"CW COMPLEXES
AND
OBSTRUCTION THEORY"

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Chap. I : CW Complexes

Given two topological spaces X and Y and a continuous mapping, $f: A \longrightarrow Y$, where A is a subset of X, we can form the adjunction space, adjoining X to Y by means of f as follows: Consider the disjoint union of Y and X, with the weak topology (ie. a subset U off $X \cup Y$ is open iff $U \cap X$ and $U \cap Y$ are open in X and Y respectively). The map f generates an equivalence relation on $X \cup Y$: x and x' in A are equivalent if $f(x) = f(x')$, and x is equivalent to y in Y, if $y = f(x)$, and x is equivalent to x for each x in $X \cup Y$. The adjunction space, $X \cup_f Y$, is the quotient space of $X \cup Y$ determined by this equivalence relation. Since, y is equivalent to y' in Y iff $y = y'$, $Y \longrightarrow X \cup Y \longrightarrow X \cup_f Y$ is a homeomorphism and under this identification we will speak of Y as a subset of $X \cup_f Y$. In which case the inclusion map $X \longrightarrow X \cup Y \longrightarrow X \cup_f Y$, i_f, is an extension of the map f.

An important example of an adjunction space is the mapping cylinder of a map $f: X \longrightarrow Y$. f determines a mapping of the subset Xx1 of XxI, by $f'(x,1) = f(x)$. The adjunction space $X\times I \cup_{f'} Y$ is called the mapping cylinder of f, written M_f. Also important is the mapping cone of f, $\mathfrak{M}_f$, obtained from the mapping cylinder by smashing Xx0 to a point, ie. $\mathfrak{M}_f = M_f / X\times 0$, or else as the adjunction space $\mathfrak{M}_f = CX \cup_{f'} Y$, where CX is the cone on X, ie. $CX = X\times I / X\times 0$.

This method of adjoining spaces to each other via continuous maps enables us to construct a rather general class of spaces, namely CW complexes, out of very simple spaces, namely cells and spheres.

A CW complex, or cell complex, introduced by J.H.C. Whitehead (1), is a union of subspaces: $K^0 \subseteq \ldots K^n \subseteq K^{n+1} \ldots$ where the space $K = \bigcup_{n=0}^{\infty} K^n$ is given the weak topology (ie. U is open in K iff $U \cap K^n$ is open in K^n for each n), where the K^n are defined inductively as follows: K^0 is a discrete set of points, or 0-cells. Assuming K^n is defined, K^{n+1} is defined by attaching n+1 cells to K^n by means of continuous maps of their boundary spheres. That is, we are given an index set I^{n+1} and for each $\sigma \in I^{n+1}$ we have a continuous map of S^n_σ, the n-sphere which bounds the n+1-cell e^{n+1}_σ, into K^n, ie. $f_\sigma : S^n_\sigma \longrightarrow K^n$. If we let $\cup f_\sigma$ be the map of the subset $\bigcup S^n_\sigma$ of $\bigcup e^{n+1}_\sigma$ (weak, disjoint union) into K^n defined as f_σ on S^n_σ, then we define $K^{n+1} = (\bigcup e^{n+1}_\sigma) \cup_{\cup f_\sigma} K^n$. Equivalently, we can look only at S^n_σ and f_σ and define $K^{n+1} = \bigcup \mathfrak{M}_{f_\sigma}$, where $\mathfrak{M}_{f_\sigma} \cap \mathfrak{M}_{f_{\sigma'}} = K^n$ if $\sigma \neq \sigma'$ with the weak topology on the union, since CS^n_σ is homeomorphic to e^{n+1}_σ (that the two definitions are equivalent follows from lemma 1, below). To avoid cumbersome notation, we will "name" the image of e^{n+1}_σ in K: σ, or if we want to make explicit the dimension, σ^{n+1}. The inclusion, $i_\sigma : e^{n+1} \longrightarrow \sigma \subseteq K$ is a continuous map which extends $f_\sigma : S^n \longrightarrow \dot{\sigma}$ (we shall write $\dot{\sigma}$ for the image of S^n under f_σ, and call it "the boundary of σ".) and is a homeomorphism of $e^{n+1}-S^n$ onto $\sigma - \dot{\sigma}$. Note that we will, in general, drop the index from e^{n+1} and S^n.

1. Lemma- The topology of K is the weak topology on

Proof: Since K has the weak topology on $\{K^n\}$, it suffices to prove the theorem for each K^n. This we do inductively. Since K^0 is discrete the theorem is trivial for dimension 0. Assuming the theorem for dimension n, we prove it for dimension n+1. Let $U \subset K^{n+1}$ such that $U \cap \sigma$ is open in σ for each $\sigma \subset K^{n+1}$ and hence by inductive assumption, $U \cap K^n$ is open in K^n. K^{n+1} is a quotient space of $(\bigcup e_\sigma^{n+1}) \cup K^n$ and so to prove that U is open in K^{n+1} it suffices to prove that its inverse image in this union is open, and since the union is disjoint and with the weak topology, we need only prove that the inverse intersects K^n and each e_σ^{n+1} in an open set. Its intersection with K^n is just $U \cap K^n$ which is open and its intersection with e_σ^{n+1} is $i_\sigma^{-1}(U \cap \sigma)$ which is open since $U \cap \sigma$ is.

In dealing with CW complexes there are a few lemmas about the weak topology, which it helps to have at hand. First, a lemma from general topology:

2. Lemma (Wallace) - If X and Y are topological spaces, A and B are compact subsets of X and Y respectively, and W is a neighborhood of AxB in the product space XxY, then there are neighborhoods U of A and V of B such that $U \times V \subset W$.

Proof: An easy exercise (or see Kelley page 142).

Throughout the following five lemmas, assume that $X = \bigcup A_\alpha$ with the weak topology, the α's elements of some index set J.

3. Lemma. If X ...

finite union of A_α's. Hence, if, in addition, the A_α's are linearly ordered by inclusion, then C is contained in some A_α.

Proof: Assume the contrary, and let $x_n \in C - \bigcup_{i=0}^{n} A_{\alpha_i}$. For each n, $J(n) = \{x_m : m \geq n\}$ intersects each A_α in a finite set and is hence closed, in X since X is T_1. $\bigcap_{n=0}^{\infty} J(n)$ thus contains all of the cluster points of the sequence (see Kelley, page 72), but this intersection is empty contradicting the assumption that C is compact. If the A_α's are linearly ordered by inclusion and C is contained in some finite union, then C is contained in the largest member of the subcollection.

4. Lemma– If X' is an open or closed subset of X, then the relative topology on X' is the weak topology of $\{X' \cap A_\alpha\}$.

Proof: If X' is open (resp. closed) in X, then let B be a subset of X' intersecting each $X' \cap A_\alpha$ in an open (resp. closed) subset of $X' \cap A_\alpha$ and hence of A_α, and thus B is open (resp. closed) in X and therefore in X'.

5. Lemma– If X' is an open or closed subset of X, and f is a map defined on X' and continuous on each $X' \cap A_\alpha$, then f is continuous on X'.

Proof: Lemma 4 reduces this to the case X' = X, for which it is trivial since $f^{-1}(V) \cap A_\alpha = f|A_\alpha^{-1}(V)$, where V is any subset of the range of f.

6. Lemma– If Y is a locally compact, regular space then XxY has the weak topology on $\{A_\alpha xY\}$.

Proof: Every set open in XxY is certainly open in the weak topology. Conversely, if $U \cap A_\alpha xY$ is open in $A_\alpha xY$, for each α in J, and $(x_0,y_0) \in U$, we will find neighborhoods V and V' of y_0 and x_0 respectively, such that $V' x V \subset U$. Assume $x_0 \in A_{\alpha_0}$, and consider $\{y \in Y : (x_0,y) \in U\}$. This is certainly a neighborhood of y_0 since $U \cap A_{\alpha_0} xY$ is open in $A_{\alpha_0} xY$. y_0 therefore has a neighborhood V, which is open, and such that $\bar{V}$ is a compact subset of the above neighborhood. Then, $x_0 x \bar{V} \subset U$ and we can consider $V' = \{x \in X : x \times \bar{V} \subset U\}$. We assert that V' is open in X. It suffices to show that $V' \cap A_\alpha$ is open in A_α. But since $V' \cap A_\alpha = \{x \in A_\alpha : x \times \bar{V} \subset U \cap A_\alpha xY\}$ and, since $x \times \bar{V}$ is compact, this follows from Wallace's lemma.

