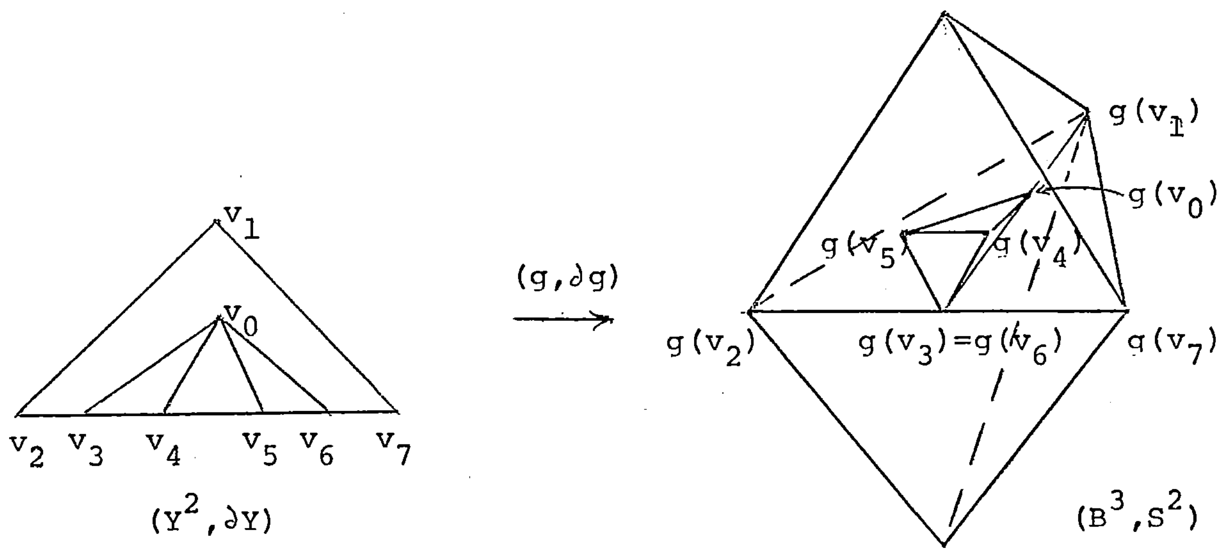


Figure 8.3. $(g,\partial g): (Y^2,\partial Y) \longrightarrow (B^3,S^2)$ spreadout. Note: To avoid confusion not all the simplices have been illustrated.

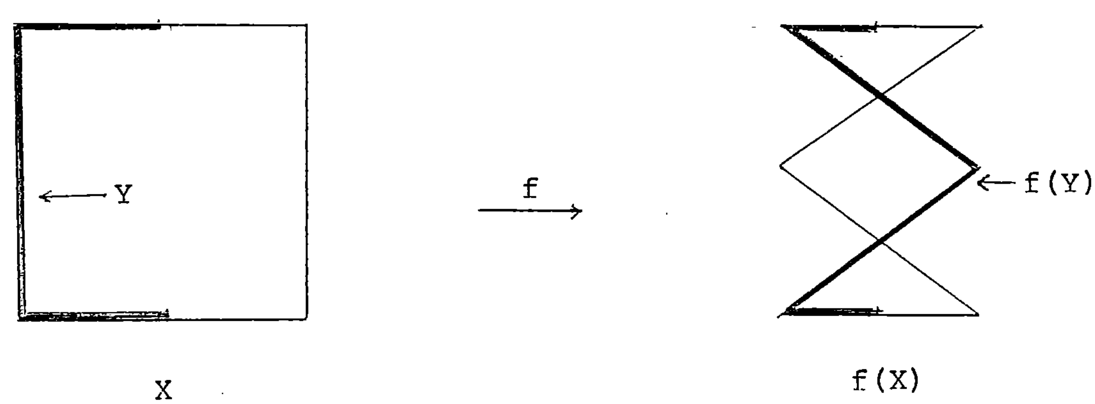


Figure 8.4. $Y^1 \subset X^1$ such that $\mathbb{D}(f|Y) \neq \mathbb{D}(f) \cap Y$

§9. The double-branch homology

The final major result of this chapter will be that $\mathbb{D}^i(X) = \mathbb{B}^i(X)$ for all $i \geq 0$. In the proof of this theorem we will need the following lemma.

Lemma 9.1 Let X^k and Y^ℓ be cycles, and $f : X^k \to R^n$, $g : Y^\ell \to R^n$, $n = k + \ell + 1$, be PL maps such that $f(X) \cap g(Y) = \emptyset$. Let $p : R^n \to R^{n-1}$ be defined by $p(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$, and let $\pi : R^n \to R$ be defined by $\pi(x_1, \ldots, x_n) = x_n$. Suppose that $\{(x,y) \in X \times Y | pf(x) = pg(y)\}$ is finite. Then

$$\mathcal{L}(f,g) = \sum_{x,y} \mathcal{L}(Lpf_x, Lpg_y), \quad \begin{array}{l} pf(x) = pg(y), \\ \pi f(x) < \pi g(y). \end{array}$$

(Figure 9.1).

proof. Choose $T \in R$ such that $\pi g(y) - \pi f(x) < T$ and $\pi g(y) < T$ for all $x \in X$ and $y \in Y$. Let $W = (X \times [0,T]) \cup C_+(X)$, with the identification of $X \times \{T\}$ with $X \subset C_+(X)$. Define $F : W \to R^n$ as follows:

$$F(x,t) = f(x) + (0, \ldots, 0, t), \quad (x,t) \in X \times [0,T],$$
$$F(\lambda c + \mu x) = \mu f(x) + (0, \ldots 0, T), \quad \lambda c + \mu x \in C_+(X).$$

By theorem 6.1, $\mathcal{L}(f,g) = \mathcal{I}(F,g)$. To complete the proof, we will show that if $F(x,t) = g(y)$, then

$$\mathcal{L}(Lpf_x, Lpg_y) = \mathcal{L}(LF_{(x,t)}, Lg_y).$$

Let $z = pf(x)$. Since $F(x,t) = g(y)$, $z = pg(y)$ as well. Let $h : R^n \to R^n$ be the map $h(s_1, \ldots, s_n) = (s_1, \ldots, s_{n-1}, s_n + \pi g(y))$, and let $i : R^{n-1} \to R^n$ be the inclusion $i(s_1, \ldots, s_{n-1}) = (s_1, \ldots, s_{n-1}, 0)$. Let $h' = Lh_{i(z)}$ and $i' = Li_z$. By proposition 5.1(f),

$$\begin{aligned}\mathcal{L}(Lpf_x , Lpg_y) &= \mathcal{L}(SLpf_x , i' \circ Lpg_y) \\ &= \mathcal{L}(h' \circ SLpf_x , h' \circ i' \circ Lpg_y) .\end{aligned}$$

Now $h' \circ SLpf_x$ is homotopic to $LF_{(x,t)}$, and $h' \circ i' \circ Lpg_y$ is homotopic to Lg_y , by homotopies with disjoint images. So proposition 5.1(b) implies that

$$\mathcal{L}(h' \circ SLpf_x , h' \circ i' \circ Lpg_y) = \mathcal{L}(LF_{(x,t)} , Lg_y) .$$

(Figure 9.2)

<u>Theorem</u> 9.2 If X^k is a k-chain, then $\mathbb{D}^i(X) = \mathbb{B}^i(X) \in H_{k-i}(X, \partial X)$ for $i \geq 0$.

(Figure 9.3)

<u>proof</u>. This is true by definition for $i = 0$. We prove the theorem for $i > 1$. The case $i = 1$ requires a separate argument. Let $n = k + i$, and let $p : R^n_+ \to R^{n-1}_+$, $p(s_1, \ldots, s_n) = (s_2, \ldots, s_n)$, and $\pi : R^n_+ \to R$, $\pi(s_1, \ldots, s_n) = s_1$. By theorem 2.2(c), there exist spread-out maps $(f, \partial f) : (X, \partial X) \to (R^n_+ , R^{n-1})$ and $(g, \partial g) : (X, \partial X) \to (R^{n-1}_+ , R^{n-2})$ such that $g = p \circ f$

and $g^{-1}(R^{n-2}) = f^{-1}(R^{n-1}) = \partial X$. Let K be a triangulation of X with respect to which f and g are simplicial.

Let σ, σ' be $(k-i+1)$-simplices of K such that $g(\sigma') = g(\sigma)$. Since $f(\sigma') \neq f(\sigma)$, it follows that either $\pi f(s) < \pi f(s')$ for all $s \in \sigma^\circ$ and $s' \in (\sigma')^\circ$ with $g(s) = g(s')$, or $\pi f(s) > \pi f(s')$ for all $s \in \sigma^\circ$ and $s' \in (\sigma')^\circ$ with $g(s) = g(s')$. We will abbreviate these two conditions to $\pi f(\sigma) < \pi f(\sigma')$ or $\pi f(\sigma) > \pi f(\sigma')$.

If σ is a $(k-i+1)$-simplex of K, we define

$$d_g^-(\sigma) = \sum_{\sigma'} \mathcal{L}(Lg_{\sigma'}, Lg_\sigma) \quad , \quad \begin{array}{l} \dim(\sigma') = k-i+1 \quad , \quad \sigma' \neq \sigma \quad , \\ g(\sigma') = g(\sigma) \quad , \quad \pi f(\sigma') > \pi f(\sigma) \quad . \end{array}$$

Let $\mathbb{D}^-(g) \subset D(g)$ be the union of all the $(k-i+1)$-simplices σ of K such that $d_g^-(\sigma) \neq 0$.

