Chap. III : Obstruction Theory

In this chapter we consider the following problem: L is a subcomplex of a CW complex K, and $f : L \to Y$. We would like to extend this map to a map of $K \to Y$. Since K is a CW complex, the obvious thing to do is to extend the map skeleton-by-skeleton and on each skeleton, cell-by-cell. So the crucial special case is where $L = S^n$ and $K = e^{n+1}$. We see that f need not extend if $\pi_n(Y) \neq 0$ and in fact, f will extend if and only iff f represents the trivial element of $\pi_n(Y)$. Using this observation we will define below the obstruction to extending a map from the n skeleton to the n+1 skeleton.

In general, we define a map $f : L \to Y$ to be n-extensible if there exists an extension of f mapping $\overline{K^n} = K^n \cup L \to Y$. If we assume, as we will from now on, that Y is pathwise connected, then the following is obvious:

1. Lemma- Every map $L \to Y$ is 1-extensible.

We define the extension index of a map $f : L \to Y$ to be the least upper bound of the set of integers n such that f is n-extensible.

2. Lemma- Homotopic maps of $L \to Y$ have the same extension index.

Proof: For any n this follows from the homotopy extension theorem applied to the pair $(\overline{K^n}, L)$.

3. Lemma- Let $g : (K',L') \longrightarrow (K,L)$ and $e : Y \longrightarrow Y'$, then if $f : L \longrightarrow Y$ is n-extensible, then so is $efg : L' \longrightarrow Y'$.

Proof: This is obvious, if g is cellular, and if g is not we can find $g_t : (K',L') \longrightarrow (K,L)$ where $g_0 = g$ and g_1 is cellular, by the cellular approximation theorem (applied twice). $efg \sim efg_1$ and hence by the previous lemma if efg_1 is n-extensible then so is efg.

Now assume that K and Y are pathwise connected and that Y is n-simple. Consider a map $f : \overline{K}^n \longrightarrow Y$. To define the obstruction we will require the following form of the relative Hurewicz theorem, which we will not prove:

Theorem- If $n \geq 2$ and (X,A) is an $(n-1)$ connected CW pair, then the Hurewicz homomorphism $h : \pi_n(X,A,x_0) \longrightarrow H_n(X,A)$ is an epimorphism and its kernel is the subgroup generated by elements of the form $\alpha - w\alpha$, where $\alpha \in \pi_n(X,A,x_0)$ and $w \in \pi_1(A,x_0)$.

By lemma 1, we can assume that $n \geq 1$. By theorem 22 of chap. 1, (K,K^n) is n connected and hence by considering the homotopy sequence of the triple $(K,\overline{K}^n,K^n)$ we get that $\pi_m(K,\overline{K}^n) = 0$ for $n \geq m > 1$ and from this and the homotopy sequence of the pair $(K,\overline{K}^n)$ we get that $\pi_1(K,\overline{K}^n) = 0$. We use that $\pi_0(\overline{K}^n) \longrightarrow \pi_0(K)$ is mono, which, since $n \geq 1$, is true.

Definition- If $f : \overline{K}^n \longrightarrow Y$, with $n \geq 1$, and K and Y pathwise connected, with Y n-simple, then we define $c_f^{n+1} \in C^{n+1}(K,L;\pi_n(Y)) = \mathrm{Hom}(C_{n+1}(K,L);\pi_n(Y))$ by the switchback:

$$\begin{array}{ccc}
\pi_{n+1}(\overline{K}^{n+1},\overline{K}^n,k_0) & \xrightarrow{h} & H_{n+1}(\overline{K}^{n+1},\overline{K}^n) \\
\downarrow \partial & & \downarrow c_f^{n+1} \\
\pi_n(\overline{K}^n,k_0) & \xrightarrow{f_*} & \pi_n(Y)
\end{array}$$

We note that the kernel of h is of the form $\{ \alpha - w\alpha \}$ by the Hurewicz theorem. $f_* \ (\alpha - w\alpha) = f_*(\partial \alpha - w \partial \alpha) = 0$ since Y is m-simple. Hence the switchback does define a homomorphism.

Intuitively, what we wish the obstruction to be is the following: Let σ be a cell of $I_K^{m+1} - I_L^{m+1}$, with attaching map $i_\sigma : (e^{m+1}, S^m) \to (\overline{K}^m \cup \sigma, \overline{K}^m)$. In order, that f extend to $\overline{K}^m \ \sigma$, it is necessary an sufficient that cls $f(i_\sigma | S^m)$ of $w_m(Y) = 0$. Thus, we would like to define c_f^{m+1} on the m+1 cells of K, by $c_f^{m+1}(\sigma) = \text{cls } f(i_\sigma \mid S^m)$. We can make this meaniful by using the decomposition of chap. II theorem 4, ie. use the switchback:

$$\begin{tikzcd}
\sum_\sigma w_{m+1}(e_\sigma^{m+1}, S_\sigma^m) \arrow[r, "\sum h_\sigma", "\approx"'] \arrow[d, "\Sigma\partial"] & \sum_\sigma H_{m+1}(e_\sigma^{m+1}, S_\sigma^m) \arrow[r, "\sum i", "\approx"'] & H_{m+1}(\overline{K}^{m+1}, \overline{K}^m) \\
\sum_\sigma w_m(S_\sigma^m) \arrow[rr, dashed, "\sum_\sigma (f(i_\sigma | S_\sigma^m))_*"'] & & w_m(Y)
\end{tikzcd}$$

To prove these two definitions are the same, pick a base point p_σ in each S_σ^m, and let w_σ be a path in $\overline{K}^m$ joining $i_\sigma(p_\sigma)$ and k_0. Then the fact that the following diagram commutes proves the two definitions equivalent.

$$\begin{tikzcd}
\sum_\sigma w_{m+1}(e_\sigma^{m+1}, S_\sigma^m, p_\sigma) \arrow[rr, dashed, "\Sigma h_\sigma"] \arrow[d, "\Sigma\, \omega_\sigma i_{\sigma *}"] \arrow[dd, bend right, "\Sigma\partial"'] & & {}_\sigma H_{m+1}(e_\sigma^{m+1}, S_\sigma^m) \arrow[d, "\Sigma\, i_{\sigma *}"] \\
w_{m+1}(\overline{K}^{m+1}, \overline{K}^m, k_0) \arrow[rr, dashed, "h"] \arrow[d, "\partial"] & & H_{m+1}(\overline{K}^{m+1}, \overline{K}^m) \\
w_m(S_\sigma^m, p_\sigma) \arrow[r, "\Sigma \omega_\sigma (i_\sigma | S^n)_*"] \arrow[rr, bend right, "\Sigma g \omega_\sigma (g(i_\sigma | S^n))_* = \Sigma (g(i_\sigma | S^n))_* \text{ by } n\text{-simplicity}"'] & w_n(\overline{K}^m, k_0) \arrow[r, dashed, "g_*"] & w_m(Y)
\end{tikzcd}$$

From the second definition the following is obvious:

4. Theorem- $f : \overline{K}^m \to Y$, extends iff $c_f^{m+1} = 0$.

$$
\begin{tikzcd}
C_*(K,L)\otimes C_*(I) \arrow[r, "\times", "\cong"'] & C_*((K,L)\times I) \arrow[r, "c_F"] & \pi_n(Y) \\
C_*(K,L)\otimes C_*(0) \arrow[u, "1\otimes i_{0*}"] \arrow[r, "\times", "\cong"'] & C_*((K,L)\times 0) \arrow[u, "i_{0*}"] \arrow[ur, "c_{F_0}"] \arrow[d, "p_*", "\cong"'] & \\
 & C_*(K,L) \arrow[uur, "c_{g_0}"'] &
\end{tikzcd}
$$

identification using augmentation

$\therefore$ If $0 \in C_*(I) \to Z$ is dual to the augmentation generator of $C_*(0)$ then above implies that $c_F(1\otimes i_{0*}) = c_{g_0}\otimes 0$. Similarly, $c_F(1\otimes i_{1*}) = c_{g_1}\otimes 1$. Since $C_*(K\times \dot{I} \cup L\times I, L\times I) = \operatorname{Im} 1\otimes i_{0*} + 1\otimes i_{1*}$, $c_F - c_{g_0}\otimes 0 - c_{g_1}\otimes 1$ vanishes on it.

5. Theorem- $\delta c_f^{n+1} = 0$, ie. c_f^{n+1} is a cocycle.

Proof: We show that $c_f^{n+1}\partial = 0$. This follows from the diagram:

$$\begin{array}{ccc}
\pi_{n+2}(\overline{K}^{n+2},\overline{K}^{n+1},k_o) & \xrightarrow[\text{epi}]{h} & H_{n+2}(\overline{K}^{n+2},\overline{K}^{n+1}) \\
\partial\downarrow & & \downarrow\partial \\
\pi_{n+1}(\overline{K}^{n+1},k_o) & \xrightarrow{h} & H_{n+1}(\overline{K}^{n+1}) \\
\downarrow & & \downarrow \\
\pi_{n+1}(\overline{K}^{n+1},\overline{K}^{n},k_o) & \xrightarrow{h} & H_{n+1}(\overline{K}^{n+1},\overline{K}^{n}) \\
\partial\downarrow & & \\
\pi_n(\overline{K}^n,k_o) \xrightarrow{f_*} \pi_n(Y) & &
\end{array}$$

Since the column on the left contains two successive terms from the homotopy sequence of $(\overline{K}^{n+1},\overline{K}^n)$, it follows that the composite is zero.