7. Lemma- If X' is an open or closed subset of X, and f_t is a map such that on each $X' \cap A_\alpha$, the restriction of f_t is a homotopy, then f_t is a homotopy on X'.

Proof: This follows from lemmas 5 and 6 with Y = I, the unit interval.

The restriction in lemma 3, to the case where J is countable is necessary in general. For example, the unit interval, I, is a compact space, but because it is first countable, its topology is the weak topology on countable sets. For if F were closed in the weak topology and not in the usual topology, then there would be a sequence in F converging to a point of the complement, which there cannot be since the two topologies certainly agree on countable sets. However, for CW complexes we will be able to improve upon lemma 3. First, an important definition:

Definition. A subset L of K is called a subcomplex of K, if L is a union of cells, σ, of K and such that if $\sigma \subset L$, and $\dot\sigma$ meets $\sigma'-\dot\sigma'$, then $\sigma' \subset L$. Equivalently, L is a union of open cells $\sigma - \dot\sigma$, such that if $\sigma - \dot\sigma \subset L$, then $\dot\sigma \subset L$. Equivalently, L is a CW-complex such that $I^n_L \subset I^n_K$ and such that $\sigma \subset L$ implies that $f^L_\sigma = f^K_\sigma$. (That L is a CW complex is proved below.)

For each n, $K^n = \{\sigma \subset K : \dim \sigma \leqq n\}$, the n-skeleton of K, is a subcomplex of K. Note that if L is a subcomplex of K, then $L \cap K^n = L^n$. We prove more generally that if L and L' are subcomplexes of K, then $L \cap L' = \{\sigma : \sigma \subset L \cap L'\}$. The set on the right is certainly a subcomplex and certainly a subset of $L \cap L'$. But if $x \in L \cap L'$, then x is an element of a unique open cell $\sigma - \dot\sigma$ and consequently $\sigma \subset L \cap L'$, and we have the opposite inclusion. From this we can prove that a subcomplex of K is a closed subset of K. By induction on the skeletons: $L \cap K^0$ is certainly closed in the discrete set K^0. Assume $L \cap K^n$ is closed, we prove that $L \cap K^{n+1} = L^{n+1}$ is closed in K^{n+1}. It suffices to prove, as in lemma 1, that the inverse image of L^{n+1} under i_σ, for σ an n+1-cell, is closed in e^{n+1}. If $L^{n+1} \cap \sigma - \dot\sigma \neq \emptyset$, then since L^{n+1} is a union of open cells, $\sigma \subset L^{n+1}$ and the inverse under i_σ is e^{n+1}. Otherwise, L^{n+1} intersects σ as a subset of $\dot\sigma$ which is contained in K^n and hence $i_\sigma^{-1}(L^{n+1}) = i_\sigma^{-1}(L^n)$, which is closed since L^n is closed by inductive hypothesis. In particular, each K^n is closed in K. It follows that each subcomplex L, has, by lemma 4, the weak topology on cells, and is consequently, a CW complex itself.

8. Lemma— If C is a compact subset of the CW complex K, then C meets only a finite number of open cells.

Proof: By lemma 3, $C \subseteq K^n$ for some n. Assume the theorem fails for C, and let N be the smallest positive integer such that the theorem fails for $C \cap K^N$. We assert that C intersects an infinite number of open N-cells, since otherwise the theorem, since it fails for N, would fail for N-1. But picking a point from each such open N-cell we get an infinite discrete subset of $C \cap K^N$, which is impossible since C is compact and K^N is closed. (Note: $C \subseteq K^n$ implies $\sigma^m \cap C \subseteq \dot{\sigma}^m$, $m > n$.)

9. Lemma— Let X' be open or closed in K, and let f (resp. f_t) be a continuous map (resp. a homotopy) defined on $X' \cap K^n$. f (f_t) extends to $X' \cap K^{n+1}$ iff each of the maps ff_σ ($f_t f_\sigma$) defined on $S^n \cap f_\sigma^{-1}(X')$ extends to $e^{n+1} \cap i_\sigma^{-1}(X')$.

Proof: Since i_σ is a quotient map and a homeomorphism on $e^{n+1} - S^n$, ff_σ extends as stated, iff $f|\dot{\sigma} \cap X'$ extends to $\sigma \cap X'$. Since $\dot{\sigma}$ is the boundary of σ, in K^{n+1}, these extensions define continuous maps on each $\sigma \cap X'$, which is continuous by lemma 5. ?
On the other hand, if f extends to $X' \cap K^{n+1}$, then it certainly extends to each $\sigma \cap X'$, for σ in K^{n+1}. Similarly for f_t using lemma 7, instead of lemma 5.

Armed with this extension lemma, we can examine the character of CW complexes as topological spaces. As is to be expected from their construction, they satisfy rather strong normality conditions.

10. Theorem. A CW complex, K, as a topological space is a perfectly normal and completely normal Hausdorff space.

Note: A topological space is perfectly normal, if it is normal and if every closed subset is a G_δ, ie. a countable intersection of open sets, or equivalently, a zero-set, ie. the set of zeroes of some continuous real-valued function. A topological space is completely normal if every subspace is normal, or equivalently if every pair of subsets A and B such that $A^- \cap B = \emptyset$ and $B^- \cap A = \emptyset$, can be separated by disjoint open neighborhoods, or equivalently, if every open subspace is normal.

Proof: Since K is obviously T_1, normality implies that K is Hausdorff, so it suffices to prove the first two conditions. Given U an open subset of K, and A and B disjoint, relatively closed subsets of U, we will construct a continuous real-valued function, $f : U \dashrightarrow \mathbb{R}$, such that $f(B) = 1$ and the zero-set of f is precisely A. This will prove the two normality requirements. We ponstruct f, inductively on the skeletons. Let $f(U \cap K^o \cap A) = 0$ and $f(U \cap K^o - A) = 1$. This defines f on $U \cap K^o$. Assume f is defined with the required properties on $U \cap K^n$; we wish to extend f to $U \cap K^{n+1}$. By lemma 9, this reduces to the problem of extending maps in Euclidean space, ie. let $U' = i_\sigma^{-1}(U)$, $A' = i_\sigma^{-1}(A)$, $B' = i_\sigma^{-1}(B)$, and $S' = U' \cap S^n$. We have a map f' defined on S', which is 1 on $B' \cap S'$ and has as zero-set $A' \cap S'$, and we want to extend f' to a similar map defined on U'. $A' \cup S'$ is a closed subset of the metric space U', and it is hence a G_δ, $A' \cup S' = \bigcap_{m=0}^{\infty} G_m$. Define f_m on the theorem, and the required function is $f'^{\,\text{extension}} = \sum_m f'_m/2^m$.

Next we generalize an important property of subcomplexes of simplicial complexes.

11. Theorem- If L is a subcomplex of a CW complex K, and U is an open neighborhood of L, then there exists an open neighborhood V of L in U, such that L is a strong deformation retract of V^-, by a homotopy which induces a strong deformation retraction of V.

Proof: We define, inductively on skeletons, the following: Open set V^n, $L^n \subset V^n \subset V^{n-} \subset U \cap K^n$, with V^n open in K^n; with $V^{n-1} = V^n \cap K^{n-1}$ homotopy f^n_t, a strong deformation retraction homotopy of V^{n-} onto $L^n \cup V^{n-1-}$, and thus, $f^n_1 = 1_{V^{n-}}$ and f^n_0 a retraction of V^n onto $L^n \cup V^{n-1-}$, with the homotopy rel $L^n \cup V^{n-1-}$ and which induces a homotopy of V^n, ie. Image $f^n_t \mid V^n \subset V^n$.