If τ is a $(k-i)$-simplex of K with $\tau \not\subset \partial X$, let $\partial d_g^-(\tau) = \sum_{\tau<\sigma} d_g^-(\sigma)$. We must show that

$$\partial d_g^-(\tau) = d_f(\tau) + b_g(\tau) \quad .$$

By definition,

$$\partial d_g^-(\tau) = \sum_{\sigma', \sigma} \mathcal{L}(Lg_{\sigma'}, Lg_\sigma) \quad , \quad \begin{array}{l} \tau < \sigma \quad , \quad \sigma' \neq \sigma \quad , \\ g(\sigma') = g(\sigma) \quad , \\ \pi f(\sigma') > \pi f(\sigma) \end{array}$$

If $g(\sigma') = g(\sigma)$ and $f(\sigma') \neq f(\sigma)$, then $f(\sigma') \cap f(\sigma)$ is either empty, or a common face of $f(\sigma')$ and $f(\sigma)$. Thus this last expression equals

$$\text{(i)} \quad \sum_{\sigma',\sigma} \mathcal{L}(Lg_{\sigma'}, Lg_{\sigma}) \ , \quad \tau < \sigma \ , \ \tau < \sigma' \ , \ \sigma' \neq \sigma \ , \ g(\sigma') = g(\sigma) \ , \ \pi f(\sigma') > \pi f(\sigma)$$

$$\text{(ii)} \quad + \sum_{\substack{v',v \\ \tau'}} \mathcal{L}(Lg_{\tau'v'}, Lg_{\tau v}) \ , \quad \tau' \neq \tau \ , \ f(\tau') = f(\tau) \ , \ g(v') = g(v) \ , \ \pi f(v') > \pi f(v)$$

$$\text{(iii)} \quad + \sum_{\substack{v',v \\ \tau'}} \mathcal{L}(Lg_{\tau'v'}, Lg_{\tau v}) \ , \quad \tau' \neq \tau \ , \ f(\tau') \neq f(\tau) \ , \ g(v') = g(v) \ , \ \pi f(\tau') > \pi f(\tau) \ , \ g(\tau') = g(\tau) \ .$$

The summand (i) equals $b_g(\tau)$. Lemma 9.1 applied to $Lf_{\tau'}$ and Lf_{τ} shows that the summand (ii) equals

$$\sum_{\tau'} \mathcal{L}(Lf_{\tau'}, Lf_{\tau}) \ , \quad \tau' \neq \tau \ , \ f(\tau') = f(\tau) \ ,$$

which is $d_f(\tau)$. The summand (iii) equals

$$\sum_{\tau'} \mathcal{A}(Lg_{\tau'}, Lg_{\tau}) \ , \quad \tau' \neq \tau \ , \ f(\tau') \neq f(\tau) \ , \ \pi f(\tau') > \pi f(\tau) \ , \ g(\tau') = g(\tau) \ .$$

Since $\tau \not\subset \partial X$, LX_{τ} and $LX_{\tau'}$ are cycles. By corollary 6.2, $\mathcal{A}(Lg_{\tau'}, Lg_{\tau}) = 0$, so (iii) is zero. ∎

Exercises

1. Prove $\mathbb{D}^1 = \mathbb{B}^1$. Illustrate this for the Klein bottle.

2. Discuss the integral version of lemma 9.1.

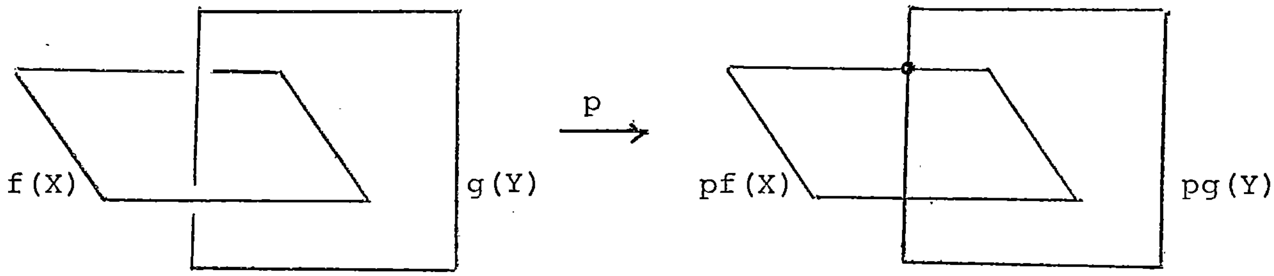


Figure 9.1. $\mathcal{L}(f,g) = \sum_{x,y} \mathcal{L}(pf_x, pg_y)$ $\quad pf(x) = pg(y),$

$\pi f(x) < \pi g(y)$

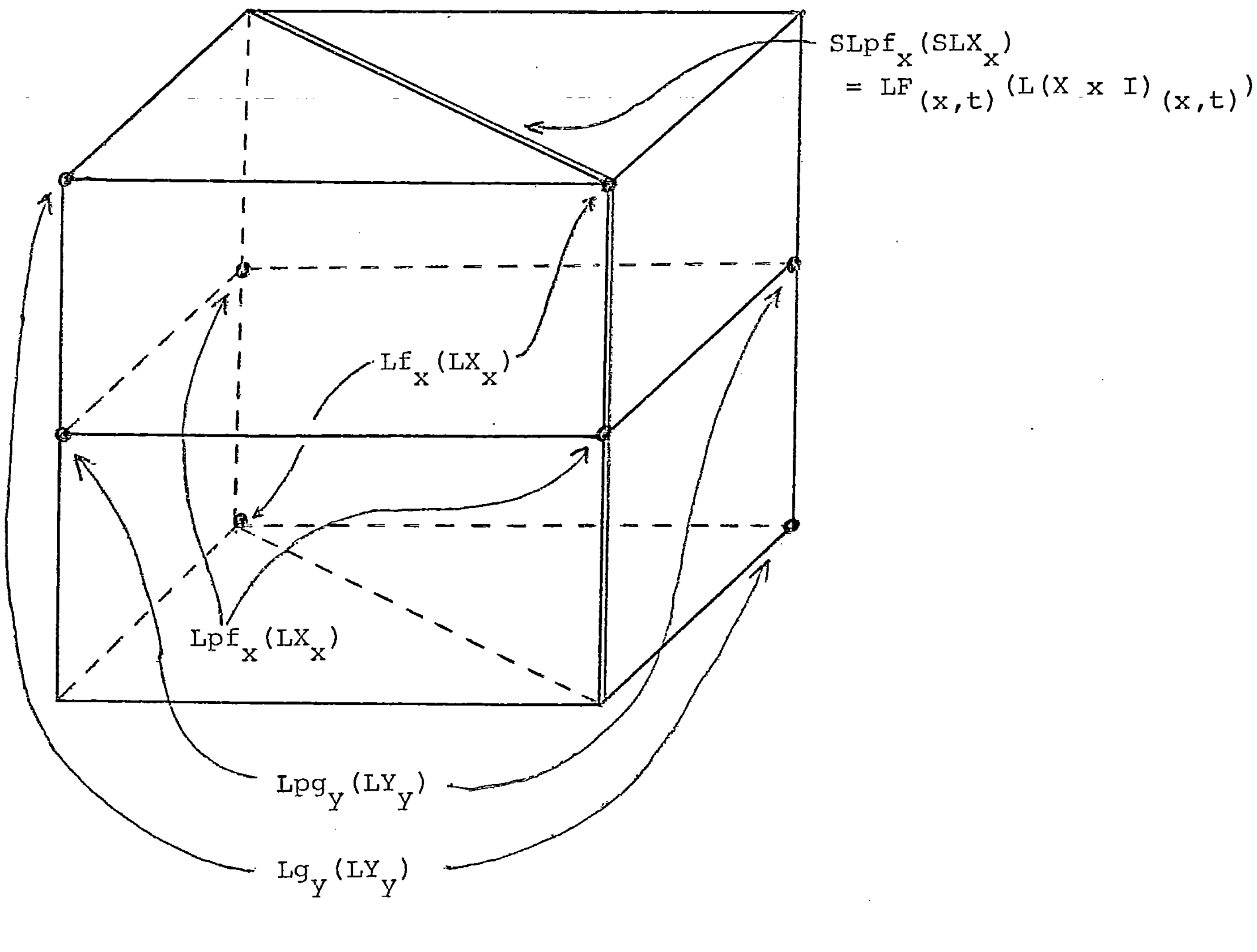


Figure 9.2. $\mathcal{L}(LF_{(x,t)}, Lg_y) = \mathcal{L}(Lpf_x, Lpg_y)$

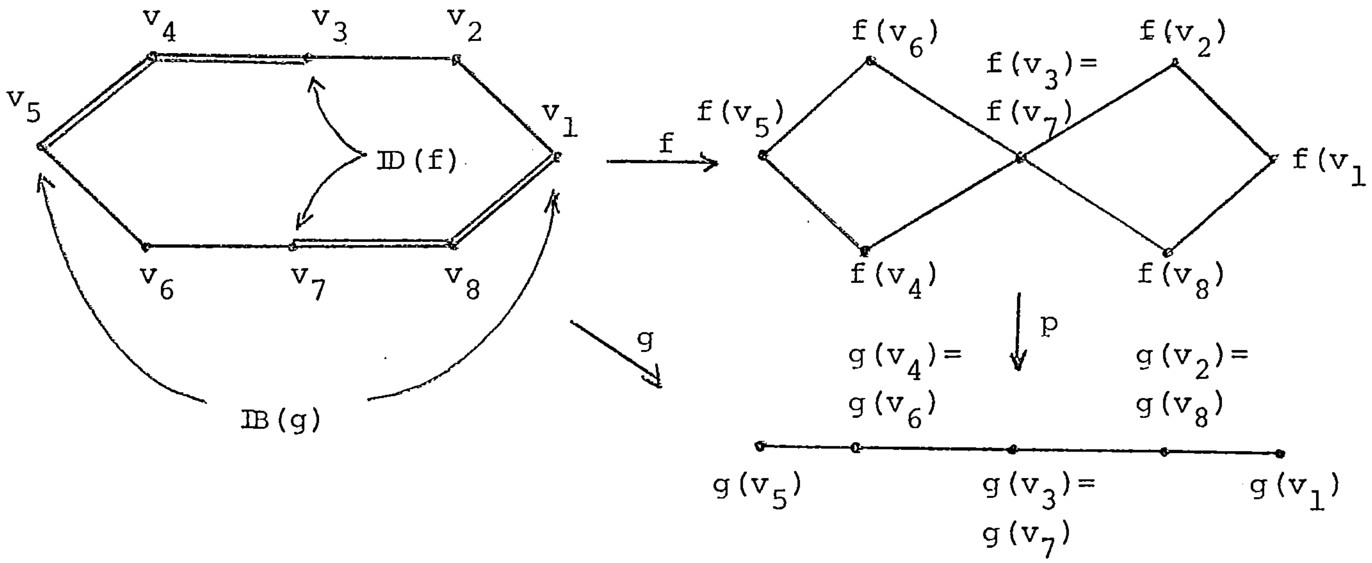


Figure 9.3. $\mathbb{D}(f)$ is homologous to $\mathbb{B}(g)$.

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Lectures on Homology Operations
Chapter 3

by

Clint McCrory and Kent Johnson

§1. Homology operations

Let (H,∂) be a homology theory satisfying the first six axioms of Eilenberg and Steenrod [ES, p. 10-11], for example mod 2 geometric homology (section 3 of chapter 2).

A <u>homology operation</u> θ is a natural transformation from the functor H to itself. That is, for each pair of spaces (A,B) there is a function $\theta(A,B) : H(A,B) \to H(A,B)$ such that for every continuous map $f : (A,B) \to (C,D)$ the diagram

$$\begin{tikzcd} H(A,B) \arrow[r, "H(f)"] \arrow[d, "\theta(A,B)"] & H(C,D) \arrow[d, "\theta(C,D)"] \\ H(A,B) \arrow[r, "H(f)"] & H(C,D) \end{tikzcd}$$

commutes. The operation θ is <u>stable</u> if it commutes with the boundary transformation; that is, if the diagram

$$\begin{tikzcd} H(A,B) \arrow[r, "\partial"] \arrow[d, "\theta(A,B)"] & H(B) \arrow[d, "\theta(A)"] \\ H(A,B) \arrow[r, "\partial"] & H(B) \end{tikzcd}$$

commutes for all (A,B) . θ has <u>degree</u> i if it lowers degrees of all homology classes by i , that is, if

$$\theta(A,B)(H_k(A,B)) \subset H_{k-i}(A,B)$$

for all k .