Because of our convenient global definition, the following naturality theorem is obvious.

6. Theorem- If $\phi : (K',L') \longrightarrow (K,L)$ is cellular then $c_{f\phi}^{n+1} = \phi^{\#}c_f^{n+1}$.

We also note the following lemma:

7. Lemma- If $f,f' : \overline{K}^n \longrightarrow Y$ are homotopic, then $c_f^{n+1} = c_{f'}^{n+1}$.

Now we consider a slightly different situation. Let g_o , g_1 : $\overline{K}^n \longrightarrow Y$ and let $h_t : \overline{K}^{n-1} \longrightarrow Y$ be a homotopy between $g_o|\overline{K}^{n-1}$ and $g_1|\overline{K}^{n-1}$. We now recall that $C^*((K,L)\times I,\pi_n(Y))$ is isomorphic to $C^*(K,L;\pi_n(Y))\otimes C^*(I)$. $C^*(I)$ is free on three generators 0, 1, and I, with $\delta 0 = -I$ and $\delta 1 = I$. Define $F : \overline{K}^{n-1}\times I \cup \overline{K}^n\times \dot{I} \longrightarrow Y$ by h_t on $\overline{K}^{n-1}\times I$, g_i on $\overline{K}^n\times i$ for $i = 0,1$. $c_F^{n+1} - c_{g_o}^{n+1}\otimes 0 - c_{g_1}^{n+1}\otimes 1$ is thus an element of $C^{n+1}((K,L)\times I)$.

However, more than that, it pulls back to $C^{n+1}((K,L)\times(I,\dot{I});\pi_n(Y))$. Consider the map: $C^{n+1}((K,L)\times I) \xleftarrow{j} C^{n+1}((K,L)\times(I,\dot{I});\pi_n(Y))$ $\xrightarrow{\gamma} C^n(K,L;\pi_n(Y))\otimes C^1(I,\dot{I}) \xrightarrow{k} C^n(K,L;\pi_n(Y))$, where j is the inclusion and k is the composite of $C^1(I,\dot{I}) \longleftarrow C^0(\dot{I},1) \longrightarrow C^0(0) = Z$, tensored with the identity on $C^n(K,L)$. We define $d^n(g_0,g_1;h_t) = (-1)^{n+1}k\gamma j^{-1}(c_F^{n+1} - c_{g_0}^{n+1}\otimes 0 - c_{g_1}^{n+1}\otimes 1)$, which is thus an element of $C^n(K,L;\pi_n(Y))$. It is easily verified that the meaning of the difference cochain is as follows: let σ be a cell of $I_K^n - I_L^n$ then $(-1)^{n+1}d(g_0,g_1;h_t)(\sigma)$ is represented by the map of $S^n = \partial(e^n\times I) = e^n\times\dot{I}\cup S^{n-1}\times I$, which is $g_0 i_\sigma$ on $e^n\times 0$, $g_1 i_\sigma$ on $e^n\times 1$, and $h_t(i_\sigma|S^{n-1})$ on $S^{n-1}\times t$. From this it follows that the analogue of theorem 4 holds:

9. Theorem- h_t extends to a homotopy of g_0 and g_1 iff $d(g_0,g_1;h_t) = 0$.

10. Theorem- $\delta d^n(g_0,g_1;h_t) = (c^{n+1}(g_0) - c^{n+1}(g_1))(-1)^{n+1}$

Proof: This follows from the fact that

$\delta(-1)^{n+1}(c_F^{n+1} - c_{g_0}^{n+1}\otimes 0 - c_{g_1}^{n+1}\otimes 1) = c_{g_0}^{n+1}\otimes I - c_{g_1}^{n+1}\otimes I$, whose pull back under $k\gamma j^{-1}$ is the required difference.

Next, we prove an important existance theorem:

11. Theorem- For every n-cochain c in $C^n(K,L;\pi_n(Y))$, $g_0 : \overline{K}^n \longrightarrow Y$ and $h_t : \overline{K}^{n-1} \longrightarrow Y$, with $h_0 = g|\overline{K}^{n-1}$, there exists $g_1 : K^n \longrightarrow Y$ such that $h_1 = g_1|K^{n-1}$ and $c = d(g_0,g_1;h_t)$.

Proof: Let the n-sphere $S = (e^n x0) \cup (S^{n-1} xI) \cup (e^n x1)$. There exists a map $f_\sigma : S \longrightarrow Y$ representing $(-1)^{n+1} c(\sigma)$ (ie. $c(i_{\sigma *}(g))$ where g is a chosen fixed generator of $H_n(e^n, S^{n-1})$), which since $(e^n x0) \cup (S^{n-1} xI)$ is contractible, we can assume $= g_o i_\sigma$ on $e^n x0$ and $h_t i_\sigma$ on $S^{n-1} xt$. Define $g_1 = \begin{cases} h_1(x) & x \in \overline{K}^{n-1} \\ f_\sigma^1(x,1) & x \in \sigma \end{cases}$. It is obvious that $d(g_o, g_1; h_t) = c$.

The following are two easy, but important naturality theorems:

12. Theorem- $d^n(g_o, g_2; h'_t \cdot h_t) = d^n(g_o, g_1; h_t) + d^n(g_1, g_2; h'_t)$ (where $h'_t \cdot h_t$ is shorthand for the homotopy j_t with $j_t = h_{2t}$ $t \leq \frac{1}{2}$, and $j_t = h'_{2t-1}$ $t \geq \frac{1}{2}$).

13. Theorem- If $\phi : (K', L') \longrightarrow (K, L)$ is cellular, then $d^n(g_o\phi, g_1\phi; h_t\phi) = \phi^{\#} d^n(g_o, g_1; h_t)$.

Our first important application of these constructions is the following extension theorem of Eilenberg:

14. Theorem- If $f : \overline{K}^n \longrightarrow Y$, then $\{c_f\} = 0$ in $H^{n+1}(K, L; \pi_n(Y))$ iff there exists a map $g : \overline{K}^{n+1} \longrightarrow Y$ such that $g|\overline{K}^{n-1} = f|\overline{K}^{n-1}$.

Proof: We let $d(g_o, g_1; h_t) = d(g_o, g_1)$ if $h_t = g_o|\overline{K}^{n-1}$ for all t.

If g exists, as described, then $c_g^{n+1}(\sigma) = 0$ and hence $c_f^{n+1}(\sigma)$ is the coboundary of $d^n(f, g)$.

If c_f^{n+1} is homologous to 0, then there exists c in $C^n(K, L; \pi_n(Y))$ such that $c_f^{n+1} = \delta c$. By theorem 11, there exists $g : \overline{K}^n \longrightarrow Y$ such that $d(f, g) = c$ and hence, by theorem 10, $c_g^{n+1} = 0$ and hence g extends to $\overline{K}^{n+1}$.

This result has the following important corollary:

15. Theorem- If Y is r-simple and $H^{r+1}(K,L;\pi_r(Y)) = 0$ for every $r : n \leq r < m \leq \infty$, then the n-extensibility of $f : L \longrightarrow Y$ implies its m-extensibility. (Where ∞-extensibility means that f can be extended to all of K.)

Related to the extension problem, that we have been discussing, is the homotopy problem. Given two maps $f, g : K \longrightarrow Y$, such that $f|L = g|L$ for a subcomplex L, we would like to determine whether or not these two maps are homotopic, rel L. Our method is again one of extending th homotopy skeleton by skeleton. If $h_t : \overline{K}^{n-1} \longrightarrow Y$ is a homotopy of $f|\overline{K}^{n-1}$ with $g|\overline{K}^{n-1}$ (rel L), then $d^n(f,g;h_t) \in C^n(K,L;\pi_n(Y))$ as before (assuming Y is n-simple). However, now we note that $c_f^{n+1} = c_g^{n+1} = 0$ since f and g are defined on K. Hence, we have that $d^n(f,g;h_t)$ is by definition $(-1)^{n+1}(k\gamma j^{-1})(c_F^{n+1})$ and $\delta\, d^n(f,g;h_t) = 0$, ie. the difference cochain is, in this case, a cocycle. By a direct application of theorem 14, we get the following analogue, with the corollary, analogous to theorem 15:

16. Theorem- $\{d^n(f,g;h_t)\} = 0$ iff there exists a homotopy $k_t : \overline{K}^n \longrightarrow Y$, such that $k_0 = f|\overline{K}^n$, $k_1 = g|\overline{K}^n$, $k_t|\overline{K}^{n-2} = h_t|\overline{K}^{n-2}$.

17. Theorem- If Y is r-simple and $H^r(K,L;\pi_r(Y)) = 0$ for every $r : n < r < m+1 \leq \infty$, if $f|\overline{K}^n$ is homotopic to $g|\overline{K}^n$ (rel L), then $f|\overline{K}^m$ is homotopic to $g|\overline{K}^m$ (rel L).