For $n = 0$, let $V^0 = L^0$ and $f^0_t = 1_{L^0}$.

Assuming that everything is defined for n, we attempt the inductive step. By lemma 9, we reduce the question of extending everything to the question ~~reduces~~ OF EXTENDING to one n+1 cell σ, and for σ not in L then pulled back by i_σ to e^{n+1}. $i_\sigma^{-1}(U)$ is a neighborhood of $i_\sigma^{-1}(V^{n-})$ and since e^{n+1} is compact, we have that there exists $\varepsilon < d(i_\sigma^{-1}(V^{n-}), e^{n+1} - i_\sigma^{-1}(U))$, and $\varepsilon > 0$. If c is the center of e^{n+1}, and $r : e^{n+1} - c \longrightarrow S^n$ is radial projection, which is a retraction, We can define $V^{n+1} \cap \sigma =$ $i_\sigma\{x : f_\sigma r(x) \in V^n \text{ and } d(x, r(x)) < \varepsilon(\sigma)\}$. This defines V^{n+1} since for σ in L, $V^{n+1} \cap \sigma$ must equal σ. V^{n+1} is open in K^{n+1}, and for σ not in L, $V^{n+1-} \cap \sigma = i_\sigma\{x : f_\sigma r(x) \in V^{n-}$

For σ not in L, $f_t^{n+1} \mid V^{n+1-} \cap \sigma\,(\tilde{x}) = i_\sigma(tr(i_\sigma^{-1}(\tilde{x})) + (1-t)i_\sigma^{-1}(\tilde{x}))$. This always has image in V^{n+1-} because, since $r(x)$ and x lie on a line with c, so does the line segment between them. This completes our inductive construction.

Let $V = \cup V^n$, which is open in K and has closure $\cup V^{n-}$, since $(\cup V^n) \cap K^m = V^m$ and $(\cup V^{n-}) \cap K^m = V^{m-}$. For the homotopy, we define:

$$F_t^n \text{ on } K^{n-} = \begin{cases} 1_{V^{n-}} & 0 \leq t \leq 1/2^{n+1} \\ f^n_{2^{n+1}t-1} & 1/2^{n+1} \leq t \leq 1/2^n \\ f^{n-1}_{2^nt-1} f_1^n & 1/2^n \leq t \leq 1/2^{n-1} \\ \quad \cdots & \\ f^o_{2t-1} f_1^1 f_1^2 \ldots f_1^n & 1/2 \leq t \leq 1 \end{cases}$$

F_t^n is certainly continuous on K^n and $F_o^n = 1_{V^{n-}}$ and $F_1^n = f_1^o f_1^1 \ldots f_1^n$ which is a retraction of V^{n-} onto L, which is all right on V^n. And furthermore, since $f_t^n \mid V^{k-} = 1_{V^{k-}}$ if $k < n$, we have that $F_t^n \mid V^{k-} = F_t^k$, and hence the F_t^n's define a homotopy F_t by lemma 7, on V^- and also on V.

Given two complexes, K and L, we can consider the topological space KxL. In general, this need not be a CW-complex. However, we can define a complex Kx_cL, by giving the product, instead of the product topology, the weak topology on the products of cells, ie., the weak topology on the set $\dot{\sigma} x \dot{\sigma}'$ for σ, and σ' cells of K and L respectively. As usual, we define the skeleton inductively. Let $(Kx_cL)^o = K^oxL^o$, which is discrete since K^o and L^o are. Assume we have defined $(Kx_cL)^{n-1}$, we define $(Kx_cL)^n$ as follows:

First, the index set: $I^n_{Kx_cL} = \bigcup_{i+j=n} I^i_K \times I^j_L$. For (σ,σ') in $I^i_K \times I^j_L$ we attach the n-cell $e^i \times e^j$ by the map defined on the boundary, that is on $S^{i-1} \times e^j \cup e^i \times S^{j-1}$ as the union of maps: $f^K_\sigma \times i^L_{\sigma'} \cup i^K_\sigma \times f^L_{\sigma'} = h_{\sigma\sigma'}$. We see that this is a CW complex which is precisely the product of K and L with the weak topology on the productsof cells. We obviously can always define a natural, continuous bijection $Kx_cL \dashrightarrow KxL$ but this map may not always be a homeomorphism.

12-Lemma- A CW complex is closure-finite, that is, every finite collection of closed cells of K is contained in some finite subcomplex of K.

Proof: If $\sigma_0,\ldots,\sigma_m$ is such a finite family, assume the indexing is such that $i < j$ implies $\dim \sigma_i \leq \dim \sigma_j$. We define thd complex inductively using $\dim \sigma_m$ steps. If $\dim \sigma_m = n$, then consider all of the n-1 cells σ such that $\sigma - \dot{\sigma}$ meets some $\dot{\sigma}_j$ of the collection. All of these, plus our original collection forms a larger collection which is still finite by lemma 8. Add all the open n-2 cells which meet the boundary of some member of the new collection. After n steps thės process yields the required finite subcomplex.

13. Theorem- The following conditions on K are equivalent:

a- K is locally compact,

b- each point of K has a neighborhood which is a union of finitely many closed cells,

c- each point of K is an interior point of some finite subcomplex,

d- each point of K has a neighborhood which meets only

Proof; a) --> b) If C is a compact neighborhood of x then by lemma 8, C meets only finitely many open cells, $\sigma - \dot{\sigma}$. The union of these σ's must contain C and is hence a neighborhood of x. b)--> c) Immediate from lemma 12.

c)--> a) A finite somplex is a finite union of compact sets and hence compact.

a)--> d) If U is an open neighborhood of x such that U^- is compact, then U^- meets only finitely many open cells, and if U meets a closed cell σ, then it meets $\sigma-\dot{\sigma}$ and hence U^- meets $\sigma-\dot{\sigma}$, and thus U meets only finitely many closed cells.

d)--> b) Obvious.

If K satisfies the conditions of the previous theorem it is called locally finite.

14. Theorem- If K and L are CW complexes and L is locally finite, then KxL is homeomorphic to Kx_cL and hence KxL is a CW-complex.

Proof: By lemma 6, KxL has the weak topology on $\{\sigma xL : \sigma$ a cell of K$\}$. But by another application of lemma 6, since each σ is compact, each σxL has the weak topology on $\{\sigma x\sigma' : \sigma'$ a cell of L$\}$. Hence, KxL has the weak topology on $\{\sigma x\sigma'\}$.

15. Corollary- If K is a CW complex and I is the unit interval, then KxI is a CW complex.

Just as CW complexes have certain preferred subspaces, namely subcomplexes, there are also certain preferred maps, called cellular maps. A continuous map f : K --> L is called two cellular maps is a cellular map. We will see many examples of why cellular maps are interesting. As one example, we have:

16. Theorem- If $f : K' \dashrightarrow L$ is a cellular map and K' is a subcomplex of K, then $K \cup_f L$ is a CW complex.

Proof: We will define inductively $(K \cup_f L)^n$ so that it equals $K^n \cup_{f|K'^n} L^n$. For $n = 0$, let $(K \cup_f L)^0 = (K^0 - K'^0) \cup L^0$. For n, we let $I^n_{K \cup_f L} = (I^n_K - I^n_{K'}) \cup I^m_L$ and if $j_{n-1} : K^{n-1} \cup L^{n-1} \dashrightarrow K^{n-1} \cup_{f|K'^{n-1}} L^{n-1}$ then let $f^{K \cup_f L}_\sigma = jf_\sigma$ for each σ in the index set. And for each σ in this index set, $\sigma - \dot\sigma$ is involved in no, non-trivial identifications. We now have to prove homeomorphic the two spaces: $K^n \cup_{f|K'^n} L^n$ and $(K \cup_f L)^n = (\text{cells}) \cup K^{n-1} \cup_{f|K'^{n-1}} L^{n-1}$. There are obvious maps of K^n and L^n into $(K \cup_f L)^n$, which factor through the adjunction space. The inverse map is the identity on the $n-1$ skeleton, and is j_n on each of the attached cells.