Example. Let (H,∂) be mod 2 simplicial homology theory. The Bockstein operation β is a stable homology operation of degree one,

$$\beta = \beta(K,L) : H_k(K,L) \to H_{k-1}(K,L) \quad .$$

It can be defined as follows (we take $L = \emptyset$ for simplicity). If c is any simplicial mod 2 cycle in K , choose an integral chain $\tilde{c}$ such that $\tilde{c}_2 = c$, where $\tilde{c}_2$ is the mod 2 reduction of $\tilde{c}$. Then $(\partial\tilde{c})_2 = \partial(\tilde{c}_2) = \partial c = 0$, so all the coefficients in $\partial\tilde{c}$ are even, and therefore $\frac{1}{2}\partial\tilde{c}$ is still an integral chain. Furthermore $\partial(\frac{1}{2}\partial\tilde{c}) = \frac{1}{2}\partial\partial\tilde{c} = 0$, i.e. $\frac{1}{2}\partial\tilde{c}$ is an integral cycle. The Bockstein operations sends the class of c to the class of the mod 2 reduction of $\frac{1}{2}\partial\tilde{c}$:

$$\beta[c] = [(\tfrac{1}{2}\partial\tilde{c})_2] \quad .$$

It is very easy to check that $\beta[c]$ is independent of the choice of $\tilde{c}$.

Remark. The operation β can be defined similarly in mod 2 singular homology. In the next section we shall see that β has a geometric interpretation in terms of double points or branch points.

For example if K is a triangulation of the Klein bottle and $\langle K\rangle \in H_2(K)$ is represented by the sum of all the 2-simplices in K, then $\beta\langle K\rangle \neq 0$. On the other hand if T is a triangulation of the torus then $\beta\langle T\rangle = 0$. (Figure 1.1)

To illustrate how homology operations can be used, we are now able to prove that there is no continuous map from the torus to the Klein bottle with nonzero mod 2 degree. For if there were such a map f, then $f_*\langle T\rangle = \langle K\rangle$, so we would have $\beta f_*\langle T\rangle = \beta\langle K\rangle \neq 0$. But $\beta f_*\langle T\rangle = f_*\beta\langle T\rangle$ by naturality, and $f_*\beta\langle T\rangle = 0$ since $\beta\langle T\rangle = 0$, giving a contradiction.

Exercises

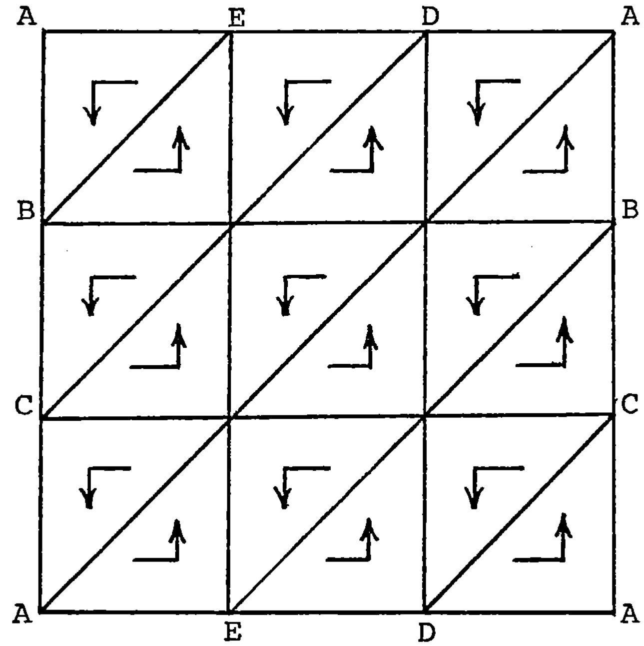
1. Show that a homology operation is stable if and only if it commutes with the suspension homomorphism.

2. Show that every homology operation θ is additive, i.e. $\theta(A,B)$ is a homomorphism for all (A,B). (This observation is due to Steenrod.)

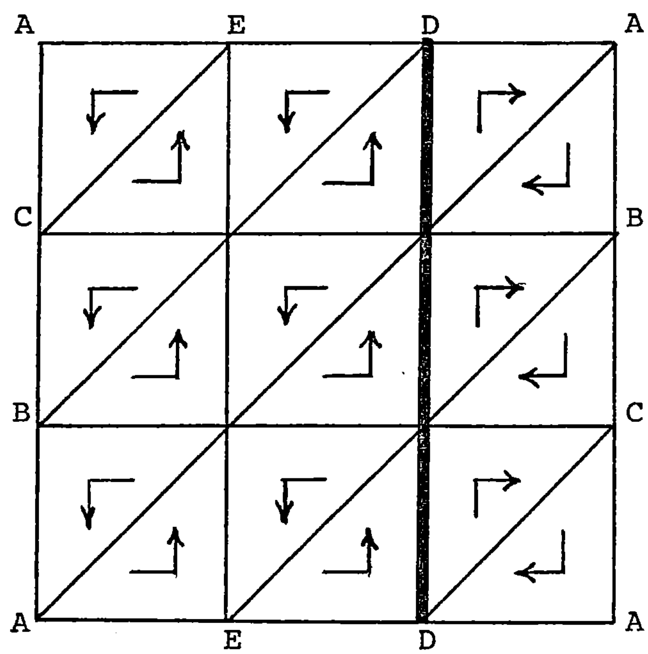
3. Check that the simplicial Bockstein homomorphism β is well-defined. Show that $\beta \circ \beta = 0$, and that β is stable.

4. Let $X = |K|$ be a polyhedral k-cycle, and let $\langle X\rangle \in H_k(K)$ be

the simplicial fundamental class of X . Show that $\mathbf{B}(f)$ represents $\beta\langle X\rangle$, where $f : X \to R^k$ is a spread-out map which is simplicial with respect to K .



Torus



Klein Bottle

Figure 1.1. The Bockstein operation.

§2. The geometric operations Φ^i

If X is a geometric k-chain then the homology class $\mathbb{D}^i(X) = \mathbb{B}^i(X) \in H_{k-i}(X,\partial X)$ of chapter 2 can be interpreted as an operation on mod 2 geometric homology as follows. We define

$$\Phi^i = \Phi^i(A,B) : H_k(A,B) \to H_{k-i}(A,B)$$

for all k by sending the homology class of the element $f : (X,\partial X) \to (A,B)$ of $Z_k(A,B)$ to the class of $f_*\mathbb{D}^i(X)$:

$$\Phi^i[f] = f_*\mathbb{D}^i(X) \quad .$$

To show that this is a good definition, we have to check that Φ^i does not depend on the choice of f in its homology class and that Φ^i is a natural transformation.

<u>Proposition</u> 2.1 If $f : (X,\partial X) \to (A,B)$ is in $B_k(A,B)$ then $f_*\mathbb{D}^i(X) = 0$ for all i .

<u>proof</u>: By definition of $B_k(A,B)$ there exists a (k+1)-chain W and a commutative diagram

$$\begin{tikzcd}
(e(X),e(\partial X)) \arrow[r, "i"] & (\partial W, \mathrm{c}\ell\,\partial W \setminus e(X)) \arrow[r, "j"] & (W, \mathrm{c}\ell\,\partial W \setminus e(X)) \arrow[d, "F"] \\
(X,\partial X) \arrow[u, "\wr"] \arrow[ur, "e"] \arrow[rr, "f"] & & (A,B)
\end{tikzcd}$$

where e is a polyhedral embedding and i and j are inclusions. Let $r : H_{k-i}(\partial W) \to H_{k-i}(e(X), e(\partial X))$ be the restriction homomorphism (see proposition 7.4 of chapter 2). Then we have

$$\begin{aligned} f_* \mathbb{D}^i(X) &= F_* j_* e_* \mathbb{D}^i(X) \\ &= F_* j_* i_* \mathbb{D}^i(e(X)) \\ &= F_* j_* i_* r\, \mathbb{D}^i(\partial W) && \text{by proposition 7.4 of chapter 2} \\ &= F_* j_* i_* r \partial\, \mathbb{D}^i(W) && \text{by theorem 7.1 of chapter 2} \\ &= 0 \end{aligned}$$

since j_* and $(i_* r \partial)$ are consecutive homomorphisms in the long exact sequence of the triple $(W, \partial W, (\partial W \setminus e(X)))$. ∎

<u>Proposition</u> 2.2 Φ^i is a natural transformation.

<u>proof</u>: Let $g : (A,B) \to (C,D)$ be a continuous map, and let $f : (X, \partial X) \to (A,B)$ be in $Z_k(A,B)$. Then $g_* \Phi^i[f] = g_* f_* \mathbb{D}^i(X) = (g \circ f)_* \mathbb{D}^i(X) = \Phi^i[g \circ f] = \Phi^i g_*[f]$. ∎

<u>Proposition</u> 2.3 The operation Φ^i has the following properties:

(1) Φ^i is stable.

(2) Φ^0 is the identity transformation.

(3) Φ^1 is the Bockstein transformation.

(4) If $x \in H_k(A,B)$ then $\Phi^i(x) = 0$ for all $i \geq k$.

<u>proof</u>: (1) follows from theorem 7.1 of chapter 2. (2) follows from

the definition of $\mathbb{D}^i(X)$, as does (4) for $i > k$. For $i = k$, (4) is true since double points occur in pairs in the source (cf. figure 7.2 of chapter 2). And (3) comes from exercise 4 of section 1 and the definition of $\mathbb{B}^1$. ∎

Exercises

1. Prove (4) of proposition 2.3 for $i = k$ by using (1) and the cone on a polyhedral chain.

2. Prove (3) of proposition 2.3 by using the double point cycle.

3. Prove that Φ^i commutes with suspension by using the double point cycle.

4. Give a geometric proof of proposition 2.1.

§3. The intersection axiom

The homology operation Φ^i is characterized by its relation to the self-intersection of a homology class in a manifold.

Let M^n be a PL n-manifold (without boundary), and let $X^k \subset M^n$ be a k-chain and $Y^\ell \subset M^n$ an ℓ-chain, with $k + \ell \geq n$, $k > 0$, and $\ell > 0$. Suppose that X and Y are in general position, i.e. $\dim(X \cap Y) \leq k + \ell - n$, $\dim(X \cap \partial Y) \leq k + \ell - n - 1$, and $\dim(\partial X \cap Y) \leq k + \ell - n - 1$. Then an intersection chain $X.Y \subset X \cap Y$ can be defined as follows.