We now consider a specialization of the theory which we have been developing, called primary obstruction theory. We now assume that Y is m-1-connected. If $m \geq 2$, then Y is simply connected and hence r-simple for all r. For the case m = 1, we also assume that Y is 1-simple, ie. that $\pi_1(Y)$ is abelian.

Given a map $f : L \longrightarrow Y$, with L a subcomplex of K, then, by theorem 15, f is m-extensible. Furthermore, by theorem 17, any two such extensions f and f' a m-1-homotopic, ie. $f|\overline{K}^{m-1}$ is homotopic to $f'|\overline{K}^{m-1}$ (rel L). If $h_t : \overline{K}^{m-1} \longrightarrow Y$ is such a homotopy, then by theorem 10, $\delta\, d(f,f';h_t) = (c_f^{m+1} - c_{f'}^{m+1})(-1)^{m+1}$. This shows that c_f^{m+1} and $c_{f'}^{m+1}$ are cohomologous, and thus represent the same element of $H^{m+1}(K,L;\pi_m(Y))$. This unique element determined by f is called the primary obstruction of f and is denoted by $\omega^{m+1}(f)$.

Pick a base point y_0 of Y, and let $0 : L \longrightarrow y_0$ in Y. By theorem 17, $f|L^{m-1}$ and $0|L^{m-1}$ are homotopic and hence by the homotopy extension theorem, f is homotopic to a map of $L \longrightarrow Y$ which when restricted to $L^{m-1} = 0|L^{m-1}$. Hence, by theorem 7, we can assume that $f|L^{m-1} = 0|L^{m-1}$, since the two homotopic maps have the same primary obstruction elements. We then define $\kappa^m(f) = \{d(f,0)\} \in H^m(L;\pi_m(Y))$. If we let $\delta^* : H^m(L;\pi_m(Y)) \longrightarrow H^{m+1}(K,L;\pi_m(Y))$ be the coboundary homomorphism of the pair (K,L), then we have the following:

18. Theorem- $\omega^{m+1}(f) = \delta^*\kappa^m(f)$.

Proof: Since $f(L^{n-1}) = y_0$, it follows that f has an extension $f^* : \overline{K}^n \longrightarrow Y$ such that $f^*(\overline{K}^n - L) = y_0$. If we let $0^* : \overline{K}^n \longrightarrow Y$ denote the constant map such that $0^*(\overline{K}^n) = y_0$, then $\delta^* \gamma d^n(f) = \delta^*\{d(f,0)\}$ is represented by $\delta d(f^*, 0^*)$ which $= c^{n+1}_{f^*}$ by theorem 10, and $c^{n+1}_{f^*}$ represents $\omega^{n+1}(f)$.

Akin to this theorem, is the corresponding result for homotopy. If $f, g : K \longrightarrow Y$, with $f|L = g|L$ we can consider the elements of $H^n(K,L;\pi_n(Y))$ represented by cocycles of the form $d(f,g;h_t)$ with h_t a homotopy rel L. Since these correspond under $(-1)^{n+1} k \gamma j^{-1}$ to elements c_F^{n+1}, it follows from the above that these cocycles are all cohomologous and hence represent a single element $\omega^n(f,g) \in H^n(K,L;\pi_n(Y))$, the primary obstruction to homotopy (rel L) of the pair (f,g). If $j : K \subset (K,L)$ is the inclusion map, then we obtain an induced map, $j^* : H^n(K,L;\pi_n(Y)) \longrightarrow H^n(K;\pi_n(Y))$.

19. Theorem— $\gamma^n(f) - \gamma^n(g) = j^* \omega^n(f,g)$.

Proof: We may assume that $f(K^{n-1}) = g(K^{n-1}) = y_0$, by altering f and g by a homotopy if need be. The result the follows from the fact, by theorem 12, that $d^n(f,0) - d^n(g,0) = d^n(f,g)$, where 0 is the constant map.

From these results we get the primary extension theorems and primary homotopy theorems.

An element of the cohomology group $H^n(L;G)$ is said to be extensible over K if it is contained in the image of the homomorphism $i^* : H^n(K;G) \longrightarrow H^n(L;G)$ induced by the inclusion map $i : L \subset K$.

20. Theorem- For a given map $f : L \to Y$, the following tements are equivalent:

(1) f is $n+1$ extensible over K.

(2) $\omega^{n+1}(f) = 0$.

(3) $\kappa^n(f)$ is extensible over K.

Proof: The equivalence of (1) and (2) follows from eorem 14. The equivalence of (2) and (3) follows from eorem 18 and the exactness of the cohomology sequence of e pair (K,L).

21. Corollary- If Y is r-simple and $H^{r+1}(K,L;\pi_r(Y)) = 0$ for every r satisfying $n < r < \dim(K-L)$, then a necessary and sufficient condition for a given map $f : L \to Y$ to have an extension is that the characteristic element $\kappa^n(f)$ is extensible over K.

Proof: Follows from theorems 20 and 15.

22. Theorem- For any two given maps $f, g : K \to Y$ such that $f|L = g|L$, the following are equivalent

(1) f and g are n-homotopic rel L.

(2) $\omega^n(f,g) = 0$

If we also have $H^{n-1}(L;\pi_n(Y)) = 0$, we can add:

(3) $\kappa^n(f) = \kappa^n(g)$.

Proof: That (1) and (2) are equivalent follows from theorem 16. The equivalence of (2) and (3) given the added condition again follows from the cohomology sequence of the pair and theorem 19.

Note that the extra condition is always fulfilled if $L = \emptyset$.

23. Corollary- If Y is r-simple and $H^r(K,L;\pi_r(Y)) = 0$ for each r satisfying $n < r < \dim(K-L)+1$, and $H^{n-1}(L;\pi_n(Y)) = 0$, then a necessary and sufficient condition for a given pair of map $f,g : K \to Y$ with $f|L = g|L$, to be homotopic rel L is that $\varkappa^n(f) = \varkappa^n(g)$.

Proof: Follows from theorems 22 and 17.

With these theorems as motivation, we now examine $\varkappa^n(f)$ more closely, for $f : K \to Y$ and Y n-1-connected and n-simple. To calculate $\varkappa^n(f)$, we replace f by a map g which is homotopic to f and such that $g(K^{n-1}) = y_o$, and then $\varkappa^n(f) = \{d(g,0)\}$. $d(f,0)$ is defined as follows:

We define $F : K^{n-1}xI \cup Kx0 \cup Kx1 \to Y$ by $F(x,t) = y_o$ for $x \in K^{n-1}$ or $t = 1$, and $F(x,0) = g(x)$. F is a map defined on the augmented n-skeleton of the pair of complexes: $(KxI, Kx0 \cup Kx1)$ or, letting $\dot{I} = \{0,1\}$, the pair $Kx(I,\dot{I})$. Thus, $c_F^{n+1} \in C^{n+1}(Kx(I,\dot{I});\pi_n(Y))$, ie:

$c_F^{n+1} : H_{n+1}(K^n x I \cup Kx\dot{I}, K^{n-1}xI \cup Kx\dot{I}) \to \pi_n(Y)$, or preceding by an excision inclusion, it maps $H_{n+1}(K^n x I, K^{n-1}xI \cup K^n x\dot{I}) \to \pi_n(Y)$.

$(-1)^{n+1}d(g,0)$ is this map preceded by the composite function:

$H_n(K^n,K^{n-1}) \to H_n((K^n,K^{n-1})x0) \longleftarrow H_n(K^n,K^{n-1}) \otimes H_o(0) \to$
$H_n(K^n,K^{n-1}) \otimes H_o(\dot{I},1) \xleftarrow{1\otimes\partial} H_n(K^n,K^{n-1}) \otimes H_1(I,\dot{I}) \to H_{n+1}((K^n,K^{n-1}) \times (I,\dot{I}))$.

This is, more or less, a review of the definition. However, we assert that the above map is the same as the simpler composite: (which is a switchback)

$H_n(K^n,K^{n-1}) \xleftarrow{q_*} H_n(K^n x0 \cup K^{n-1}xI, K^{n-1}x\dot{I}) \to H_n(K^{n-1}xI \cup K^n x\dot{I}, K^n xI \cup K^{n-1}x\dot{I}$
$\longleftarrow H_n(K^{n-1}xI \cup K^n x\dot{I}) \xleftarrow{\partial} H_{n+1}((K^n,K^{n-1})x(I,\dot{I}))$, where q is the restriction of the projection of KxI onto K.