It remains to prove that $K \cup_f L$ has the weak topology on the skeletons. But if j is the projection from $K \cup L$, then if A is a subset of $K \cup_f L$ which intersects $(K \cup_f L)^n$ in a closed set for each n, then $j^{-1}(A) \cap (K^n \cup L^n) = j^{-1}(A \cap (K \cup_f L)^n) \cap (K^n \cup L^n) = j_n^{-1}(A \cap (K \cup_f L)^n)$, which is closed in $K^n \cup L^n$ since j_n is continuous there. Hence $j^{-1}(A)$ is closed in $K \cup L$ and hence A is closed in $K \cup_f L$.

17. Corollary- If $f : K \dashrightarrow L$ is a cellular map then the mapping cylinder M_f and the mapping cone $\hat{M}_f$ are CW complexes.

Proof: By corollary 15, KxI is a CW complex. We note that Kx0 and Kx1 are subcomplexes. Since f is cellular, the mapping $f' : K\times 1 \longrightarrow L$, by $f'(x,1) = f(x)$ is also. If we take a CW complex consisting of a single zero-cell, e^0, then the mapping of $K\times 0 \xrightarrow{c} e^0$ is also cellular. $M_f = K\times I \cup_{f'} L$ is a CW complex by theorem 16, and $\tilde{M}_f = M_f \cup_c e^0$ is also a CW complex by theorem 16.

We note that if K'' and L' are subcomplexes of K and L in Theorem 16, then the map $j|K'' \cup L'$ is a cellular map. We also note that the image of a cellular map need not be a subcomplex of the range complex:

A cellular map of S^1 into itself.

We call a map $h : K' \longrightarrow K$ a subdivision of K if h is a homeomorphism and for each open cell $\sigma' - \dot{\sigma}'$ of K', $h(\sigma' - \dot{\sigma}')$ is contained in some, necessarily unique, open cell of K, $\sigma - \dot{\sigma}$. Since h is a homeomorphism, it follows that dimension $\sigma' \leqq$ dimension σ. From this we can prove that $h^{-1} : K \longrightarrow K'$ is a cellular map, by induction on the skeletons. For $K^{-1} = \emptyset$, the result is trivial. If σ is a cell of K^n, then by inductive hypothesis $h^{-1}(\dot{\sigma})$ is in K'^{n-1} and so it suffices to show that $h^{-1}(\sigma - \dot{\sigma})$ is in K'^n. If not then this set meets some $\sigma'^m - \dot{\sigma}'^m$ for $m > n$ and hence $h(\sigma'^m - \dot{\sigma}'^m)$ meets and is thus contained in $\sigma - \dot{\sigma}$, which is impossible if $m > n$. This works for the case $n = 0$

An application of the use of subdivisions is given in the following result.

18. Theorem- A CW complex is locally contractible and hence locally pathwise connected.

Proof (suggested by D. Stone): By theorem 11, if x is any point of the complex K, it suffices to find a subdivision of K such that x is a zero-cell of the subdivision. [cf. Theorem 11] This we prove by induction on the dimension of the open cell $\sigma - \dot{\sigma}$, containing x. Let the dimension of $\sigma = n$.

$n=0$: x is already a zero-cell.

$n=1$: Let $K'^0 = K^0 \cup x$ and let $I^1_{K'} = (I^1_K - \sigma) \cup \{\sigma_1, \sigma_2\}$, where $f_{\sigma_1}(0) = f_\sigma(0)$ and $f_{\sigma_1}(1) = x$, and $f_{\sigma_2}(0) = x$ and $f_{\sigma_2}(1) = f_\sigma(1)$. The map $h : K'^1 \to K^1$ is obvious. We shall show that this suffices.

Assuming the result for $n-1$, we prove it for n (> 1): It suffices to find a subdivision for K^n with the required property, since whenever $h : K' \to K^n$ is a subdivision, we can always extend this to a subdivision $h' : K'' \to K^{n+1}$ where $K''^n = K'$, by using the same index set I^{n+1} and for each σ, using the attaching map $h^{-1} f_\sigma$. Let $y = i_\sigma^{-1}(x) \in e^n - S^{n-1}$ and let P be a line joining y to some point of S^{n-1} $\ni P(1) \in S^{n-1}$. Since the composition of two subdivisions is a subdivision, the inductive hypothesis implies that we can assume that $f_\sigma(P(1))$ is a zero-cell of K. Let $K'^0 = K^0 \cup x$ and $K'^1 = K^1 \cup i_\sigma P$, and so on so that $K'^{n-1} = K^{n-1} \cup i_\sigma P$. Let S^{n-1} be given a metric d, so that the distance from the point $P(1)$, considered as the south pole, to each point of the equator is 1, and the distance from any point of E^{n-1}_- to $P(1)$ is less than 1. Let $k : S^{n-1} \to S^{n-1} \cup P$, be a map is just a relative homeomorphism of e_+^{n-1} mod the equator, onto S^{n-1} mod $P(1)$. This extends to a relative homeomorphism, $k : (e^n, S^{n-1}) \longrightarrow (e^n, S^{n-1} \cup P)$. Attach all of the n cells to K' just as they were attached to K except for σ. For σ we substitute a new n-cell σ' which is attached by the composite map: $S^{n-1} \xrightarrow{k} S^{n-1} \cup P \xrightarrow{f_\sigma \cup i_\sigma} K'^{n-1} = K^{n-1} \cup i_\sigma P$. The only remaining question is how to define h on $\sigma' - \dot{\sigma}'$, since its definition is immediate anywhere else in K'^n. On $\sigma' - \dot{\sigma}'$ we define h as $i_\sigma k\, i_{\sigma'}^{-1}$. Since $f_{\sigma'}$ is by definition the restriction of $i_\sigma k$ this definition is consistent with h on the rest of K'^n.

19. Theorem (Homotopy Extension Theorem)* Let K be a CW complex, and L a subcomplex. If $f : K \longrightarrow X$ is a map into some topological space and $h : L \times I \longrightarrow X$ is a homotopy such that $h_0 = f|L$, then there exists a homotopy $H : K \times I \longrightarrow X$ such that $H|L \times I = h$ and $H_0 = f$.

Proof: While most of the following proofs can be done, as before, by induction on the skeletons, we will use instead the slightly faster method of Zorn's lemma. We note that the union of a chain of subcomplexes of a given complex is also a subcomplex, under ordering by inclusion, and that the "union" of maps which agree under restrictions is continuous since we are dealing with the weak topology on cells.

Let L" be a maximal subcomplex of K containing L such that h extends to H" having the required properties on L".

We assert that L'' is in fact K, for if it were not, then let σ be a cell of minimum dimension which is not in L''. Let $L' = L'' \cup \sigma = L'' \cup_{f_\sigma} e^n$ where $n = \dim \sigma$. By lemma 9, this reduces the problem of extending the homotopy H'' to the problem of extending the homotopy $H''(f_\sigma x I)$: $S^{n-1} x I \longrightarrow X$ to a homotopy on e^n, H, such that $H_0 = fi_\sigma$. Thus, we have reduced the problem to the homotopy extension theorem for simplicial complexes. (See Hu p. 14, or Hilton and Wylie p. 33.)

An example of the uses of the homotopy extension theorem (though the following is actually just the simplicial version).

20. Lemma- $f : S^{n-1} \longrightarrow X$ extends to a map of $e^n \longrightarrow X$ iff f is homotopic to a constant map.

Proof: If $H : S^{n-1} x I \longrightarrow X$ is the homotopy of f with the constant map c, let c' be an extension of the constant map. By the homotopy extension theorem H_1 is an extension of f. Conversely, if f extends to $f' : e^n \longrightarrow X$ then, using vector space notation, $H(x,t) = f'(tx)$ is a homotopy of f to a constant map.