Triangulate M so that X and Y are subcomplexes. For each $(k+\ell-n)$-simplex σ of $X \cap Y$, define

$$i(\sigma) = \mathcal{L}(LX_\sigma, LY_\sigma) \quad ,$$

the linking number of the $(n-k-1)$-cycle LX_σ and the $(n-\ell-1)$-cycle LY_σ in the $(2n-k-\ell-1)$-sphere LM_σ. Let $X.Y$ be the union of all the $(k+\ell-n)$-simplices σ of $X \cap Y$ such that $i(\sigma) \neq 0$.

Proposition 3.1 If X and Y are chains in general position in the manifold M, then

$$\partial(X.Y) = (\partial X).Y + X.(\partial Y) \quad .$$

This is an easy consequence of the linking theorem (6.1 of chapter 2). (Cf. figure 3.1)

Lemma 3.2 If $\alpha \in H_k(M)$ and $\beta \in H_\ell(M)$, where M is a manifold, there is a cycle $X \subset M$ representing α and a cycle $Y \subset M$ representing β such that X and Y are in general position.

This follows from the spreading out technique of chapter 2, §2.

Now if $\alpha \in H_k(M^n)$ and $\beta \in H_\ell(M^n)$, we define $\alpha.\beta \in H_{k+\ell-n}(M^n)$ to be the homology class of $X.Y$, for X and Y as in lemma 3.2. It is not hard to show that this intersection product is a well-defined associative, commutative bilinear pairing

$$H_k(M^n) \times H_\ell(M^n) \to H_{k+\ell-n}(M^n) \quad .$$

This definition is due to Lefschetz [L, ch. IV §6]. (Alexandroff and Hopf probably would have developed this definition in volume two of their book [AH], had it been written. They developed linking in Euclidean space in volume one.) The definition can be extended to non-embedded geometric chains (cf. §6 of chapter 2), and to a manifold with boundary, modulo a (PL) subset:

$$H_k(M^n,A) \times H_\ell(M^n,A) \to H_{k+\ell-n}(M^n,A) \quad .$$

Theorem 3.3

1) Let M be a compact n-dimensional PL submanifold (with boundary) of R^n, and let A be a subpolyhedron of M. If $\alpha \in H_{n-i}(M,A)$ then $\Phi^i(\alpha) = \alpha.\alpha \in H_{n-2i}(M,A)$.

2) If Ψ^i is any mod 2 homology operation of degree i such that $\Psi^i(\alpha) = \alpha.\alpha$ for all α as in (1), then $\Psi^i = \Phi^i$.

proof of (1). Replacing A by a regular neighborhood, we can assume that $A = N$, a compact n-manifold with boundary. Let $(X,\partial X) \subset (\text{int } M, \text{int } N)$ be an embedded chain representing α.

For each vector $v \in R^n$, define $g_v : X \to R^n$ by $g_v(x) = x + v$. Let $W = (X \times 0) \cup (\partial X \times I) \cup (X \times 1)$, and define $h_v : W \to R^n$ by

$$h_v(x,0) = x \;,$$
$$h_v(x,t) = x + tv \;, \quad x \in \partial X \;, \quad 0 \le t \le 1 \;,$$
$$h_v(x,1) = g_v(x) \;.$$

There exists a vector $v \in R^n$ such that

a) $g_{tv}(X,\partial X) \subset (\text{int } M, \text{int } N)$ for all $t \in [0,1]$,

b) X and $g_v(X)$ are in general position,

c) h_v is spread-out.

Choose such a vector v, and let $Y = g_v(X)$, $h = h_v$. It follows from the definition of the double point cycle $\mathbb{D}$ that

$$[X.Y] = i_* r(W, X \times 0)\mathbb{D}(h) \in H_{n-2i}(M,N) ,$$

where $i : (X \times 0, \partial X \times 0) \to (M,N)$ is the inclusion and $r(W, X \times 0)$ is the restriction homomorphism. By proposition 7.4 of chapter 2, $r(W, X \times 0)\mathbb{D}(h)$ represents $\mathbb{D}(X \times 0)$, so $i_* r(W, X \times 0)\mathbb{D}(h)$ represents $\Phi^i(\alpha)$, and (1) is proved.

proof of (2). It suffices to show that $\Psi^i[X] = \Phi^i[X]$ for any k-chain X. For suppose $\alpha \in H_k(A,B)$ is represented by $f : (X, \partial X) \to (A,B)$, so that $\alpha = f_*[X]$. Then $\Psi^i(\alpha) = \Psi^i f_*[X] = f_* \Psi^i[X] = f_* \Phi^i[X] = \Phi^i f_*[X] = \Phi^i(\alpha)$. In fact it suffices to show that $\Psi^i[Z] = \Phi^i[Z]$ for any k-cycle Z. For if X is a k-chain, then the space dX, obtained by gluing together two copies of X along ∂X, is a cycle. Letting $r : H_*(dX) \to H_*(X, \partial X)$ be the restriction homomorphism, we then have $\Psi^i[X] = \Psi^i r[dX] = r\Psi^i[dX] = r\Phi^i[dX] = \Phi^i r[dX] = \Phi^i[X]$.

If Y is a k-chain which embeds piecewise-linearly in R^{k+i}, then $\Psi^i[Y] = \Phi^i[Y]$. For let (M,N) be a regular neighborhood of $(Y, \partial Y)$ in R^{k+i}, and let $i : (Y, \partial Y) \to (M,N)$ be the inclusion. Then $i_* \Psi^i[Y] = \Psi^i i_*[Y] = (i_*[Y]).(i_*[Y]) = \Phi^i i_*[Y] = i_* \Phi^i[Y]$, so $\Psi^i[Y] = \Phi^i[Y]$ since i_* is an isomorphism.

Now let Z be a k-cycle. We shall construct a k-chain $Y \subset Z$ such that

a) $Fr(Z,Y) = \partial Y$,

b) the restriction homomorphism $r : H_{k-i}(Z) \to H_{k-i}(Y,\partial Y)$ is injective, and

c) Y embeds in R^{k+i} .

This will prove (2), because $r\Psi^i[Z] = \Psi^i r[Z] = \Psi^i[Y] = \Phi^i[Y] = \Phi^i r[Z] = r\Phi^i[Z]$, so $\Psi^i[Z] = \Phi^i[Z]$.

So it remains to construct the k-chain $Y \subset Z$. Let K be a triangulation of Z , and let Y be the closure of the complement of a regular neighborhood of the (k-i-1)-skeleton K_{k-i-1} of K . Then (a) is clear. To see (b) consider the following commutative diagram:

$$\begin{array}{ccc} H_{k-i}(Z) & \xrightarrow{r} & H_{k-i}(Y,\partial Y) \\ \downarrow u & & \downarrow v \\ H_{k-i}(Z,|K_{k-i-1}|) & \xrightarrow{w} & H_{k-i}(Z,C\ell(Z\setminus Y)) \end{array}$$

Here $u,v,$ and w are induced by inclusions. The map u is injective since $H_{k-i}(|K_{k-i-1}|) = 0$. The map v is an excision isomorphism, and w is an isomorphism because $|K_{k-i-1}|$ is a strong deformation retract of $C\ell(Z\setminus Y)$. Therefore r is injective.

Finally we show that (c) holds, for an appropriate choice of triangulation K. Choose K so that there is a spread-out map $f : Z \to R^{k+i}$ which is simplicial on K. Then $D(f) \subset |K_{k-i}|$. Now modify the second barycentric subdivision K'' as follows. Move the barycenters of the (k-i)-simplices σ of K to points $b(\sigma)$ such that $f(b(\sigma_1)) = f(b(\sigma_2))$ only if $\sigma_1 = \sigma_2$. Call the resulting subdivision $\tilde{K}'$. For each simplex ω' of $\tilde{K}'$ there is a unique simplex σ of K such that the interior of ω' is contained in the interior of σ. For each such ω', move the barycenter of ω' to a point $b(\omega')$ very close to $b(\sigma)$. Call the resulting shifted second barycentric subdivision $\tilde{K}''$. Choose the points $b(\omega')$ so close to the corresponding points $b(\sigma)$ that $f(\mathrm{Star}(b(\sigma_1),\tilde{K}'')) \cap f(\mathrm{Star}(b(\sigma_2),\tilde{K}'')) \neq \emptyset$ only if $\sigma_1 = \sigma_2$.

Now let Y (respectively $\tilde{Y}$) be the union of all the simplices of K'' (respectively $\tilde{K}''$) which have no vertices in the subdivision of K_{k-i-1}. Then Y is PL homeomorphic to $\tilde{Y}$, and $f|\tilde{Y}$ is an embedding, by construction of Y. ∎

Exercises

1. Prove proposition 3.1.

2. What are the signs in the integral version of proposition 3.1?

3. Carry out the proof that the intersection product is well-defined in homology. (You will need a stronger version of lemma 3.2.)

4. Go through the proof of 3.3(2) for Z the Klein bottle. In other words, show that if Ψ^1 has the intersection property then $\Psi^1[Z] = \Phi^1[Z]$.

5. Let L be a subcomplex of the simplicial complex K . Construct a strong deformation retraction of N to $|L|$, where N is the stellar neighborhood of L'' in K'' (second barycentric subdivisions).

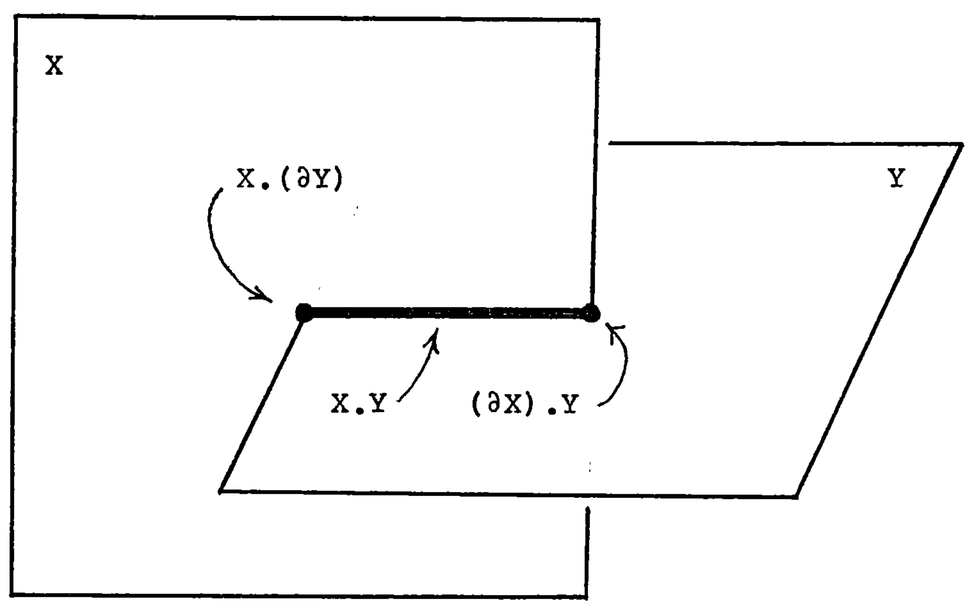


Figure 3.1. $\partial(X.Y) = (\partial X).Y + X.(\partial Y)$.