This result comes from the diagram:

$$\begin{tikzcd}
H_n(K^n,K^{n-1})\otimes H_1(I,\dot I) \arrow[r, "\gamma"] \arrow[d, "\partial\otimes 1 + 1\otimes\partial"] & H_{n+1}((K^n,K^{n-1})x(I,\dot I)) \arrow[d, "\partial'"] & \\
 & H_n(K^n x\dot I \cup K^{n-1}xI, K^n x1 \cup K^{n-1}x\dot I) & H_n(K^{n-1}xI \cup K^n x0, K^{n-1}x\dot I) \arrow[l, "j_*"'] \arrow[dd, bend left, "q_*"] \\
H_n(K^n,K^{n-1})\otimes H_0(\dot I,1) \oplus H_{n-1}(K^{n-1})\otimes H_1(I,\dot I) \arrow[r, "\gamma'_0+\gamma'_1"] & H_n(K^n x\dot I, K^{n-1}x\dot I \cup K^n x1) \oplus H_n(K^{n-1}x(I,\dot I)) \arrow[u, "i_0+i_1"] & \\
H_n(K^n,K^{n-1})\otimes H_0(0) \arrow[u, "i_*+0"] \arrow[r, "\gamma''"] & H_n((K^n,K^{n-1})x0) \arrow[u, "i_*+0"] \arrow[uur, "k_*"] &
\end{tikzcd}$$

Where the following identities are easily verified:

$q_*k_* = 1$, $j_*k_* = i_o+i_*$, $q_*j_*^{-1}i_1 = 0$, $\gamma_o' = i_*\gamma'' i_*^{-1}$.

Now using the naturality for γ that was proved in chapter 2, we have: $(i_o+i_1)(\gamma_o'+\gamma_1')(\partial_o+\partial_1)\gamma^{-1} = \partial'$. Therefore, $q_*j_*^{-1}\gamma' = q_*j_*^{-1}(i_o+i_1)(\gamma_o'+\gamma_1')(\partial_o+\partial_1)\gamma^{-1} = q_*j_*^{-1}i_o\gamma_o'\partial_o\gamma^{-1} = q_*j_*^{-1}i_oi_*\gamma'' i_*^{-1}\partial_o\gamma^{-1} = q_*j_*^{-1}j_*k_*\gamma'' i_*^{-1}\partial_o\gamma^{-1} = \gamma'' i_*^{-1}\partial_o\gamma^{-1}$. The latter is an isomorphism and hence the former is. The result that we want is the equality of the inverse maps.

Using this identification we prove the following:

24. Lemma- $d(g,0)$ is given by the switchback:

$$H_n(K^n,K^{n-1}) \xleftarrow{h} \pi_n(K^n,K^{n-1}) \xrightarrow{g_*} \pi_n(Y,y_0) \longrightarrow \pi_n(Y).$$

Proof: This follows from the diagram:

$$\begin{tikzcd}
 & \pi_{n+1}((K^n,K^{n-1})x(I,\dot I)) \arrow[r, "h"] \arrow[d, "\partial'"] & H_{n+1}((K^n,K^{n-1})x(I,\dot I)) \arrow[d, "\partial'"] \\
 & \pi_n(K^n x\dot I \cup K^{n-1}xI) \arrow[r, "h"] \arrow[d] \arrow[ddl, bend right, "F_*"'] & H_n(K^n x\dot I \cup K^{n-1}xI) \\
 & \pi_n(K^n x\dot I \cup K^{n-1}xI, K^n x1 \cup K^{n-1}x\dot I) \arrow[r, "h"] & H_n(K^n x\dot I \cup K^{n-1}xI, K^n x1 \cup K^{n-1}x\dot I) \\
\pi_n(Y,y_0) & \pi_n(K^{n-1}xI \quad K^n x0, K^{n-1}xI) \arrow[r, "h"] \arrow[u] \arrow[d, "g_*"] \arrow[l, "F_*"'] & H_n(K^{n-1}xI \quad K^n x0, K^{n-1}xI) \arrow[u] \arrow[d, "g_*"] \\
 & \pi_n(K^n,K^{n-1}) \arrow[r, "h"] \arrow[ul, "f_*"] & H_n(K^n,K^{n-1})
\end{tikzcd}$$

We next consider the situation that arises when Y is itself a CW complex. In that case we can assume that $K = Y$ and look at the priviledged map 1_Y. $\varkappa^n(1_Y)$ is called the fundamental class of Y, and is written $\varkappa^n(Y)$. Its important in the classification of maps comes from the following theorem, which follows directly from theorem 13:

25. Theorem- If $f : K \longrightarrow Y$, then $\varkappa^n(f) = f^* \varkappa^n(Y)$.

If Y is n-1-connected and $n \geq 2$, then we have the following result:

26. Theorem- The image of $\varkappa^n(Y)$ under the isomorphism: $H^n(Y;\pi_n(Y)) \xrightarrow{\alpha} \mathrm{Hom}(H_n(Y),\pi_n(Y)) \xrightarrow{\mathrm{Hom}(h,1)} \mathrm{Hom}(\pi_n(Y),\pi_n(Y))$ is $(-1)^{n+1} 1_{\pi_n(Y)}$.

Proof: This follows from lemma 24 and the commutativity of the following diagram, where $i : Y \longrightarrow Y$ is homotopic to 1_Y and $i(Y^{n-1}) = y_0$,

$$\begin{tikzcd}
 & \pi_n(Y) \arrow[r, "\approx"] \arrow[dl, "i_* = 1_*"'] & H_n(Y) \\
 & \pi_n(Y^n) \arrow[r, "\approx"] \arrow[u] \arrow[d] \arrow[l, "i_*"'] & H_n(Y^n) \arrow[u] \arrow[d] \\
\pi_n(Y, y_0) & \pi_n(Y^n, Y^{n-1}) \arrow[l, "i_*"] \arrow[r, "h"] & H_n(Y^n, Y^{n-1}).
\end{tikzcd}$$

The case where we would expect the most success with primary obstruction theory, would be the case where the primary obstructions are the only ones.

We say that a CW complex is a $K(\pi,n)$ if the homotopy groups of the space are 0 except for dimension n, where the homotopy group π_n is just π. If $n \neq 1$, then π must, of course, be abelian. We are only interested in these notes in the case π abelian, since if $n = 1$, we will want 1-simplicity.

If X and Y are topological spaces then we shall denote by $[X,Y]$ the set of homotopy classed of maps of X to Y.

27. Theorem- If X is a CW complex, then there is a natural, one-to-one, onto correspondence:
$[X, K(\pi,n)] \longrightarrow H^n(X;\pi)$, By the correspondence: $\{f\} \longrightarrow f^* \iota^n(K(\pi,n))$.

Proof: If c is a cocycle representing an element of $H^n(X;\pi)$ then by theorem 11, there exists a map $f : X^n \longrightarrow K(\pi,n)$ with $f(X^{n-1}) = k_0$, a base point of $K(\pi,n)$, and such that if O is the constant map $O(X) = k_o$, then $d(f,O) = c$. $c_f^{n+1} = \delta c = 0$, and hence f extends to X^{n+1}, and hence to all of X by theorem 15. Hence, f represents an element of $[X,K(\pi,n)]$ and $\{f\} \longrightarrow c$, by theorem 25. The correspondence is hence onto.

The correspondence is easily seen to be one-to-one. For if $f^* \iota^n(K(\pi,n)) = g^* \iota^n(K(\pi,n))$, then by theorem 15, $\iota^n(f) = \iota^n(g)$ and hence f is homotopic to g by theorem 23.

The correspondence is obviously natural in X.

Now as to the existence of $K(\pi,n)$'s. We note that since X and the singular complex $S(X)$ have the same homotopy groups, the restriction that the $K(\pi,n)$'s be CW complexes is no restriction at all in an existence proof. We will explicitly construct $K(\pi,n)$ for $n \geq 2$ and π abelian. It follows that $K(\pi,1)$'s exist for π abelian, since $\pi_i(\Omega K(\pi,n)) = \pi_{i+1}(K(\pi,n))$, where $\Omega K(\pi,n)$ is the loop space of $K(\pi,n)$.

Present π as F/R for F a free abelian group on a set of generators I and R a subgroup of F. Let X be a wedge of n-spheres S_i for $i \in I$. Since $n > 1$, $\pi_i(X) = \begin{cases} F & i = n \\ 0 & i < n \end{cases}$

Under the correspondence of $\pi_n(X)$ with F, the elements of a set of generators for R are represented by a collection of maps $f_\alpha : S^n \longrightarrow X$. We define X^{n+1} by attaching an n+1-cell with each of these maps. We assert that $\pi_i(X^{n+1}) = \begin{cases} \pi & i = n \\ 0 & i < n \end{cases}$ and to prove this, it obviously suffices to show that $\pi_{n+1}(X^{n+1},X) \longrightarrow \pi_n(X)$ has image precisely R. The map is certainly onto R and that it is included in R follows from the fact that $\pi_{n+1}(X^{n+1},X)$ is generated by elements of the form $\{i_\alpha\}$ where $i_\alpha : (e^{n+1},S^n) \longrightarrow X^{n+1}$ is the attaching map of the cell e_α^{n+1}. This in turn follows from the Hurewicz theorem. We now proceed inductively, to construct X^{n+r+1} by attaching n+r+1 cells using a generating set for $\pi_{n+r}(X^{n+r})$ and thus killing the n+rth homotopy group. It is easily seen that $\bigcup X^{n+r}$ is a $K(\pi,n)$.

We now show that any two $K(\pi,n)$'s are homotopically equivalent.

28. Theorem- If X and Y are CW complexes such that each is a $K(\pi,n)$ (same π, same n), then X is homotopically equivalent to Y.