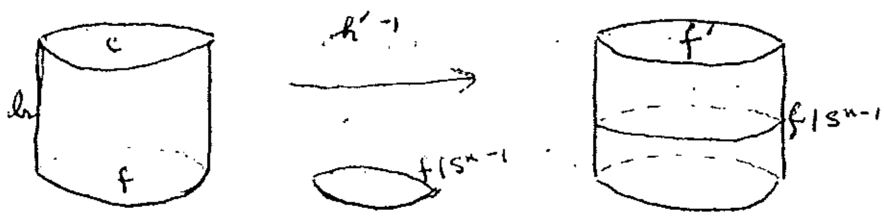
Finally, we will prove the cellular approximation theorem and the Whitehead characterization of homotopy equivalences. Preliminary to these two results we require certain lemmas.

21. Lemma- Let $f : (e^n, S^{n-1}) \longrightarrow (X, X')$ where $\pi_n(X,X') = 0$, $n \geq 1$, then f is homotopic, rel S^{n-1}, to a map into X'.

Proof: Pick a base point s_0 in S^{n-1} and let $x_0 = f(s_0)$. Since $\pi_n(X,X',x_0) = 0$, it follows that there exists a homotopy $h_t : (e^n,S^{n-1},s_0) \longrightarrow (X,X',x_0)$ such that $h_0 = f$ and $h_1 = c$, the constant map of e^n into x_0. However, we are looking for a homotopy which keeps S^{n-1} fixed. We proceed in two steps: first, we will define our candidate $f' : e^n \longrightarrow X'$, which is the required extension of $f|S^{n-1}$, and then we will find a homotopy, rel S^{n-1}, of f and f'.

$h|S^{n-1}xI \cup e^n x1$ has image in X' and hence by the homotopy extension theorem, this extends to a homotopy h' mapping $e^n xI \longrightarrow X'$. Let $f' = h'_0$.

To define the required homotopy we define a homotopy of the "cylinder" $e^n xI$ as shown in the diagram:

H : 

ie. H is defined on $(e^n xI)x0 \cup (S^{n-1}xI \cup e^n x1)xI$ as follows: $H(x,t,0) = h(x,t)$, $H(x,1,s) = h'(x,1-s)$ for $x \in e^n$ and $H(x,t,s) = h(x,t(1-s)) = h'(x,t(1-s))$ for $x \in S^{n-1}$. Note that if $t = 0$ or $s = 1$, $H_{(t,s)}|S^{n-1} = f|S^{n-1}$. By another application of the homotopy extension theorem we extend H to a map of $(e^n xI)xI \longrightarrow X$. Now we read off the homotopy of f and f' by "going across the "bottom" and up the right cylinder". The required homotopy is h" defined:

$$h''(x,t) = \begin{cases} H(x,0,2t) & 0 \leq t \leq \frac{1}{2} \\ H(x,2t-1,1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

22. Theorem- Let K be a CW complex, then if $m \geqq n \geqq 1$ $\pi_n(K,K^m) = 0$.

Proof: First, we assume that we have proved that $\pi_n(K^{m+1},K^m) = 0$ for all $m \geqq n \geqq 1$. By the homotopy sequence of the pair, (K^{m+1},K^m), we get that $\pi_n(K^m) \xrightarrow{i_\#} \pi_n(K^{m+1})$ is an onto map, and that $\pi_{n-1}(K^m) \xrightarrow{i_\#} \pi_{n-1}(K^{m+1})$ has kernel zero, for m and n as above. We prove inductively that $\pi_n(K^m) \longrightarrow \pi_n(K^{m+r})$ is onto and $\pi_{n-1}(K^m) \longrightarrow \pi_{n-1}(K^{m+r})$ has kernel zero, for $m \geqq n \geqq 1$ and $r \geqq 1$. This is immediate since the two maps factor into : $\pi_n(K^m) \longrightarrow \pi_n(K^{m+r-1}) \longrightarrow \pi_n(K^{m+r})$ and $\pi_{n-1}(K^m) \longrightarrow \pi_{n-1}(K^{m+r-1}) \longrightarrow \pi_{n-1}(K^{m+r})$, in each case the first map is the inductive hypotheses map and the second is the case $r = 1$, which we know. We then have, by the homotopy sequence of the pair (K^{m+r},K^m) that $\pi_n(K^{m+r},K^m) = 0$. The proof from this that $\pi_n(K,K^m) = 0$ is just a special case of the proof that π_n preserves weak direct limits; thus, if $f : (e^n,S^{n-1},s_o) \longrightarrow (K,K^m,k_o)$, then because $f(e^n)$ is compact, lemma 8, implies that f represents an element of $\pi_n(K^{m+r},K^m,k_o)$, for some $r \geqq 1$, and since cls f is zero in this group, it is zero in $\pi_n(K,K^m,k_o)$.

To prove that $\pi_n(K^{m+1},K^m) = 0$: let $f : (e^n,S^{n-1},s_o) \longrightarrow (K^{m+1},K^m,k_o)$ represent a typical element of $\pi_n(K^{m+1},K^m,k_o)$. By lemma 8, since $f(e^n)$ is compact, the image of f is contained in $(\bigcup e^{m+1}_{\sigma_i}) \cup \bigcup f_{\sigma_i} K^m$ for $\{\sigma_i\}$ a finite subcollection of the index set of K : I^{m+1}. We will show that f can be pushed off each $e^{m+1}_{\sigma_i}$, ie. f is homotopic (rel S^{n-1}) to a map into $K^{m+1} - i_{\sigma_i}(\text{Int } e^{m+1}_{\sigma_i})$, and is consequently homotopic to a representative of $\pi_n(K^m,K^m)$ which is zero. By induction on the number of cells in $\{\sigma_i\}$ we reduce the problem to pushing f off of some $e^{m+1}_{\sigma_o}$. Assume that $i_{\sigma_o}^{-1}(\text{Image } f)$ does not contain the center of the cell $e^{m+1}_{\sigma_o}$. Then if $r : e^{m+1}_{\sigma_o} - \text{center} \longrightarrow S^{m}_{\sigma_o}$ is radial projection, ~~then~~ we can define a homotopy r_t deforming $K^{m+1} - i_{\sigma_o}(\text{center})$ onto $K^{m+1} - (\sigma_o - \dot{\sigma}_o)$ rel $K^{m+1} - (\sigma_o - \dot{\sigma}_o)$, by defining r_t on $\sigma_o - (\dot{\sigma}_o \cup \text{center})$ to be $r_t(x) = i_{\sigma_o}(t\,i_{\sigma_o}^{-1}(x) + (1-t)r(i_{\sigma_o}^{-1}(x)))$. So finally we are left with the question of how to make sure that f does not hit $i_{\sigma_o}(\text{center})$. Consider (E,S) a closed m+1 cell contained in the interior of $e^{m+1}_{\sigma_o}$ and containing the center in its interior. Let $U \subset e^n$ be defined as $U = f^{-1}(i_{\sigma_o}(E-S))$ and $X = \bar{U}$ with $A = \bar{U} - U$ = boundary U. $i_{\sigma_o}^{-1} f : (X,A) \longrightarrow (E,S)$. The space A being the boundary of an open set in e^n has dimension n-1 (Hurewicz and Wallman page 46), while S is a sphere of dimension m which is > n-1, by hypothesis. Hence the map $i_{\sigma_o}^{-1} f | A$ mapping A into S has an extension g mapping X into S, by the Hopf extension theorem (Hurewicz and Wallman page 146). $f|X$ is homotopic to $g|X$ rel A, by the homotopy $h_t(x) = i_{\sigma_o}(tg(x) + (1-t)i_{\sigma_o}^{-1} f(x))$. Because this homotopy is constant on A, it extends to a homotopy of f with a map which agrees with f except on X, where it dquals g. Hence, we have succeeded in pushing f away from the center and can consequently push f entirely off the cell σ_o. Proceeding inductively we obtain a homotopy of f with a map which represents an element of $\pi_n(K^m,K^m)$, as promised.

The above proof is somewhat streamlined by the use of the Hopf extension theorem. Its ...