§4. Steenrod operations

The set $\mathcal{O}$ of all operations on mod 2 homology is a ring, with addition $(\theta + \Psi)(\alpha) = \theta(\alpha) + \Psi(\alpha)$, multiplication $(\theta\Psi)(\alpha) = \theta(\Psi(\alpha))$ and multiplicative identity 1 equal to the identity transformation. Furthermore $\mathcal{O}$ is graded by degree. Any $\theta \in \mathcal{O}$ can be written uniquely as a sum $\theta = \theta^0 + \theta^1 + \theta^2 + \ldots$, where θ^i has degree i . (If α is a homology class of dimension k then $\theta^i(\alpha) = 0$ for $i > k$, so $\theta(\alpha) = \theta^0(\alpha) + \theta^1(\alpha) + \ldots + \theta^k(\alpha)$, a finite sum.)

Define $\Phi \in \mathcal{O}$ by $\Phi = \Phi^0 + \Phi^1 + \Phi^2 + \ldots$. Since $\Phi^0 = 1$ there exists a unique operation $\bar{\Phi} \in \mathcal{O}$ such that $\Phi\bar{\Phi} = 1 = \bar{\Phi}\Phi$. The components $\bar{\Phi}^i$ of $\bar{\Phi}$ can be calculated inductively from those of Φ from the identity $(1 + \Phi^1 + \Phi^2 + \ldots)(\bar{\Phi}^0 + \bar{\Phi}^1 + \bar{\Phi}^2 + \ldots) = 1$.

Since mod 2 cohomology $H^k(A,B)$ can be identified with $\mathrm{Hom}(H_k(A,B), \mathbb{Z}/2)$, the dual of the $\mathbb{Z}/2$ vector space $H_k(A,B)$, we can define a pairing $\langle , \rangle$ between cohomology and homology as follows. If $\alpha \in H^k(A,B)$ and $\beta \in H_\ell(A,B)$ then

$$\langle \alpha, \beta \rangle = \begin{cases} \alpha(\beta) & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases} .$$

Furthermore, we can define an action of Φ^i on cohomology by the identity

$$<\Phi^i(\alpha),\beta> = <\alpha,\bar{\Phi}^i(\beta)> \quad .$$

Let Sq^i be the ith Steenrod operation on mod 2 cohomology [SE].

Theorem 4.1 $\quad \Phi^i = Sq^i$.

proof. We can just as well make Sq^i act on homology, by requiring that

$$<\alpha,Sq^i(\beta)> = <\overline{Sq}^i(\alpha),\beta>$$

(cf. [MS, p.136]), and the theorem is equivalent to the statement that $\Phi^i = Sq^i$ on homology. To show this we need only check that the action of Sq^i on homology is natural and satisfies the intersection axiom, by theorem 3.1.

a) Sq^i is natural, i.e. $Sq^i f_*(\beta) = f_* Sq^i(\beta)$ for all homology classes β and maps f . This follows from the naturality of $\overline{Sq}^i$ on cohomology: $<\alpha,Sq^i f_*\beta> = <\overline{Sq}^i\alpha,f_*\beta> = <f^*\overline{Sq}^i\alpha,\beta> = <\overline{Sq}^i f^*\alpha,\beta> = <f^*\alpha,Sq^i\beta> = <\alpha,f_*Sq^i\beta>$.

b) Sq^i satisfies the intersection property of 3.3(1) To prove this we need a lemma.

Let $Sq = Sq^0 + Sq^1 + Sq^2 + \ldots$, and let $\frown$ be the cap product pairing.

<u>Lemma</u> 4.2 (1) $Sq(\alpha \frown \beta) = Sq\alpha \frown Sq\beta$.

(2) If M is a compact n-dimensional submanifold of R^n , then $Sq[M] = [M]$.

The proof of this lemma is left as an exercise. (Hints are given below.)

Now let $\alpha \in H^i(M \setminus N, (M \setminus N) \cap \partial M)$ be the Poincaré dual of β , i.e. $\alpha \frown [M] = \beta$. Then

$$\begin{aligned} Sq^i\beta &= Sq^i(\alpha \frown [M]) \\ &= \sum_j Sq^j\alpha \frown Sq^{i-j}[M] && \text{by (1)} \\ &= Sq^i\alpha \frown [M] && \text{by (2)} \\ &= (\alpha \smile \alpha) \frown [M] && \text{by [SE, p.1, (3)]} \\ &= \beta.\beta \ . \end{aligned}$$

This completes the proof of theorem 4.1. ∎

The proof of the following theorem illustrates how the geometry of Φ and the algebra of Sq can be used together.

<u>Theorem</u> 4.3 Let n be a positive integer. The real projective space P^{2^n} cannot be polyhedrally embedded in $R^{2^{n+1}-1}$ or polyhedrally immersed in $R^{2^{n+1}-2}$.

proof. Let α be a generator of $H^1(P^{2^n})$. Then $\langle\alpha, Sq^{2^n-1}[P^{2^n}]\rangle = \langle\bar{S}q^{2^n-1}\alpha, [P^{2^n}]\rangle = \langle\alpha^{2^n}, [P^{2^n}]\rangle \neq 0$. That $\bar{S}q^{2^n-1}\alpha = \alpha^{2^n}$ is proved by a simple induction (cf. [SE, p.34]). Therefore $Sq^{2^n-1}[P^{2^n}] \neq 0$. But $Sq^{2^n-1}[P^{2^n}] = \Phi^{2^n-1}[P^{2^n}] = \mathbb{D}^{2^n-1}[P^{2^n}]$, the class of the double point cycle of a spread-out map

$$P^{2^n} \to R^{2^{n+1}-1} ,$$

so P^{2^n} can't be embedded in $R^{2^{n+1}-1}$. Since $Sq^{2^n-1}[P^{2^n}] = \mathbb{B}^{2^n-1}[P^{2^n}]$, the class of the branch point cycle of a spread-out map

$$P^{2^n} \to R^{2^{n+1}-2} ,$$

neither can P^{2^n} be immersed in $R^{2^{n+1}-2}$. (Clearly any immersion can be approximated by a spread-out immersion.) ∎

Remark. This theorem can be strengthened to the nonexistence of topological embeddings and immersions (cf. [M2,§7], [HM,§9]).

A challenging open problem is to give geometric proofs of the Cartan formula [SE, p. 1] and the Adem relations [SE, p. 2] for the Steenrod operations using double points and branch points.

Exercises

1. Let $x_1, x_2, \ldots$ be noncommuting symbols. Define $\bar{x}_1, \bar{x}_2, \ldots$

by the identity $(1 + x_1 + x_2 + \dots)(1 + \bar{x}_1 + \bar{x}_2 + \dots) = 1$. Show that $(1 + \bar{x}_1 + \bar{x}_2 + \dots)(1 + x_1 + x_2 + \dots) = 1$.

2. Assuming the Steenrod axioms for Sq^i [SE, p. 1], show that $Sq(\alpha \frown \beta) = Sq\alpha \frown Sq\beta$ [MS, p. 136, problem 11-F].

3. Using Thom's formula for the Stiefel-Whitney classes of a bundle [MS, p. 91] and the preceding exercise, show that $Sq^i[M]$ is Poincaré dual to $\bar{w}^i(M)$, the ith Stiefel-Whitney class of the stable normal bundle of M .

4. Using exercise 3, prove part (2) of lemma 4.2.

5. Prove geometrically that $Sq^1 Sq^{2i+1} = 0$.

6. Prove that if the polyhedron X embeds piecewise-linearly in R^n , or immerses piecewise-linearly in R^{n-1} , and $\alpha \in H_k(X)$, then $Sq^i(\alpha) = 0$ for $i \geq n - k$.

7. Do exercise 6 with "piecewise-linearly" replaced by "topologically".

§5. A combinatorial formula for Φ^i (cf. [BM])

Let K be a triangulation of the k-cycle X, and choose an ordering of the vertices of K. For each ordered pair (σ,τ) of top dimensional simplices of K, define $\mu(\sigma,\tau) \in \mathbb{Z}/2\mathbb{Z}$ as follows:

Let $v_1, \ldots, v_s$ be the vertices of σ and τ in the given order. (If a vertex belongs to both σ and τ, it should be listed only once.) Let $\mu(\sigma,\tau) = 1$ if $v_1, v_3, v_5, \ldots$ are in σ and $v_2, v_4, v_6, \ldots$ are in τ, and $\mu(\sigma,\tau) = 0$ otherwise.

Let $c_{k-i}(K)$ be the simplicial mod 2 $(k-i)$-chain

$$c_{k-i}(K) = \sum_{(\sigma,\tau)} \mu(\sigma,\tau)\sigma \cap \tau \quad ,$$

summed over all ordered pairs of simplices (σ,τ) of K such that $\dim(\sigma \cap \tau) = k - i$. Let $[X] \in H_k(X)$ be the fundamental class of X.

Theorem 5.1 The chain $c_{k-i}(K)$ is a mod 2 cycle, and its homology class is $\Phi^i[X]$.

For example, let K be the octahedron, with vertices ordered as in figure 5.1. With this ordering,

$$c_1(K) = \langle 1,2\rangle + \langle 2,3\rangle + \langle 3,4\rangle + \langle 4,5\rangle + \langle 5,6\rangle + \langle 6,1\rangle \quad .$$

To see that the coefficient of $\langle 1,2\rangle$ is 1, we calculate $\mu(\sigma,\tau)$ for all ordered pairs (σ,τ) of 2-simplices such that $\sigma \cap \tau = \langle 1,2\rangle$. The only 2-simplices with $\langle 1,2\rangle$ as a face are $\sigma = \langle 1,2,3\rangle$ and $\tau = \langle 1,2,6\rangle$. The list of vertices occuring in σ and τ is $1,2,3,6$. Since 1 and 3 are in σ, while 2 and 6 are in τ, we have $\mu(\sigma,\tau) = 1$; on the other hand, $\mu(\tau,\sigma) = 0$. The other coefficients of $c_1(K)$ are determined similarly.

If we choose a different ordering for the vertices of K, we may get a different chain c_{k-i}. For example, if we use the ordering of figure 5.2 for the octahedron, then

$$c_1(K) = \langle 1,2\rangle + \langle 2,6\rangle + \langle 6,5\rangle + \langle 5,1\rangle + \langle 2,4\rangle + \langle 4,5\rangle + \langle 5,3\rangle + \langle 3,2\rangle .$$

The reader can also check that $c_0(K) = 0$ using ordering #1, while $c_0(K) = \sum_{i=1}^{6} \langle i\rangle$ using ordering #2.