Proof: Let f represent the element of $[X,Y]$ such that $f^*\varkappa^n(Y) = \varkappa^n(X)$, and let g represent the element of $[Y,X]$ such that $g^*\varkappa^n(X) = \varkappa^n(Y)$. Since $(fg)^*\varkappa^n(Y) = \varkappa^n(Y)$ and $(gf)^*\varkappa^n_\pi(X) = \varkappa^n_\pi(X)$, it follows from theorem 27, that fg is homotopic to 1_Y and gf is homotopic to 1_X. is homotopic to 1_Y and gf is homotopic to 1_X.

Finally, we prove a theorem which extends Whitehead's theorem, (chap. 1, theorem 27).

29. Theorem. Let $f : X \to X'$ be a continuous mapping of connected, n-simple topological spaces, such that $f_* : \pi_i(X,x_0) \to \pi_i(X',x_0')$ is an isomorphism for $1 \leq i < n+1 \leq \infty$, then if K is any n-dimensional CW complex, the induced map: $f_* : [K,X] \to [K,X']$, is one-to-one and onto.

Proof: By replacing X' by the mapping cylinder M_f if necessary, we can assume that f is an inclusion.

f_* is onto: Let $h : K \to X'$ represent an element of $[K,X']$. Since the relative groups $\pi_i(X',X) = 0$ for $1 \leq i < n+1$, corollary 25 of chapter 1 implies that there exists $h' : K \to X$ which is homotopic to h in X'. This implies that f_* is onto since $f_* \mathrm{cls}\, h' = \mathrm{cls}\, h$.

f_* is one-to-one: Let $h_0', h_1' : K \to X$ such that such that there is a homotopy $H : K\times I \to X'$ such that $H_0 = h_0'$ and $H_1 = h_1'$, ie. $f_* \mathrm{cls}\, h_0' = f_* \mathrm{cls}\, h_1'$. Again applying corollary 25 of chapter 1, we may assume that $H((K\times I)^n) \subset X$. Thus, the restriction of H is a map $H' : K\times 0 \cup K^{n-1}\times I \cup K\times 1 \to X$, which we would like to extend to a map of $K\times I$ into X. Consider the obstruction to such an extension: $c^{n+1}_{H'} \in C^{n+1}(K\times I;\pi_n(X))$. $f_\# : C^{n+1}(K\times I;\pi_n(X)) \to C^{n+1}(K\times I;\pi_n(X'))$ is an isomorphism. Since $f_\# c^{n+1}_{H'} = c^{n+1}_H = 0$, we have that $c^{n+1}_{H'} = 0$ and hence H' extends as required. Thus, $\mathrm{cls}\, h_0' = \mathrm{cls}\, h_1'$ and f_* is therefore one-to-one.

Chap. IV : Fiber Spaces and Postnikov Systems

We now consider the concept of fiber space or fibration, due to Serre. For reference, we refer the feader to the treatment in Hu, chapter III.

A map $p : E \longrightarrow B$ is said to satisfy the covering homotopy property with respect to a space X, if for every homotopy $h_t : X \longrightarrow B$ and every map $f : X \longrightarrow E$ such that $pf = h_0$, there exists a homotopy $f_t : X \longrightarrow E$ such that $f_0 = f$ and $pf_t = h_t$. A map p which satisfies the covering homotopy property for every (finitely) triangulable space X, is said to be a fibration and the triple (E,B,p) is a fiber space.

A strengthening of this property is given by the covering homotopy extension property (CHEP). A map $p : E \longrightarrow B$ is said to satisfy the covering homotopy extension property, the CHEP, with reppect to a pair (X,A) of spaces if for every homotopy $h_t : X \longrightarrow B$ and map $f : X \longrightarrow E$, and partial homotopy $g_t : X \longrightarrow E$ such that $g_0 = f|A$ and $pg_t = h_t|A$, there exists a homotopy $f_t : X \longrightarrow E$ such that $f_0 = f$, $pf_t = h_t$ and $f_t|A = g_t$. The two properties are related by the following theorem:

1. Theorem- For a map $p : E \longrightarrow B$, the following are equivalent:

i) p is a fibration

ii) For each $m \geq 0$, p has the covering homotopy property with respect to the m-cell e^m.

iii) For each $m \geq 0$, p has the CHEP with respect to the pair m-cell mod boundary, (e^m, S^{m-1}).

iv) p has the CHEP with reppect to every CW-pair (K,L).

v) If (K,L) is a CW-pair such that L is a strong deformation retract of K and if $f : K \longrightarrow B$ and $g : L \longrightarrow E$ such that $pg = f|L$, then g has an extension $g' : K \longrightarrow E$ such that $pg' = f$.

Proof: i) --> ii) A fortiori.

ii) --> iii) see Hu pages 63-64.

iii) --> iv) Using the standard Zorn's lemma argument we can extend $g_t : L \longrightarrow E$ to a maximal pair (g_t', L') with L' a subcomplex of K containing L and $g_t' : L' \longrightarrow E$ such that $g_0' = f|L'$, $g_t'|L = g_t$ and $pg_t' = h_t|L'$. We assert that by maximality $L' = K$. If not, then let σ be a cell of smallest dimension of K-L'. To extend g_t' to σ, it suffices to extend $g_t' f_\sigma : S^{m-1} \longrightarrow E$ (where m = dimension σ), to a homotopy $f_t : e^m \longrightarrow E$ such that $f_0' = fi_\sigma$, $pf_t = h_t i_\sigma$, and this extension exists by iii.

iv) --> v) Let $k_t : K \longrightarrow K$ be a strong deformation retraction of K to L, ie. k_0 retracts K to L and $k_t(x) = x$ if $x \in L$ or $t = 1$. Let $\gamma = gk_0 : K \longrightarrow E$, $\tilde{h}_t = fk_t : E \longrightarrow B$ and $\tilde{g}_t = g : L \longrightarrow E$. If $\gamma_t : K \longrightarrow E$ is the homotopy produced by an application of the CHEP, then γ_1 is the required extension g'.

v) --> i) If X is triangulable (in fact, if X is a CW complex) then (XxI,Xx0) is a CW pair with Xx0 a strang deformation retract of XxI, whence i) from v).

Note that either the proof that v) --> i) or property iv) specified down to the case $L = \emptyset$, prove that a fibration has the covering homotopy property with respect to every CW complex.

We will assume from now on that the base is pathwise connected, from which it easily follows that p is onto. If we choose a base point b_0 in B, we define $p^{-1}(B_0) = F$ to be the fiber of p. Choosing a base point x_0 in F, we get (see Hu page 152) the homotopy sequence of the fibering, p, which is an exact sequence:

$$\cdots \longrightarrow \pi_{n+1}(B,b_0) \xrightarrow{d} \pi_n(F,x_0) \xrightarrow{i_*} \pi_n(E,x_0) \xrightarrow{p_*} \pi_n(B,b_0) \longrightarrow \cdots$$

With a base point chosen, we shall sometimes write a fiber space as $E \xrightarrow[F]{p} B$, with the base point $b_0 = p(F)$.

Given a fibering $p : E \longrightarrow B$ and a map $f : B' \longrightarrow B$ we define $f^*E = \{(x,y) \in B' \times E : f(x) = p(y)\}$ which is the topological "pull-back" of p and f, ie. given a space X and maps $g_1 : X \longrightarrow B'$ and $g_2 : X \longrightarrow E$ such that $fg_1 = pg_2$, then there exists a map $h : X \longrightarrow f^*E$ such that the following diagram commutes ($h(x) = (g_1(x), g_2(x))$ and is obviously unique):

$$\begin{tikzcd}
X \arrow[dr, "h"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& f^*E \arrow[r, "q_2"] \arrow[d, "q_1"] & E \arrow[d, "p"] \\
& B' \arrow[r, "f"] & B
\end{tikzcd}$$

Using this property, it is easily proved that the triple (f^*E, B', q_1) is a fiber space (where q_1 and q_2 are the projections of f^*E to B' and E, respectively). It is called the fiber space induced by f.

We call two fiber spaces over the same base, isomorphic if there is a homeomorphism of spaces between them which commutes with projections, ie.

$$\begin{tikzcd}
E \arrow[rr, "h", "\approx"'] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\
& B &
\end{tikzcd}$$

In general, if the map is understood, we shall refer to E as the fiber space, and B as the base space. We shall say E is isomorphic to E' and mean p is isomorphic to p'. We leave to the reader the easy proofs that if $g : B'' \longrightarrow B'$, then $g^*(f^*E)$ is isomorphic to $(fg)^*E$, and that if f is the inclusion of a subset into B, then f^*E is isomorphic to $p^{-1}(B')$.

Note that if $f : (B',b_o') \longrightarrow (B,b_o)$, then $q_2 : (f^*E,F') \longrightarrow (E,F)$ where F' and F are the fibers over b_o' and b_o, respectively. Since $F' = b_o' \times F$, with the restriction of q_2 just projection on the second factor, we see that $q_2|F' : F' \longrightarrow F$ is a homeomorphism and we will usually identify the two fibers by this isomorphism.