Motivated by the preceding theorem we define, for any topological space X, a skeletal decomposition of X as an ascending sequence of subspaces: $X^0 \subset X^1 \subset \ldots \subset X^n \subset \ldots$ with the property, that if $n \leq m$, then $\pi_n(X,X^m) = 0$ (or equivalently by the relative Hurewicz theorem, $H_n(X,X^m) = 0$ (singular homology)) ($\pi_0(K,K^0) = 0$ is interpreted to mean $\pi_0(K^0) \xrightarrow{0} \pi_0(X)$ is epi.). Note that we do not demand that $X = X^n$. Theorem 22 states that for any CW complex the sequence, $K^0 \subset K^1 \subset \ldots \subset K^n \subset \ldots$ is a skeletal decomposition of K. We call a map f of a CW complex into a space with a given skeletal decomposition, cellular if $f(K^n) \subset X^n$. With these definitions we can prove the following general form of the cellular approximation theorem:

23. Theorem- Let K be a CW complex with subcomplex L, and let X be a space with a given skeletal decomposition $\{X^n\}$, and $f : K \to X$ be a continuous map such that the restriction $f|L$ is cellular. There exists a map g which is cellular and is such that f is homotopic to g, rel L.

Proof: Let $f^0 = f$. We will construct inductively f^{i+1} such that $f^{i+1}|K^i \cup L$ is cellular and h^i_t a homotopy rel $K^{i-1} \cup L$ of f^i with f^{i+1}.

By the homotopy extension theorem, it suffices, given f^i to construct h^i_t on $K^i \cup L^0$. The construction of h^0_t is obvious from the definition we have given to the hypothesis that $\pi_0(K,X^0) = 0$. Assuming that f^i is defined we proceed to define h^i_t. By lemma 9, it suffices to prove that for each σ of dimension i, the map $f^i i_\sigma : (e^i, S^{i-1}) \to (X, X^{i-1})$ is homotopic rel S^{i-1} to a map into X^i. But since $X^{i-1} \subset X^i$, we have that [illegible]

$\pi_i(X,X^i) = 0$ and hence the result follows from lemma 21.

We now define $g : K \dashrightarrow X$ so that $g\, K^{i-1} = f^i$. g is continuous because we have the weak topology on skeletons. Assuming that $h_0^i = f^{i+1}$ and $h_1^i = f^i$, then we define h_t so that $h_t(x) = h^i_{2^{i+1}t-1}(x)$ for $1/2^{i+1} \leq t \leq 1/2^i$ and $h_0(x) = g(x)$. Since on each skeleton these homotopies are eventually constant, the fact that we are dealing with the weak topology implies the continuity of h.

This result has two important corollaries:

24. Corollary (Cellular Approximation Theorem)- Let $f : K \dashrightarrow K'$ be a continuous map of cell complexes and L be a subcomplex of K such that $f|L$ is cellular, then there exists a cellular map g which is homotopic to f, rel L.

Proof: As we have seen, Theorem 22 implies that the skeletons of K' form a skeletal decomposition of K', with the result by theorem 23.

25. Corollary- Let K be a CW complex, with L a subcomplex and let X' be a subspace of a topological space X such that $\pi_i(X,X') = 0 \quad 0 \leq i \leq n \leq \infty$, and $f : K \dashrightarrow X$ be a continuous map such that $f(L^n) \subset X'$. Then there exists a map g such that $g(K^n) \subset X'$ and g is homotopic to f, rel L.

Proof: Letting $X^i = X'$ for $i \leq n$ and $X^i = X$ for $i > n$ we obtain a skeletal decomposition of X (in particular, if $n = \infty$, let $X^i = X'$ for all i and the requirement is that $f(L) \subset X'$ with the result that $g(K) \subset X'$), with the result from theorem 23.

Using this corollary we will prove the Whitehead Theorem. First some preliminaries.

We shall say that a space A dominates a space X if there exist maps μ and μ' : $A \underset{\mu}{\overset{\mu'}{\rightleftarrows}} X$ so that $\mu'\mu$ is homotopic to 1_X. Similarly, we shall say that the pair (A,A') dominates the pair (X,X') if there exist maps: $(A,A') \underset{\mu}{\overset{\mu'}{\rightleftarrows}} (X,X')$ such that $\mu'\mu$ is homotopic (by a homotopy of maps of pairs) to $1_{(X,X')}$. We shall be particularly interested in spaces which are dominated by CW complexes.

The key to the proof of the Whitehead theorem is the use of mapping cylinders, for which we need one more lemma.

26. Lemma- If we have a diagram:

$$\begin{tikzcd} P \arrow[d, bend right, "\lambda'"'] \arrow[rr, "\mu f \lambda'"] & & Q \arrow[d, bend right, "\mu'"'] \\ X \arrow[u, bend right, "\lambda"'] \arrow[rr, "f"] & & Y \arrow[u, bend right, "\mu"'] \end{tikzcd} \qquad \mu'\mu \sim 1_Y, \ \lambda'\lambda \sim 1_X$$

then the pair (M_f, X) is dominated by the pair $(M_{\mu f \lambda'}, P)$ and the maps restricted to X and P are λ and λ'. (Recall that X is identified with $X \times 0$ in $M_f = X \times I \cup_{f'} Y$, with $f'(x,1) = f(x)$.)

Proof (Following J.H.C. Whitehead (2)): Let $\xi_t : X \longrightarrow X$ and $\eta_t : Y \longrightarrow Y$ be homotopies such that $\xi_0 = \lambda'\lambda$, $\xi_1 = 1_X$ and $\eta_0 = \mu'\mu$, $\eta_1 = 1_Y$. We will just give the maps and homotopy and leave the verification that they are single-valued to the reader. This is an easy exercise and implies continuity since we are dealing with identification topologies.

Let $\nu : (M_f, X) \dashrightarrow (M_{\mu f \lambda'}, P)$ be given by :

$$\nu(x,t) = (\lambda x, 2t) \qquad 0 \leqq t \leqq \tfrac{1}{2}$$
$$= \mu f \xi_{2t-1} x \qquad \tfrac{1}{2} \leqq t \leqq 1$$
$$\nu y = \mu y \qquad x \in X,\ y \in Y$$

Let $\nu' : (M_{\mu f \lambda'}, P) \dashrightarrow (M_f, X)$ be given by :

$$\nu(p,t) = (\lambda' p, 2t) \qquad 0 \leqq t \leqq \tfrac{1}{2}$$
$$= \eta_{2-2t} f \lambda' p \qquad \tfrac{1}{2} \leqq t \leqq 1$$
$$\nu' q = \mu' q \qquad p \in P,\ q \in Q.$$

The map $\nu'\nu : (M_f, X) \longrightarrow (M_f, X)$ is then given by :

$$\nu'\nu(x,t) = (\lambda'\lambda x, 4t) \qquad 0 \leqq t \leqq \tfrac{1}{4}$$
$$= \eta_{2-4t} f \lambda'\lambda x \qquad \tfrac{1}{4} \leqq t \leqq \tfrac{1}{2}$$
$$= \mu'\mu f \xi_{2t-1} x \qquad \tfrac{1}{2} \leqq t \leqq 1$$
$$\nu'\nu y = \mu'\mu y$$

Let $p(s,t) = \frac{1}{2}((4-3s)t + 3s - 2)$, then the homotopy ζ_s is given by:

$$\zeta_s(x,t) = (\xi_s x, (4-3s)t) \qquad 0 \leqq t \leqq 1/(4-3s)$$
$$= \eta_{2-(4-3s)t} f \xi_s x \qquad 1/(4-3s) \leqq t \leqq (2-s)/(4-3s)$$
$$= \eta_s f \xi_{p(s,t)} x \qquad (2-s)/(4-3s) \leqq t \leqq 1$$
$$\zeta_s y = \eta_s y$$

One sees by inspection that $\zeta_0 = 1_{M_f}$ and $\zeta_1 = \nu'\nu$. Furthermore, we note that $\zeta_s | X = \xi_s$ and $\zeta_s | Y = \eta_s$. Thus, we actually have that the triple $(M_{\mu f \lambda'}, P, Q)$ (or triad since there is no inclusion) dominates the triple (M_f, X, Y).

27. Theorem (J.H.C. Whitehead)- If $f : X \longrightarrow Y$ is a continuous mapping, with X and Y pathwise connected dominated by CW complexes K and L, respectively. If, letting $n = \max(\text{dimension } K, \text{dimension } L)$, $f_* : \pi_i(X) \longrightarrow \pi_i(Y)$ is an isomorphism for all $i : 1 \leqq i < n+1$, then f is a homotopy equivalence.