To prove the theorem, let $f : X \to R^n$, $n = k + i - 1$, be a map which is linear on each simplex of K and which sends the ith vertex of K to $(t_i, t_i^2, \ldots, t_i^n)$, where $t_i < t_j$ if $i < j$. In other words, f maps the vertices of K, in order, to points of the "moment curve" $(t, t^2, \ldots, t^n)$. [N.B. f is not necessarily simplicial with respec to K, since the images of simplices may cross each other.] The homology class of $\mathbb{B}(f)$ is $\Phi^i[X]$, provided f is spread-out. Thus it suffices to prove (1)

f is spread-out, and (2) $\mathbb{B}(f)$ is the union of the $(k-i)$-simplices with nonzero coefficient in $c_{k-i}(K)$. In other words, there is a sub-division K' of K such that f is simplicial with respect to K', and if the $(k-i)$-simplex ω' of K' is contained in the $(k-i)$-simplex ω of K, then

$$b_f(\omega') = \sum_{\sigma \cap \tau = \omega} \mu(\sigma,\tau) \quad .$$

Before discussing (1) and (2), we show some examples of such maps f. If K is the octahedron with the ordering of figure 5.1 then the image of $f : K \to R^2$ is illustrated by figure 5.3. The fold set of f is the perimeter of the image, so $\mathbb{B}(f)$ is the support of the chain $c_1(K)$ associated with the ordering of figure 5.1.

As another example, if K is the octahedron with the ordering of figure 5.2 then the vertex $\langle 1 \rangle$ is in $\mathbb{B}(f)$ for $f : K \to R^3$. The image of the top of K in R^3 is illustrated by figure 5.4. So $\langle 1 \rangle$ is a "pinch point" of the image, and hence $\langle 1 \rangle$ is in $\mathbb{B}(f)$. (In fact the images of all the vertices under f for this ordering are pinch points.)

We now proceed to prove (1) and to discuss the proof of (2).

(1) <u>f is spread-out</u>. In fact, if $v_0, \dots, v_s$ are distinct vertices of K, $s \leq n$, then the span of $f(v_0), \dots, f(v_s)$ in

R^n has dimension s . (This implies that F is spread-out, by the proof of the spreading out theorem.) For suppose $f(v_i) = (t_i, t_i^2, \ldots, t_i^n)$, where $t_i \leq t_j$ if and only if $i \leq j$. The matrix

$$\begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^n \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & t_s & t_s^2 & \cdots & t_s^n \end{bmatrix}$$

has rank s , since

$$\det \begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^s \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & t_s & t_s^2 & \cdots & t_s^s \end{bmatrix} = \prod_{i>j} (t_i - t_j) \neq 0 \quad .$$

So the dimension of $\mathrm{Span}\{f(v_0), \ldots, f(v_s)\}$ is s .

(2) <u>$\mathbb{B}(f)$ equals the support of $c_{k-i}(K)$</u> . The key to proving this is the following lemma of Arnold Shapiro:

<u>Lemma</u> 5.2 Let A and B be q-simplices in R^m , $m = 2q$ (or let A be a q-simplex, B a (q+1)-simplex, and $m = 2q + 1$), with distinct vertices lying on the moment curve C . Then $A \cap B \neq \emptyset$ if and only if the vertices of A and B alternate

along C .

Now the proof of (2) goes <u>roughly</u> as follows. First recall that the coefficient $b_f(\omega')$ of ω' in $\mathbb{B}(f)$ is $b_f(\omega') = \sum \mathcal{L}(Lf_{\alpha'}, Lf_{\beta'})$, where the sum is taken over all pairs $\{\alpha', \beta'\}$ of $(k-i+1)$-simplices in K' , such that $\omega' < \alpha'$, $\omega' < \beta'$, $\alpha' \neq \beta'$, and $f(\alpha') = f(\beta')$.

If σ and τ are k-simplices of K with $\omega < \sigma$, $\omega < \tau$, and $\mu(\sigma,\tau) = 1$, then the definition of $\mu(\sigma,\tau)$, together with lemma 5.2, implies that there are simplices $s = \langle v_1, v_3, v_5, \ldots \rangle < \sigma$ and $t = \langle v_2, v_4, v_6, \ldots \rangle < \tau$ (where $v_1, v_2, v_3, \ldots$ are the vertices of σ and τ in order) whose images under f intersect, say in a point p . Let a and b be those points in s and t respectively such that $f(a) = p = f(b)$. Then $\alpha = a\omega$ and $\beta = b\omega$ are $(k-i+1)$-simplices with $\omega < \alpha$, $\omega < \beta$, $\alpha \neq \beta$, and $f(\alpha) = f(\beta)$. Furthermore, $\mathcal{L}(Lf_\alpha, Lf_\beta) = 1$. (This follows from corollary 6.3 of chapter 2, since if D is a little $(2i-2)$-ball transverse to $f(\alpha) = f(\beta)$, with boundary $LR^n_{f(\alpha)}$, then $f(LX_\alpha)$ is the boundary of $f(\sigma) \cap D$, $f(LX_\beta)$ is the boundary of $f(\tau) \cap D$, and $f(\sigma) \cap D$, $f(\tau) \cap D$ intersect simply.) Finally, given a $(k-i)$-simplex ω' of K' , with ω' contained in ω , each such pair $\{\alpha,\beta\}$ determines a pair $\{\alpha',\beta'\}$ in K' with $\omega' < \alpha'$, $\omega' < \beta'$, $\alpha' \neq \beta'$, $f(\alpha') = f(\beta')$, and $\mathcal{L}(Lf_{\alpha'}, Lf_{\beta'}) = 1$. For further details of the proof of theorem 5.1, see [BM].

Remarks.

(1) As a corollary of theorems 4.1 and 5.1, we get a combinatorial formula for the action of the Steenrod operations Sq^i on homology. This formula is very similar to Steenrod's original (1947) formula for Sq^i in cohomology, using "cup-i products".

(2) By exercise 3 of section 4, theorem 4.1 also gives a combinatorial formula for the normal Stiefel-Whitney classes of a manifold. There is a similar formula for the tangential Stiefel-Whitney classes, due to Goldstein and Turner [GT].

Exercises

1. Prove Shapiro's lemma. Can it be generalized to the case $\dim A + \dim B = m$?

2. Find a combinatorial proof that $c_{k-i}(K)$ is a cycle. (This problem was suggested by Lee Rudolph.)

3. If K is the octahedron, which mod 2 1-cycles of K can occur as $c_1(K)$ for some ordering of the vertices of K ?

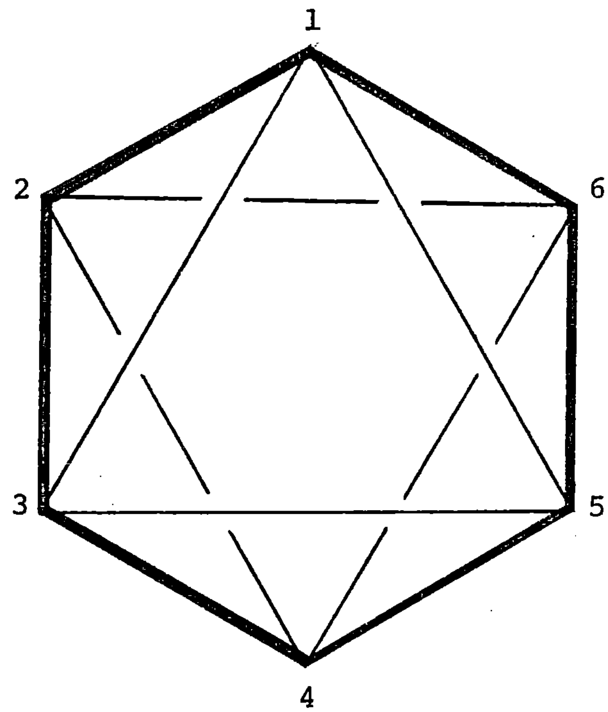


Figure 5.1.

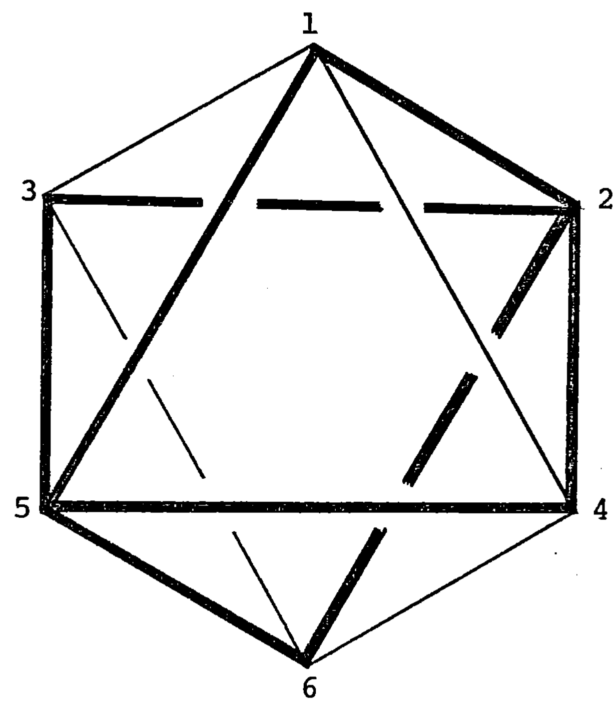


Figure 5.2.

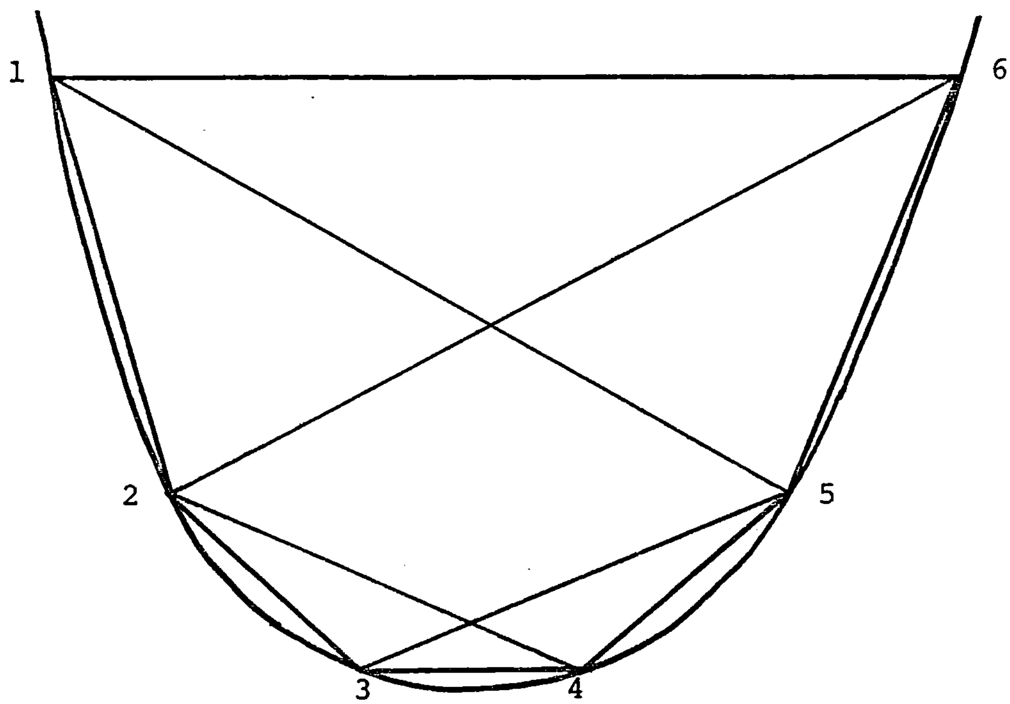


Figure 5.3.