From now on, we will assume that every fiber space has fixed base points chosen in E and B such that p is base-point preserving, ie. the base-point of E is in the fiber over the base-point of B, and we shall refer to the fiber over the base-point of B, as the fiber of p, usually denoted F. We shall summarise the statement: "$p : E \longrightarrow B$ is a fiber space with fiber F" by writing $p : E \xrightarrow[F]{} B$. For convenience we shall refer to all the base-points as $*$.

We now examine the notion of principle fiber space.

Definition- A fiber space $E \xrightarrow[F]{p} B$ is called a principle fiber space if: i- F is an H-space, ie. there exists a map $\mu : F \times F \longrightarrow F$ such that $*$ is a homotopy unit for μ, that is, if $\phi : F \vee F \longrightarrow F$ is the folding map sending $(x,*)$ and $(*,x)$ to x, then the following commutes up to a homotopy; rel $* \times *$:

$$\begin{tikzcd} F \vee F \arrow[r, "\phi"] \arrow[d, phantom, "\bigcap"] & F \\ F \times F \arrow[ur, "\mu"'] & \end{tikzcd}$$

and ii- there exists a map γ: $F \times E \longrightarrow E$ which commutes with projection, ie. the following commutes:

$$\begin{tikzcd} F \times E \arrow[rr] \arrow[dr, "pp_2"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

and are such that the following commute up to homotopy; rel $*\times*$:

$$\begin{tikzcd} * \times E \arrow[r, hook] \arrow[dr] & F \times E \arrow[d, "\gamma"] \\ & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F \times F \arrow[r, "1 \times i"] \arrow[d, "\mu"'] & F \times E \arrow[d, "\gamma"] \\ F \arrow[r, "i"] & E \end{tikzcd} .$$

The motivating example iw where E is the space of paths based at $*$ in B with the usual action $\Omega B \times E \longrightarrow E$. The important property that we will need is given by the following lemma:

2. Lemma- If f^*E is the fiber space induced by the map $f: (B',*) \longrightarrow (B,*)$ and E is a principle fiber space over B, then f^*E is a principle fiber space over B'.

Proof: We must define γ' : $F \times f^*E \longrightarrow f^*E$. By the pullback property of f^*E, it suffices to define maps into E and B' such that the two maps agree when composed with the map into B. These two maps are: $F \times f^*E \longrightarrow f^*E \longrightarrow B'$ and $F \times f^*E \longrightarrow F \times E \xrightarrow{\gamma} E$. The commutativity (up to a homotopy where required) of the three diagrams is an easy exercise, using the corresponding diagrams for .

On homotopy, γ_* corresponds to addition of elements of thw two homotopy groups. Formally, we have the lemma:

3. Lemma- Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a principle fiber space with map $\gamma : F\text{x}E \dashrightarrow E$, then the following commutes:

$$\begin{tikzcd}
\pi_q(F\text{x}E) \arrow[rr, "\gamma_*"] & & \pi_q(E) \\
& \pi_q(F)+\pi_q(E) \arrow[ul] \arrow[ur, "i_*+1_*"'] &
\end{tikzcd}$$

.

Proof: The result follows from the fact that we will produce a diagram that commutes up to homotopy with the top and bottom maps representing the two results: let $x : S^q \dashrightarrow F$ and $y : S^q \dashrightarrow E$ represent elements of $\pi_q(F)$ and $\pi_q(E)$, respectively. The following diagram commutes up to homotopy:

$$\begin{tikzcd}
& S^q\text{x}S^q \arrow[r, "x\times y"] & F\text{x}E \arrow[rr, "\gamma"] & & E \\
S^q \arrow[ur, "\Delta"] \arrow[dr, "e"'] & \cup & \cup \arrow[urr, "\gamma"] & & \\
& {}_wS^q\text{v}S^q \arrow[r, "x\text{v}y"] & F\text{v}E \arrow[r, "i\text{v}1"] & E\text{v}E \arrow[uur, "\varphi"'] &
\end{tikzcd}$$

Where e is the map pinching the equator of S^q to $*$ and Δ is the diagonal map.

That the triangle on the left commutes up to homotopy is a standard result of homotopy theory. It is a special case of the following lemma, which we include for completeness:

4. Lemma- Given a space X with base point $*$, we define the suspension of X, $\Sigma'X = X\text{x}I/X\text{x}0 \cup X\text{x}1 \cup *\text{x}I$. Let $e : \Sigma'X \dashrightarrow \Sigma'X\text{v}\,\Sigma'X$ by $e(x,t) = \begin{cases} (x,2t)\text{x}* & 0\leq t\leq \frac{1}{2} \\ *\text{x}(x,2t-1) & \frac{1}{2}\leq t\leq 1 \end{cases}$, and $\Delta : \Sigma'X \dashrightarrow \Sigma'X\text{x}\;\Sigma'X$ be the diagonal map, then the following diagram commutes up to homotopy:

$$\begin{tikzcd}
& \Sigma'X\text{x}\,\Sigma'X \\
\Sigma'X \arrow[ur, "\Delta"] \arrow[dr, "e"'] & \bigcup \\
& \Sigma'X\text{v}\,\Sigma'X
\end{tikzcd}$$

Proof: The required homotopy is given by:

$$H((x,t),s) = (x,\min((1+s)t,1))\text{x}(x,1-\min((1+s)(1-t),1)).$$

We now return to the problem which motivated the previous chapter. Assume that we have a complex K and subcomplex L and a mapping $f : L \to X$. We would like to extend f to a mapping of K into X. The method considered in chapter three was to extend f skeleton by skeleton, that is to say, use the skeletons $\overline{K}^n$ as "approximations" to K and to extend f as far as is possible, using these approximations. We now introduce the notion of the Postnikov system which "approximates" X instead of K, and approximates by using mappings which preserve homotopy groups instead of inclusions.

We define the Postnikov system of a space X to be a diagram:

$$\begin{tikzcd} & \vdots \arrow[d] \\ & X_n \arrow[d, "p_n"] \\ X \arrow[ur, "f_n"] \arrow[r, "f_{n-1}"] & X_{n-1} \arrow[d] \\ & \vdots \end{tikzcd}$$

where $f_{n*} : \pi_i(X,*) \to \pi_i(X_n,*)$ is an isomorphism, $i \leq n$, and where $\pi_i(X_n,*) = 0$ for $i > n$. (We assume that X is 1-connected from now on), and where p_n is a fiber space map.

For X a CW complex we can construct this system inductively using the same method of attaching cells to kill homotopy groups, that was used in the construction of $K(\pi,n)$ at the end of chapter three. That is, we can find a space Y such that $X \subset Y$, and $Y^{n+2} \subset X$ and $\pi_i(Y) = 0$ $i > n$. It follows that the inclusion of X into Y induces an isomorphism of homotopy groups for $i \leq n$. By theorem 15 of chapter 3, $f_{n-1} : X \to X_{n-1}$ extends to a map of $Y \to X_{n-1}$. Thus, we get, by induction, a diagram like the above except that the p_n's needn't be fiber maps. However, there is a method of "replacing" any map by a fiber map, by replacing the domain, space by a homotopy equivalent space. By thus "replacing" the p_n's by fiber maps, inductively, we can construct directly the Postnikov system for X.

There is another construction which is less geometrical but is rather cohomological. In addition to being easier to handle, this method generalizes the Postnikov system of a space to the Moore-Postnikov system of a map. However, before we give this construction, we must digress to introduce the homotopy groups of a map, and similarly homology and cohomology groups.

Given a map $f : X \to Y$, we have defined the mapping cylinder M_f and we note the following properties. X (identified with Xx0) and Y are subsets of M_f and the inclusion of Y into M_f is a homotopy equivalence, with inverse $r_f : M_f \to Y$ defined by $r_f(x,t) = f(x)$ and $r_f(y) = y$. We have the following commutative diagram, with the row being the homotopy sequence of the pair and hence exact:

$$\begin{tikzcd} \cdots \pi_n(X) \arrow[r] \arrow[dr, "f_*"'] & \pi_n(M_f) \arrow[r] & \pi_n(M_f,X) \arrow[r, "\partial"] & \pi_{n-1}(X)\cdots \\ & \pi_n(Y) \arrow[u, "\approx"] & & \end{tikzcd}$$

We assume that X and Y are 1-connected so that we need not worry about base points. We define $\pi_n(M_f,X) = \pi_n(f)$ and replace $\pi_n(M_f)$ by $\pi_n(Y)$ as shown to obtain the exact homotopy sequence of the map:

$$\cdots \pi_n(X) \xrightarrow{f_*} \pi_n(Y) \to \pi_n(f) \xrightarrow{\partial} \pi_{n-1}(X)\cdots$$

Similarly, if we define $H_n(f;G) = H_n(M_f,X;G)$ and $H^n(f;G) = H^n(M_f,X;G)$ then we obtain exact homology and cohomology sequences of f:

$$\cdots H_n(X;G) \xrightarrow{f_*} H_n(Y;G) \longrightarrow H_n(f;G) \xrightarrow{\partial} H_{n-1}(X;G) \cdots$$

$$\cdots H^n(X;G) \xleftarrow{f^*} H^n(Y;G) \longleftarrow H^n(f;G) \xleftarrow{\delta} H^{n-1}(X;G) \cdots$$

If f is a fiber map, then we also have the homotopy sequence of the fiber space:

$$\cdots \pi_n(X) \xrightarrow{f_*} \pi_n(Y) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \pi_{n-1}(X) \cdots$$

It is obvious, by exactness of the two sequences that $\pi_i(F) = 0$ iff $\pi_{i+1}(f) = 0$ for any i. We will require a sharper isomorphism result only in the following special case:

5. Lemma- Let $p : E \xrightarrow[F]{} B$ be a fiber space, with F $n-1$ connected and $(B,*)$ having the homotopy type of a CW pair. Then the following is an isomorphism: $\pi_n(F) \xleftarrow{\partial} \pi_{n+1}(cF,F) \longrightarrow \pi_{n+1}(p)$. Where cF is the cone on F, and $(cF,F) \longrightarrow (M_p,E)$ is the inclusion $(x,t) \longrightarrow (x,t)$ for x in F. We also assume that E is 1-connected.