Note: If $n = \infty$, the hypothesis requires that f_* be an isomorphism on all the homotopy groups.

Proof: If M_f is the mapping cylinder of f, $i : X \longrightarrow M_f$ the inclusion of X as $X\times 0$, and $p : M_f \longrightarrow Y$ be the projection $p(x,t) = f(x)$ and $p(y) = y$ for $x \in X$ and $y \in Y$, then the following diagram commutes up to homotopy:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & M_f \arrow[d, "p"] \\ & Y \end{tikzcd}$$

We also note that p is a homotopy equivalence, and hence to prove that f is a homotopy equivalence, it suffices to prove that i is a homotopy equivalence.

Assume that there is a homotopy $h_t : (M_f, X) \longrightarrow (M_f, X)$ such that $h_0 = 1$ and $h_1 M_f \subset X$. Let $h : M_f \longrightarrow X$ be given by h_1, then $ih = h_1 : M_f \longrightarrow M_f$ is homotopic to 1_{M_f} by h_t, but hi is homotopic to 1_X by $k_t = h_t|X$. We need only prove the existence of such a homotopy.

Since $f_* = p_* i_*$, by the above diagram, and since p is a homotopy equivalence, our hypothesis implies that the map $i_* : \pi_m(X) \longrightarrow \pi_m(M_f)$ is an isomorphism for $0 \leqq m \leqq n+1$, and hence, from the exact homology sequence of the pair (M_f, X), it follows that $\pi_m(M_f, X) = 0$ for $1 \leq m < n+1$, and trivially we have $\pi_0(M_f, X) = 0$, according to the convention which defines it.

By hypothesis we have maps : $K \underset{\nu}{\overset{\mu'}{\rightleftarrows}} X$ and $L \underset{\lambda}{\overset{\lambda'}{\rightleftarrows}} Y$ with homotopies $\xi_t : X \longrightarrow X$ and $\gamma_t : Y \longrightarrow Y$ with $\xi_0 = \lambda'\lambda$, $\xi_1 = 1_X$, $\gamma_0 = \mu'\mu$, $\gamma_1 = 1_Y$. Furthermore, since the dimension $L \leq n$, we have, by corollary 25, a homotopy $\rho_t : L \longrightarrow M_f$ with $\rho_0 = j\mu'$, where $j : Y \longrightarrow Z$ is the inclusion map, and $\rho_1(L) \subset X$.

We now spend a paragraph pushing the hypotheses a little farther by proving, if $n < \infty$, that $\pi_{n+1}(M_f, X) = 0$. By the exact homology sequence, it suffices to prove that $\pi_{n+1}(X, x) \xrightarrow{i_*} \pi_{n+1}(M_f, x)$ is onto for any x. Using $\rho_t\mu$, we know that $\rho_1\mu \sim j\mu'\mu \sim j1_Y = j$, but then $\rho_1\mu p \sim jp = p \sim 1_{M_f}$. Thus, there exists a homotopy $\delta_t : M_f \longrightarrow M_f$ such that $\delta_0 = 1_{M_f}$ and $\delta_1(M_f) \subset X$. Let w be the path defined by $w(t) = \delta_t(x_0)$ and let $w(1) = x_1$. We assert that $\pi_{n+1}(X, x_1) \xrightarrow{i_*} \pi_{n+1}(M_f, x_1)$ composed with $\pi_{n+1}(M_f, x_1) \xrightarrow{w} \pi_{n+1}(M_f, x_0)$ is onto. For if α is a map representing an element of $\pi_{n+1}(M_f, x_0)$, then w^{-1} cls α is represented by $\delta_1\alpha$ which represents an element of $\pi_{n+1}(X, x_1)$. However, w is an isomorphism and hence i_*: $\pi_{n+1}(X, x_1) \longrightarrow \pi_{n+1}(M_f, x_1)$ is onto, and hence $\pi_{n+1}(M_f, X, x_1) = 0$, which, since X is arcwise connected is true for every base point.

Let $P = K \times I \cup L$ (disjoint union). This is a CW complex by corollary 15. Let $\phi : P \longrightarrow M_{\mu f \lambda'}$ be the identification map. (Note that $M_{\mu f \lambda'}$ need not be a CW complex since $\mu f \lambda'$ need not be cellular.) Let $\nu : (M_f, X) \longrightarrow (M_{\mu f \lambda'}, K)$, $\nu' : (M_{\mu f \lambda'}, K) \longrightarrow (M_f, X)$ and $\varphi_t : (M_f, X) \longrightarrow (M_f, X)$ be defined as in lemma 20.

Let $Q = K\times 0 \cup K\times 1 \cup L \subset P$ and define $\rho_t' : Q \longrightarrow M_f$ $\rho_t'(k,0) = \psi'\phi(k,0)$, $\rho_t'(k,1) = \rho_t\phi(k,1)$, and $\rho_t' k' = \rho_t\phi k'$ for $k \in K$ and $k' \in L$. Note that $\rho_0' = \nu'\phi|Q$, since $\rho_0 = j\mu' = \nu'|L$. Note also that $\rho_1'(Q) \subset X$. Letting $\psi_0 = \nu\phi : P \longrightarrow M_f$, the homotopy extension theorem implies that ρ_t' has an extension $\psi_t : P \longrightarrow M_f$. Since $\psi_1(Q) = \rho_1'(Q) \subset X$ and since the dimension of $P \leq n+1$, the fact that $\pi_m(M_f,X) = 0$ for $0 \leq m < n+2$ implies, by corollary 25, that there is a homotopy $\psi_t' : P \longrightarrow M_f$ (rel Q) such that $\psi_0' = \psi_1$ and $\psi_1'(P) \subset X$. Let $\theta_t : P \longrightarrow M_f$ be the resultant of ψ_t followed by ψ_t'. We assert that $\chi_t = \theta_t\phi^{-1} : M_{\mu f\lambda'} \longrightarrow M_f$ is well-defined. Since the only non-trivial identifications made by ϕ are on Q, it suffices to examine $\theta_t|Q$. $\psi_t'|Q = \psi_1|Q$ and so we are left with $\psi_t|Q = \rho_t'$. $\rho_t'\phi^{-1}$ is obviously well defined. Since the topology of $M_{\mu f\lambda'}$ is the identification topology induced by ϕ this implies that χ_t is continuous. Moreover $\chi_0 = \psi_0\phi^{-1} = \nu'\phi\phi^{-1} = \nu'$ and $\chi_1(M_{\mu f\lambda'}) = \psi_1'\phi^{-1}(M_{\mu f\lambda'}) \subset X$. Hence, $\chi_t\nu$ gives a homotopy of $\nu'\nu$ and a map into X, and thus the required homotopy h_t may be defined as the resultant of ζ_{1-t} followed by $\chi_t\nu$.

The construction of the homotopy h_t would have been greatly simplified if we assumed that X and Y were CW complexes and threw away K and L. Assuming $n = \max(\text{dimension } X, \text{dimension } Y)$. Then by corollary 24, we could assume f was cellular, in which case, by corollary 17, M_f is a CW complex, of dimension n+1, so as soon as we proved that $\pi_m(M_f,X) = 0$ $\quad 0 \leq m < n+2$, the existence of the required homotopy follows from corollary 25.

28. Corollary - If X is an arcwise connected space dominated by a CW complex of dimension n, and $\pi_i(X) = 0$, $1 \leq i < n+1$, then X is contractible.

Proof: The map of X into a point, ie. a zero cell, induces the required isomorphisms of homotopy and is hence a homotopy equivalence.

1. J.H.C. Whitehead - Combinatorial Homotopy I, Bull. Amer. Math. Soc. 55 (1949), 213-245.

2. J.H.C. Whitehead - On the Homotopy Type of ANR's, Bull. Amer. Math. Soc. 54 (1948), 1133-1145.

Example

1. Due to Dowker: Let K consist of a collection of closed one cells $\{A_i\}$ with the power of the continuum, with a common vertex u_e, and let L consist of a collection of closed one cells $\{B_j\}$ $j = 1,2,\ldots$ with a common vertex v_e. Let K and L have the weak topology on compact sets, ie. the Whitehead weak topology, then $K x L \neq K x_k L$.