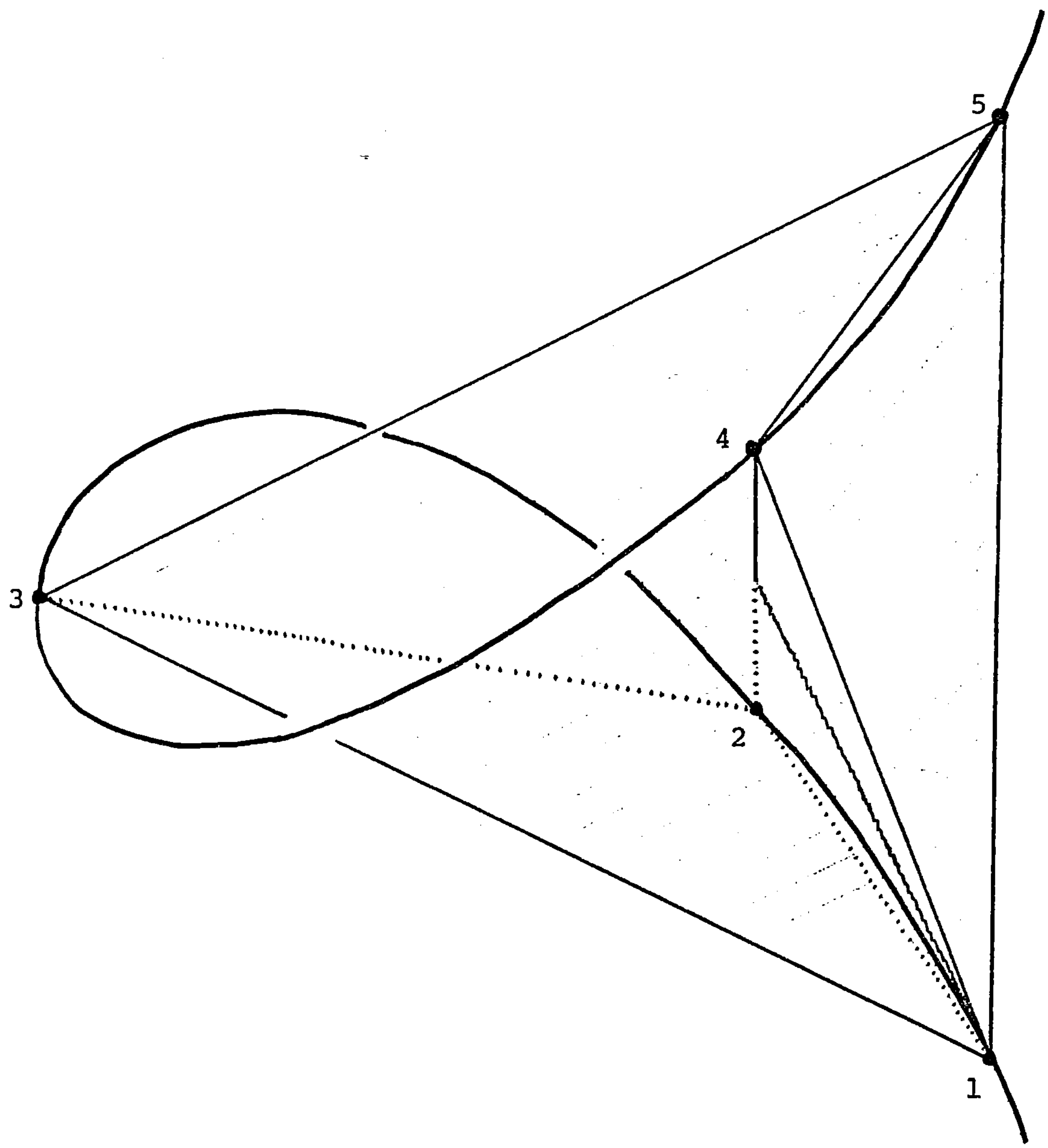


Figure 5.4.

§6. Smith operations

Let T be a piecewise-linear involution of the polyhedron X, that is, a map from X to itself which is PL and such that $T(T(x)) = x$ for all $x \in X$. Let K be a triangulation of X for which T is simplicial (that is, $\sigma \in K$ implies that $T\sigma \in K$), and such that the fixed point set X^T of T, $X^T = \{x \in X | T(x) = x\}$, supports a subcomplex of K. (Such a triangulation exists for any PL involution.)

Let C_i be the group of mod 2 simplicial i-chains of K, let $C_* = \bigoplus_i C_i$, and let $T_\# : C_* \to C_*$ be the morphism induced by T. Then $T_\#$ is a morphism of chain complexes, i.e. $T_\# \partial c = \partial T_\# c$ for all $c \in C_*$.

Let $C_i^T = \{c \in C_i | T_\# c = c\}$ be the group of equivariant i-chains of K. (N.B. This is not the same as the group generated by the equivariant i-simplices.) Then $C_*^T = \bigoplus_i C_i^T$ is a subcomplex of C_*, for if $c \in C_i$ and $T_\# c = c$, then $T_\# \partial c = \partial T_\# c = \partial c$, so that $\partial(C_i^T) \subset C_{i-1}^T$.

Let $H_*^T(X)$, the T-equivariant homology of X, be the homology of C_*^T. In other words, $H_i^T(X)$ is the group of equivariant i-cycles modulo boundaries of equivariant (i+1)-chains. $H_*^T(X)$ is independent of the triangulation K.

In the late 1930's, P. A. Smith defined homomorphisms

$$A : H_i^T(X) \to H_i(X^T)$$

$$B : H_i^T(X) \to H_{i-1}^T(X)$$

(where all coefficients are $\mathbb{Z}/2\mathbb{Z}$) as follows.

If $c = \sum_j \sigma_j$ is a chain in C_i^T , then $Ac = \sum_{T_\# \sigma_j = \sigma_j} \sigma_j$, the sum of all the simplices σ_j of c such that $T_\# \sigma_j = \sigma_j$. Choose a chain c' in C_i so that

$$c = Ac + c' + T_\# c'$$

(c' is not unique), and let $Bc = \partial c'$. (Bc is a boundary, but not necessarily the boundary of an equivariant chain, so Bc need not represent $0 \in H_{i-1}^T(X)$.)

To see that A and B make sense and that they are well-defined in homology, we need these facts:

1) $\partial Ac = A\partial c$
2) If $\partial c = 0$, then $T_\# Bc = Bc$
3) $\partial Bc = 0$
4) If $c = Ac + c'' + T_\# c''$ then $\partial c'' = \partial c' + \partial d$ for some $d \in C_i^T$.

5) If $c = \partial e$, $e \in C^T_{i+1}$, and $e = Ae + e' + T_\# e'$, then we can let $c' = \partial e'$.

We apply these statements as follows. (1) says that A is a chain map, so A induces a map in homology. (2) and (3) say that if c is an equivariant cycle, so is Bc . (4) says that the class of Bc in H^T_{i-1} doesn't depend on the choice of c' , and (4) together with (5) imply that if c is the boundary of an equivariant chain, then so is Bc . (For if $c' = \partial e'$ then $Bc = \partial c' = \partial\partial e' = 0$.)

The proofs of (1)-(5) are easy:

1) $\partial c = \partial(Ac + c' + T_\# c') = \partial Ac + \partial c' + T_\# \partial c'$, so $A\partial c = A\partial Ac + A(\partial c' + T_\# \partial c') = \partial Ac$, because $A(b + T_\# b) = 0$ for any chain b , and $T_\# \sigma = \sigma$ for every σ in ∂Ac .

2) If $\partial c = 0$, then $\partial Ac = A\partial c = A0 = 0$, so the equation $\partial c = \partial Ac + \partial c' + T_\# \partial c'$ becomes $T_\# \partial c' = \partial c'$, or $T_\# Bc = Bc$.

3) $\partial Bc = \partial\partial c'$ because $Bc = \partial c'$, and $\partial\partial c' = 0$.

4) We have $Ac + c' + T_\# c' = c = Ac + c'' + T_\# c''$, so $c' + c'' = T_\#(c' + c'')$, i.e. $c' + c'' \in C^T_i$. But $\partial c'' = \partial c' + \partial(c' + c'')$, so we can let $d = c' + c''$.

5) We have $c = \partial e = \partial Ae + \partial e' + T_\# \partial e'$, and $\partial Ae = A\partial e = Ac$, so $c = Ac + \partial e' + T_\# \partial e'$.

Having defined A and B, now define

$$F^n : H_i^T(X) \to H_{i-n}(X^T)$$

by $F^n = \underbrace{AB \cdots B}_{n}$.

In the early 1950's, Wu Wen-tsün showed that mod 2 homology operations could be defined by using these Smith homomorphisms for the canonical involution on the square of a space.

$$T : X \times X \to X \times X \quad , \quad T(x,y) = (y,x) \quad .$$

Wu observed that if $\alpha \in H_k(X)$, then $\alpha \times \alpha$ defines an element of $H_{2k}^T(X \times X)$. He defined a <u>Smith operation</u> $Sm^i : H_k(X) \to H_{k-i}(X)$ by

$$Sm^i(\alpha) = F^{k+i}(\alpha \times \alpha) \in H_{k-i}(\Delta) = H_{k-i}(X) \quad ,$$

where $\Delta \subset X \times X$ is the diagonal, which is the fixed point set of T .

It is easy to show that if c is a cycle in X , then $c \times c$ is an equivariant cycle in $X \times X$. However, it is not obvious (although it's true) that if c is a boundary in X then $c \times c$ the boundary of an <u>equivariant</u> chain in $X \times X$. (Wu overlooked the non-triviality of this in his book [Wu]!)

For example, if d is a 1-simplex and $c = \partial d$, then $c \times c$ is not the boundary of $d \times d$, because $d \times d$ has dimension two! But $c \times c$ is the boundary of an equivariant chain, namely the sum of the two diagonals in $d \times d$.