Proof: Since cF is contractible, $\pi_{n+1}(cF,F) \xrightarrow{\partial} \pi_n(F)$ is an isomorphism, and since F is $n-1$ connected, the pairs (cF,F) and (M_p,E) are $n-1$ connected and hence the Hurewicz homomorphisms are isomorphisms for $n+1$, and thus, it suffices to prove that the inclusion induces an isomorphism of homology: $H_{n+1}(cF,F) \longrightarrow H_{n+1}(p)$.

$(B,*)$ has the homotopy type of a CW pair (K,L). Since L is contractible, (K,L) has the same homotopy type as $(K/L,*)$ and hence we can assume that $L = *$. Let $h: (B,*) \longrightarrow (K,*)$ and $k : (K,*) \longrightarrow (B,*)$ be two maps such that kh is homotopic to 1_B rel $*$. By the homotopy extension property for the pair $(K,*)$, there exists a neighborhood U of $*$ in K and a homotopy $r_t : K \longrightarrow K$ (rel $*$) such that $r_0 = 1_K$ and $r_1(U) = *$. If we let r_t' be the composite of the homotopy of kh with 1_B followed by kr_th

then $r'_t : B \dashrightarrow B$ (rel $*$) and $r'_o = 1_B$ while $r'_1(U') = *$, where $U' = h^{-1}(U)$. We lift the homotopy $r'_t p : E \dashrightarrow B$ to a homotopy s_t with $s_o = 1_E$, using the covering homotopy property. Let $V = p^{-1}(U')$. Note that $s_1 : V \dashrightarrow F$. Let V_p be the subset of M_p with $(x,t) \in V_p$ whenever $x \in V$. Define $s : (V_p, V) \dashrightarrow (cF, F)$ by $s(x,t) = (s_1(x), t)$. We then factor the inclusion of (cF,F) into (M_p, E), into the inclusions: $(cF,F) \xrightarrow{i_1} (V_p, V) \xrightarrow{i_2} (M_p, E)$. i_2 is an excision and hence induces isomorphisms of homology groups. On the other hand, si_1 is homotopic to $1_{(cF,F)}$ and hence $s_* i_{1*}$ = identity, and further, $i_2 i_1 s$ is homotopic to i_2 and hence $i_{2*} i_{1*} s_* = i_{2*}$ and since i_{2*} is an isomorphism it follows that $i_{1*} s_*$ = identity. Thus, i_{1*} and i_{2*} are isomorphisms and hence, so is the composite.

If $f : X \dashrightarrow Y$ and $f' : X' \dashrightarrow Y'$, and there are maps $g_o : X \dashrightarrow X'$ and $g_1 : Y \dashrightarrow Y'$ such that $f' g_o = g_1 f$, and map is induced on mapping cylinders which in turn induces homomorphisms of the homotopy, homology and cohomology sequences.

Let $f : X \dashrightarrow B$, with X and B 1-connected, $(X, \emptyset)$ and $(B, *)$ having the homotopy type of CW pairs. The Moore-Postnikov system of the map f is the diagram:

$$\begin{tikzcd} & \vdots \\ X \arrow[r, "f_n"] \arrow[dr, "f_{n-1}"'] & X_n \arrow[d, "p_n"] \\ & X_{n-1} \\ & \vdots \end{tikzcd}$$

With: (i) $X_o = B$, $f_o = f$; each $(X_n, *)$ having the homotopy type of a CW pair, (ii) $\pi_{i+1}(f_n) = 0 \quad i \leq n$, (iii) p_n is a fiber map, with fiber a $K(\pi, n)$.

Construction by induction, with $n = 0$ from (i).

Let $\pi = \pi_{n+1}(f_{n-1})$ (note that the lower homotopy groups vanish), Let $\upsilon : \mathrm{Hom}(\pi,\pi) \dashrightarrow \mathrm{Hom}(H_{n+1}(f_{n-1}), \pi) \longleftarrow H^{n+1}(f_{n-1};\pi) \dashrightarrow H^{n+1}(X_{n-1};\pi)$

By theorem 27 of chapter 3, we can find a map $k : X_{n-1} \longrightarrow K(\pi, n+1)$ such that $k^*(\varkappa(K(\pi,n+1))) = \nu(1_\pi)$. We pick a base point $*$ of $K(\pi,n+1)$ to be the image of the basepoint of X_{n-1} under k (the base point of X_{n-1} is the image of the base point of X under f_{n-1}). We let $\eta_1 : P \longrightarrow K(\pi,n+1)$ be the space of paths based at $*$, ie. $P = \{p : I \longrightarrow K(\pi,n+1)$, with $p(0) = *\}$ and $\eta_1(p) = p(1)$. η_1 is a fiber map, and we define X_n and p_n as the induced fiber space over X_{n-1}. The diagram is:

$$
\begin{array}{ccccc}
 & & X_n & \longrightarrow & P \\
 & & \downarrow p_n & & \downarrow \eta_1 \\
X & \xrightarrow{f_{n-1}} & X_{n-1} & \xrightarrow{k} & K(\pi,n+1)
\end{array}
$$

That X_n has the homotopy type of a CW complex is a combination of two results. Milnor has proved that P has the homotopy type of a CW complex (1; thm. 3) and from this, a result of Stasheff (2; prop. 12) implies that X_n has, too. (See notes for the sharpening of Stasheff's result.)

The fiber of p_n is the same as the fiber of η_1 which is $\Omega K(\pi,n+1)$ and is thus a $K(\pi,n)$ (note: same π).

Thus, our construction will be complete when we define f_n, a lifting of f_{n-1} to X_n so that $\pi_{i+1}(f_n) = 0$ for $i \leq n$.

First, we note that liftings of f_{n-1} do exist, for $f_{n-1}{}^*k^* \varkappa(K(\pi,n+1)) = f^*_{n-1}\ \nu(1_\pi) = 0$, and hence kf_{n-1} is homotopic to the constant map, by theorem 27 of chapter 3. Since the constant map can be lifted to P, the covering homotopy property implies that kf_{n-1} can be lifted to P, and hence, using the pullback definition of X_n, f_{n-1} can be lifted to X_n. Let $f' : X \longrightarrow X_n$ be such a lifting. We consider the homotopy groups $\pi_i(f')$.

We will, ambiguously, let p_n also represent the map $M_{f'} \dashrightarrow M_{f_{n-1}}$ induced by p_n and 1_X. p_n, then induces a map of the homotopy sequence of f' into that of f_{n-1} :

$$\begin{tikzcd}
 & \pi_{i+1}(X) \arrow[r, "id"] \arrow[d, "f'_*"'] & \pi_{i+1}(X) \arrow[d, "f_{n-1*}"] & \\
\pi_{i+1}(F) \arrow[r] & \pi_{i+1}(X_n) \arrow[r, "p_{n*}"] \arrow[d] & \pi_{i+1}(X_{n-1}) \arrow[r] \arrow[d] & \pi_i(F) \\
 & \pi_{i+1}(f') \arrow[r, "p_{n*}"] \arrow[d] & \pi_{i+1}(f_{n-1}) \arrow[d] & \\
 & \pi_i(X) \arrow[r, "id"] \arrow[d, "f'_*"'] & \pi_i(X) \arrow[d, "f_{n-1*}"] & \\
\pi_i(F) \arrow[r] & \pi_i(X_n) \arrow[r, "p_{n*}"] & \pi_i(X_{n-1}) \arrow[r] & \pi_{i-1}(F)
\end{tikzcd}$$

The columns are the homotopy sequences of the two maps and the rows are the homotopy sequence of the fiber space where F, a $K(\pi,n)$, is the fiber of p_n.

Thus, for $i \leq n-2$, the two columns are isomorphic by the five-lemma, and hence $\pi_{i+1}(f') = 0$.

We note that for $i \leq n-1$, $\pi_{i+1}(f_{n-1}) = 0$ and $\pi_i(F) = 0$, and hence $\pi_{i+1}(p_n) = 0$, and thus $f'_* : \pi_{i+1}(f_{n-1}) \longrightarrow \pi_{i+1}(p_n)$ is trivially an isomorphism for $i \leq n-1$. We show that with the additional assumption that $f'_* : \pi_{n+1}(f_{n-1}) \longrightarrow \pi_{n+1}(p_n)$ is an isomorphism, we can prove that $\pi_n(f') = 0$ and $\pi_{n+1}(f') = 0$, and thus we can let $f_n = f'$.

$i = n-1$: In the bottom square of the above diagram the top, bottom and right-hand maps are isomorphisms and hence the left-hand map, $f'_* : \pi_{n-1}(X) \longrightarrow \pi_{n-1}(X_n)$ is an isomorphism. Furthermore, if we consider f' mapping the sequence of the map f_{n-1} into that of the map p_n, our assumption gives us, by the 5-lemma, that $f'_* : \pi_n(X) \longrightarrow \pi_n(X_n)$ is an isomorphism and hence, by exactness, $\pi_n(f') = 0$.

$i = n$: It suffices, by exactness to show that the map

$f'_* : \pi_{n+1}(X) \longrightarrow \pi_{n+1}(X_n)$ is epi, which is an easy diagram chase of the diagram; (it is actually a special case of the weak 4-lemma):

$$\begin{tikzcd}
\pi_{n+2}(f_{n-1}) \arrow[r, "f'_*"] \arrow[d] & \pi_{n+2}(p_n) = 0 \arrow[d] \\
\pi_{n+1}(X) \arrow[r, "f'_*"] \arrow[d] & \pi_{n+1}(X_n) \arrow[d] \\
\pi_{n+1}(X_{n-1}) \arrow[r, "id"] \arrow[d] & \pi_{n+1}(X_{n-1}) \arrow[d] \\
\pi_{n+1}(f_{n-1}) \arrow[r, "f'_*"] & \pi_{n+1}(p_n)
\end{tikzcd}$$

So the problem is reduced to finding a particular lifting f_n so that $f_{n*} : \pi_{n+1}(f_{n-1}) \longrightarrow \pi_{n+1}(p_n)$ is an isomorphism. Our method will be as follows: first, we will find a likely looking isomorphism of the two groups; we will then find a mapping of X into the fiber F which will "represent" the difference between f'_* and this isomorphism; and finally, we will "twist" f' by this map, using the principle fiber space structure induced by P, and η_1.

Consider the map: $\pi_{n+1}(p_n) \xrightarrow{k_*} \pi_{n+1}(\eta_1) \longleftarrow \pi_{n+1}(K(\pi,n+1))$. The second map is an isomorphism since P is contractible and the first map factors into $\pi_{n+1}(p_n) \longleftarrow \pi_{n+1}(cF,F) \longrightarrow \pi_{n+1}(\eta_1)$ each map of which is an isomorphism by lemma 5. Call this map $\alpha : \pi_{n+1}(p_n) \longrightarrow \pi_{n+1}(f_{n-1})$. Now consider the diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\pi_{n+1}(p_n),\pi) \arrow[r, "\approx"] & \mathrm{Hom}(H_{n+1},\pi) & H^{n+1}(p_n;\pi) \arrow[l, "\approx"'] \arrow[r] & H^{n+1}(X_{n-1};\pi) \\
\mathrm{Hom}(\pi_{n+1}(\eta_1),\pi) \arrow[u, "\alpha"'] \arrow[d, "\alpha"] \arrow[r, "\approx"] & \mathrm{Hom}(H_{n+1},\pi) \arrow[u] \arrow[d] & H^{n+1}(\eta_1;\pi) \arrow[l, "\approx"'] \arrow[u] \arrow[d] \arrow[ur, "k^*"'] & \\
\mathrm{Hom}(\pi_{n+1}(K(\pi,n+1)),\pi) \arrow[r, "\approx"] & \mathrm{Hom}(H_{n+1},\pi) & H^{n+1}(K(\pi,n+1);\pi) \arrow[l, "\approx"'] &
\end{tikzcd}$$

We will let $\overline{\alpha}$ be the element of $H^{n+1}(p_n;\pi)$ which α maps into by the above succesion of Hom of the Hurewicz map and the universal coefficient map. Similar notation for other homomorphisms will be used, Thus, for example, $T_\pi = \varkappa(K(\pi,n+1))$.

This implies that $\overline{\alpha} \longrightarrow \nu(1_\pi)$, by commutativity and the definition of k. We can consider the element of the cohomology group: $\overline{1_\pi} - f'^* \overline{\alpha}$ ($= \overline{1_\pi - \alpha f'_*}$) $\in H^{n+1}(f_{n-1};\pi)$, and we assert that it can be pulled back to $H^n(X;\pi)$. From the diagram:

$$\begin{array}{ccccc} & & H^{n+1}(X_{n-1};\pi) & & \\ & \nearrow & & \nwarrow & \\ H^{n+1}(f_{n-1};\pi) & & \xleftarrow{f'^*} & & H^{n+1}(p_n;\pi) \\ \delta\nearrow & & & & \\ H^n(X;\pi) & & & & \end{array}$$

we note that $\overline{1_\pi} - f'^* \overline{\alpha}$ goes into $\nu(1_\pi) - \nu(1_\pi)$ in $H^{n+1}(X_{n-1};\pi)$ and this $= 0$ and so the element in question pulls back to some member of $H^n(X;\pi)$, which, by theorem 27 of chapter 3, is represented by some continuous function $g : X \longrightarrow F$, since the fiber is a $K(\pi,n)$. For the definition of $\varkappa(K(\pi,n))$ we use the following map as an identification of $\pi_n(K(\pi,n))$ with $\pi_{n+1}(f_{n-1})$: $\pi_n(F) \xleftarrow{\partial} \pi_{n+1}(cF,F) \longrightarrow \pi_{n+1}(p_n) \xrightarrow{\alpha} \pi$, each map of which is an isomorphism. Applying theorem 26 of chapter 3, again, we note that $\delta\, g^*\varkappa(F) = \overline{g_*\partial} = \overline{(1_\pi - \alpha f'_*)}$ and hence $g_*\partial = 1_\pi - \alpha f'_*$.

By lemma 2, $p_n : X_n \longrightarrow X_{n-1}$ is a principal fiber space, with composition $\gamma : F x X_n \longrightarrow X_n$. So from the lifting f' of f_{n-1} and the map g, we can form a new lifting $\gamma(g x f')\Delta$, which we call f_n. The crucial lemma that we require is:

6. Lemma- Let $i : F \subset X_n$, then $f_{n*} : \pi_{n+1}(f_{n-1}) \longrightarrow \pi(p_n)$ is the sum $f'_* + (ig)_*$.

Proof: This lemma is just a relativization of lemma 3. The proof follows a similar diagram. If $\rho : (e^{n+1},S^n) \longrightarrow (M_{f_{n-1}},X)$ represnets an element of $\pi_{n+1}(f_{n-1})$, then we have the diagram:

$$
\begin{tikzcd}
 & e^{n+1}\times e^{n+1} \arrow[r, dashed, "g\beta\times f'\beta"] \arrow[d, phantom, "\bigcup"] & (M_{p_n q_2}, F\times X_n) \arrow[dr, "\gamma"] & \\
e^{n+1} \arrow[ur, "\Delta"] \arrow[dr] & {} & & (M_{p_n}, X_n) \\
 & e^{n+1}\vee e^{n+1} \arrow[r, dashed, "g\beta\vee f'\beta"] & (M_{\phi(p_n\vee *)}, X_n\vee F) \arrow[uu] \arrow[r, dashed] & (M_{\phi(p_n\vee p_n)}, X_n\vee X_n) \arrow[u]
\end{tikzcd}
$$

Where $q_2 : F\times X_n \dashrightarrow X_n$ is the projection, $*$ is the constant map $F \dashrightarrow *$ ($= p_n | F$) and ϕ is the folding map of $A\vee A \dashrightarrow A$ for any A.

In this diagram, which commutes up to homotopy, the top map represents $\gamma(g\times f')\Delta\beta$, and the bottom map represents $f'\beta + ig\beta$.

Armed with this lemma, we can show that f_{n*} is an isomorphism as required. We first consider just what $(ig)_*$ is. It factors as shown:

$$
\begin{tikzcd}
\pi_{n+1}(f_{n-1}) \arrow[r, "g_*"] \arrow[d, "\partial"] & \pi_{n+1}(cF,F) \arrow[r, "i_*"] \arrow[d, "\partial"] & \pi_{n+1}(p_n) \\
\pi_n(X) \arrow[r, "g_*"] & \pi_n(F) &
\end{tikzcd}
$$

$f_{n*} = f'_* + (ig)_*$, by the lemma. Hence, $\alpha f_{n*} = \alpha f'_* + \alpha(ig)_*$, but the above diagram shows that $\alpha(ig)_*$ is just $g_*\partial$ followed by the map identifying $\pi_n(F)$ with $\pi_{n+1}(f_{n-1})$. This, in turn, we know to be equal to $1_\pi - \alpha f'_*$. It follows that $\alpha f_{n*} = 1_\pi$ and hence f_{n*} is the isomorphism α^{-1}.

The construction is thus completed.

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