To avoid this problem, we work with a geometric k-cycle X, and just define $Sm^i[X]$, where $[X] \in H_k(X)$ is the fundamental class. If $\mathcal{X} \in H^T_{2k}(X \times X)$ is the canonical class, we set

$$Sm^i[X] = F^{k+i}(\mathcal{X}) \in H_{k-i}(X) \quad .$$

<u>Theorem</u> 6.1 (Wu) $Sm^i[X] = Sq^i[X]$.

Wu's proof of this theorem is a direct combinatorial calculation, which is difficult to read.

Here is the outline of a <u>geometric</u> proof, using branch points and double points. If $N \geq n$ let $p_{N,n} : R^N \to R^n$ be the orthogonal projection onto the first n coordinates. Given a polyhedral k-cycle X, choose an embedding $f_N : X \to R^N$ such that the maps $f_n = p_{N,n} \circ f_N : X \to R^n$ are spread-out if $n \geq k$, and have $\dim f_n^{-1}(x) \leq k - n$ for all $x \in R^n$ if $n \leq k$.

Choose a triangulation K of X so that all the maps f_n are simplicial with respect to K. Let L be the first barycentric subdivision of the cell complex $K \times K$. The diagonal of $X \times X$ is a subcomplex of L.

Now let C be the sum of all the 2k-simplices of L. Then $F^{k+i}C = AB^{k+i}C$ represents $Sm^i[X]$, by definition. I claim that the choices involved in defining B^{k+i} can be made so that $AB^{k+i}C = \mathbb{B}(f_{k+i-1})$. This implies theorem 6.1, since $\mathbb{B}(f_{k+i-1})$ represents $Sq^i[X]$, by theorem 4.1.

For each j define $\mathcal{D}(f_j)$ and $\mathcal{D}^+(f_j)$ to be mod 2 j-chains whose supports are contained respectively in the sets

$$C\ell\{(x,y) \in X \times X \,|\, x \neq y,\ f_j(x) = f_j(y)\},$$
$$C\ell\{(x,y) \in X \times X \,|\, x \neq y,\ f_j(x) = f_j(y),\ \pi_{j+1}f_{j+1}(x) > \pi_{j+1}f_{j+1}(y)\},$$

where $\pi_{j+1} : R^{j+1} \to R^1$ is projection onto the last coordinate. The actual multiplicities of $\mathcal{D}(f_j)$ and $\mathcal{D}^+(f_j)$ are defined using local linking numbers--this is left to the reader. These chains have the following properties:

1) $\mathcal{D}(f_j) = \mathcal{D}^+(f_j) + T_\# \mathcal{D}^+(f_j)$

2) $\partial \mathcal{D}^+(f_j) = \begin{cases} \mathcal{D}(f_{j+1}) & \text{if } j < k-1 \\ \mathbb{B}(f_j) + \mathcal{D}(f_{j+1}) & \text{if } j \geq k-1, \end{cases}$

where $\mathbb{B}(f_j)$ is identified with a cycle in the diagonal, and $\mathbb{B}(f_{k-1})$ is just $[X]$.

Using (1) and (2), we shall show

3) $B^jC = \begin{cases} \mathcal{D}(f_j) & \text{if } j < k \\ \mathbb{B}(f_{j-1}) + \mathcal{D}(f_j) & \text{if } j \geq k. \end{cases}$

This is trivial if $j = 0$. If (3) holds for $B^{j-1}C$, then the terms in the equation

$$B^{j-1}C = AB^{j-1}C + (B^{j-1}C)' + T_{\#}(B^{j-1}C)'$$

can be chosen using (1):

$$AB^{j-1}C = \begin{cases} 0 & \text{if } j \leq k \\ \mathbb{B}(f_{j-2}) & \text{if } j > k \end{cases} ,$$

and we can take $(B^{j-1}C)' = \mathcal{D}^{+}(f_{j-1})$, so $B^{j}C = B(B^{j-1}C) = \partial(B^{j-1}C)' = \partial\mathcal{D}^{+}(f_{j-1})$, and so (2) implies (3).

It follows from (3) that $AB^{j}C = \mathbb{B}(f_{j-1})$ if $j \geq k$, which is what we wanted. ∎

For example, let X be an octagon, with maps $f_2 : X \to R^2$, $f_1 : X \to R^1$, $f_0 : X \to R^0$ as illustrated in figure 6.1. The chains $\mathcal{D}^{+}(f_i)$ and $\mathbb{B}(f_i)$ are illustrated in figure 6.2, which shows that if C is the sum of all the 2-simplexes of the triangulation L of $X \times X$ defined above, then

$$C = \mathcal{D}^{+}(f_0) + T_{\#}\mathcal{D}^{+}(f_0) ,$$
$$\partial\mathcal{D}^{+}(f_0) = \mathbb{B}(f_0) + \mathcal{D}^{+}(f_1) + T_{\#}\mathcal{D}^{+}(f_1) ,$$
$$\partial\mathcal{D}^{+}(f_1) = \mathbb{B}(f_1) + \mathcal{D}^{+}(f_2) + T_{\#}\mathcal{D}^{+}(f_2) .$$

Exercises

1. Show that if T is a PL involution of X , there is a triangulation K of X for which T is simplicial, and such that X^T is a subcomplex of K .

2. Show that $H_*^T(X)$ is independent of the choice of triangulation K .

3. Illustrate the proof of theorem 6.1 using a PL embedding of the projective plane in R^4 .

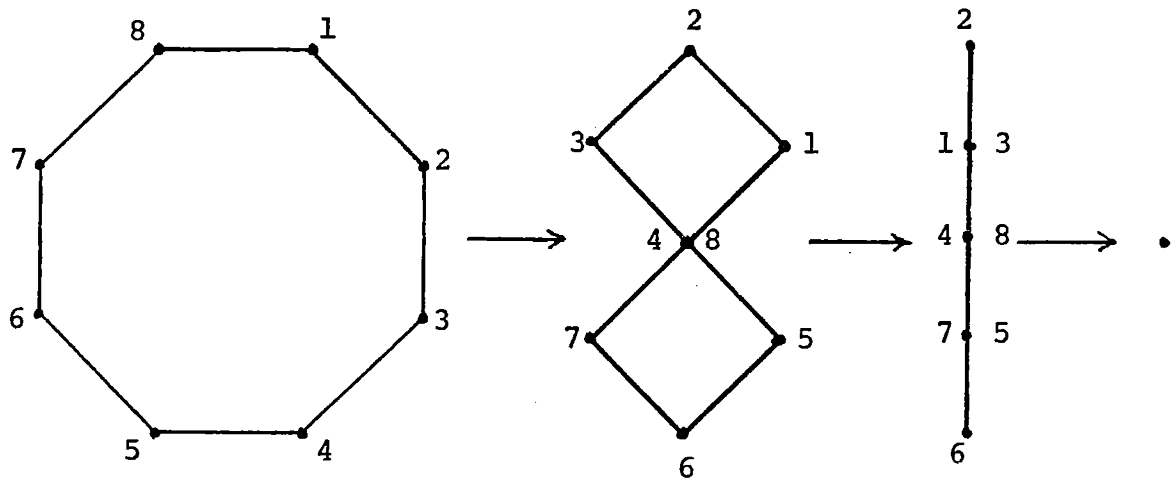


Figure 6.1.

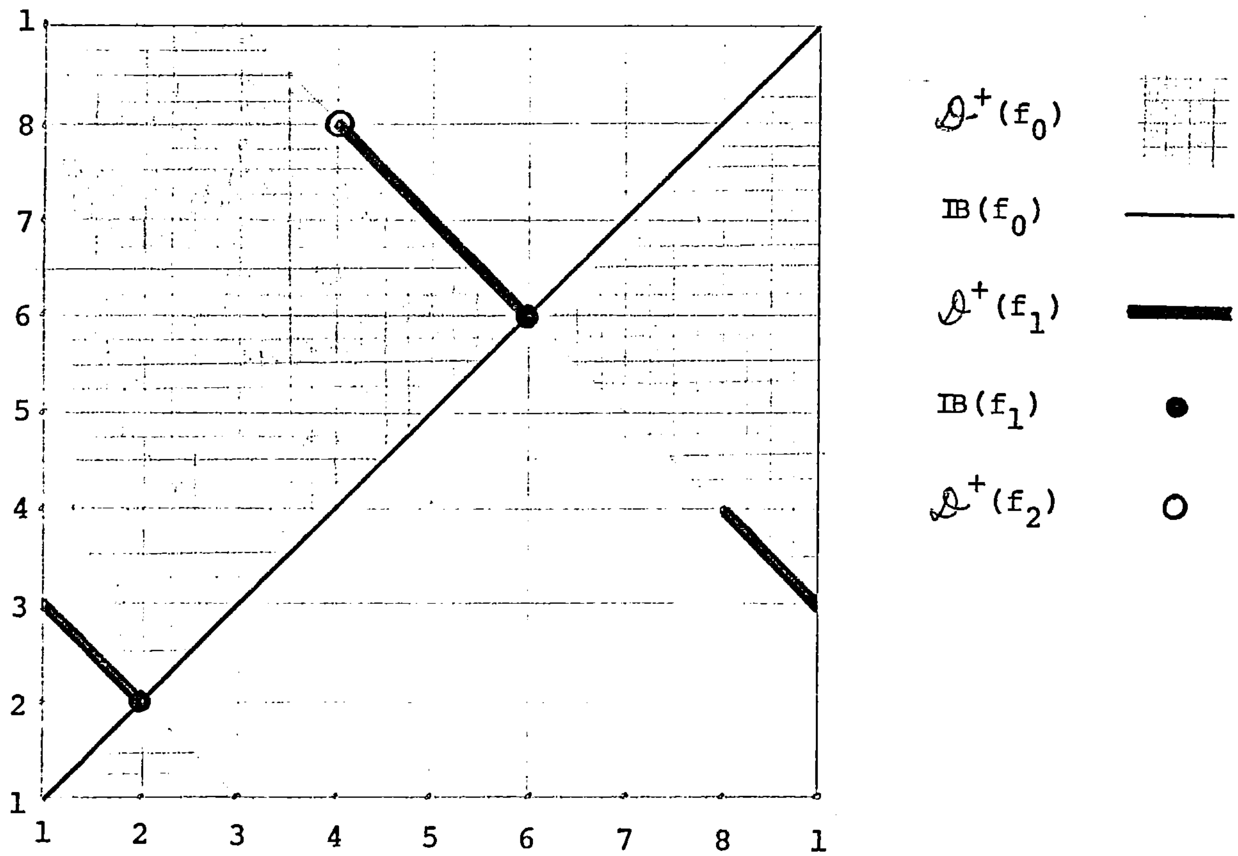


Figure 6.2.

References*

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[GT] R. Goldstein and E. Turner, A formula for Stiefel-Whitney homology classes, Proc. Amer. Math. Soc. 58 (1976), 339-342.

*Additional references can be found at the end of chapter 2.