Proof: Let A_i be indexed by sequences of integers not equal to 0, $i = \{i_1, i_2, \ldots\}$ and let A_i be parametrized by x_i with $x_i = 0$ implying $x_i = u_e$. Similarly, let B_j be parametrized by y_j, with v_e corresponding to $y_j = 0$. Let $p_{ij} = (1/i_j, 1/i_j)$ in $A_i x B_j$ and let $P = \{p_{ij}\}$. Since $P \cap A_i x B_j = \{p_{ij}\}$, P is closed in $K x_k L$. However, we assert that $(u_e, v_o) \in$ the closure of P in KxL. For a nbhd U of u_e is given by $\{x_i < a_i : a_i > 0\}$ and similarly V of v_e is given by $\{y_j < b_j : b_j > 0\}$. Then, let the sequence $\bar{I} = (\bar{I}_1, \bar{I}_2, \ldots)$ be chosen so that for each j, $\bar{I}_j > \max(j, 1/b_j)$ and let $\bar{J}$ be chosen so that $\bar{J} > 1/a_{\bar{I}}$. Then it is easily seen that $p_{\bar{I}\bar{J}} \in UxV$.

Appendix : Paracompactness of CW Complexes

The theorem of Morita (1) that a CW complex with the weak topology is a paracompact topological space, is an interesting exercise in point-set topology.

1. Lemma- If A is a closed subset of a paracompact space X, and $\mathcal{U}$ is a (relatively) open, locally finite cover of A, then there exists a locally finite collection of open sets $\{ V_U : U \in \mathcal{U} \}$ such that $V_U \cap A = U$.

Proof: By Kelley, chap. 5, lemma 31, there exists a neighborhood D, of the diagonal of XxX, such that the collection of open sets $\{ D[U] : U \in \mathcal{U} \}$ is locally finite. Let $V_U = D[U] \cap (U \cup X-A)$.

2. Lemma- If $\mathcal{A}$ is a collection of closed sets of a normal space X, and $\{ U_A : A \in \mathcal{A} \}$ is a locally finite collection of open sets, then there exists a collection $\{ G_A : A \in \mathcal{A} \}$ of open sets such that $A \subseteq G_A \subseteq G_A^- \subseteq U_A$, and if $G_{A_1}^- \cap \ldots \cap G_{A_m}^- \neq \emptyset$ for a finite subcollection of $\mathcal{A}$, then $A_1 \cap \ldots \cap A_m \neq \emptyset$.

Proof: For subcollections $\mathcal{W}$ of $\mathcal{A}$, let $G(\mathcal{W}) = \{ G_A : A \in \mathcal{A} \}$ which satisfy the required inclusion and intersection conditions. We apply Zorn's lemma to get a maximal such $G(\mathcal{W})$ and we assert that $\mathcal{W} = \mathcal{A}$. If not, then let $A \in \mathcal{A} - \mathcal{W}$. $\{ G_B^- : B \in \mathcal{W} \} \cup \{ B : B \in \mathcal{A} - \mathcal{W} \}$ is a locally finite collection of closed subsets of X, since the collection of open sets $\{U_A\}$ is locally finite. Then $\mathcal{F}$, the collection of intersections of finite subcollections is also locally finite. Hence, $H = U_A \cap (X - \bigcup\{F \in \mathcal{F} : F \cap A = \emptyset\})$ is an open neighborhood of A. Letting G_A be an open neighborhood of A, with $\overline{G_A} \subset H$ defines $G(\mathcal{W} \cup \{A\})$ which obviously has all the required properties.

3. Lemma (Morita)– If A is a closed subspace of a paracompact space X, and $\mathcal{U}$ is a (relatively) open, locally finite cover of A, with each element of $\mathcal{U}$ an F_σ, then there exists a locally finite collection of open F_σ's $\{V_U : U \in \mathcal{U}\}$ such that $V_U \cap A = U$ and if $V_{U_1} \cap \ldots \cap V_{U_m} \neq \emptyset$ for a finite subcollection of $\mathcal{U}$, then $U_1 \cap \ldots \cap U_m \neq \emptyset$.

Proof: By hypothesis, each $U = \bigcup_i F_U^i$ with the F_U^i closed. Let $\{\tilde{V}_U : U \in \mathcal{U}\}$ be the locally finite collection of lemma 1. Let $G_{\bullet,U} = \emptyset$. Then, inductively, we define, using lemma 2, $G_{i,U}$ as an open subset of X such that $\overline{G}_{i-1,U} \cup F_U^i \subset G_{i,U} \subset \overline{G}_{i,U} \subset \tilde{V}_U$. Let $V_U = \bigcup_{i=1}^{\infty} G_{i,U} = \bigcup_{i=1}^{\infty} \overline{G}_{i,U}$.

4. Theorem–– Assume that a regular normal space X is the weak union of an increasing sequence F_n of closed paracompact subspaces. Then X is paracompact.

Proof: Let $\mathcal{U}$ be an open cover of X, we will construct an open σ-locally finite refinement, with the result by Kelley chap. 5, theorem 28.

Since the space is completely regular, we can assume that the elements of $\mathcal{U}$ are F_σ's, by going to a refinement if necessary. Let $\mathcal{V}_n$ be a (relatively) open locally finite refinement by F_σ's of $\{U \cap F_n : U \in \mathcal{U}\}$. We will construct a locally finite collection of open sets, $\tilde{\mathcal{V}}_n$ of X, refining $\mathcal{U}$, and such that $\{V \cap F_n : V \in \tilde{\mathcal{V}}_n\} = \mathcal{V}_n$.

Consider $\mathcal{V}$ (since we are dealing with a fixed $\mathcal{V}_n$, we drop the subscript). We will construct simultaneously, $\tilde{\mathcal{V}}$ and a sequence $\mathcal{W}_r$, with each $\mathcal{W}_r$ an open cover of F_{n+r}, which "tests" the local finiteness of $\tilde{\mathcal{V}}$ there, ie. each memeber of $\mathcal{W}_r$ intersects only finitely many members of $\tilde{\mathcal{V}}$.

Let $\mathcal{V}^0 = \mathcal{V}$, and let $\mathcal{W}_0^0$ be a locally finite, relatively open cover of F_n, by F_σ's, each member of which intersects at most finitely many members of $\mathcal{V}^0$. We can take the union of the two collections and apply lemma 3, $X = F_{n+1}$, and separate out again to define locally finite collections of open (in F_{n+1}) F_σ's $\mathcal{V}^1$ and $\mathcal{W}_0^1$ such that $\mathcal{V}^1 \cap F_n = \mathcal{V}^0$ and $\mathcal{W}_0^1 \cap F_n = \mathcal{W}_0^0$ (where, for a collection $\mathcal{A}$ of sets we define $\mathcal{A} \cap A = \{ B \cap A : B \in \mathcal{A} \}$). Furthermore, by the second condition of lemma 3, an element of $\mathcal{W}_0^1$ intersects an element of $\mathcal{V}^1$ iff it intersects the corresponding element of $\mathcal{V}^0$.

Thus by induction, we construct collections $\mathcal{V}^r$ and $\mathcal{W}_s^r$ $(s \leq r)$ of sets which are open F_σ's of F_{n+r} and such that $\mathcal{V}^r \cap F_{n+r-1} = \mathcal{V}^{r-1}$ and $\mathcal{W}_s^r \cap F_{n+r-1} = \mathcal{W}_s^{r-1}$ $(s < r)$ and such that each element of $\mathcal{W}_s^r$ intersects an element of $\mathcal{V}^r$ iff it intersects the corresponding element of $\mathcal{V}^{r-1}$, $(s < r)$, and, finally, such that each element of $\mathcal{W}_r^r$ intersects only finitely many members of $\mathcal{V}^r$.

Thus for $V \in \mathcal{V}$, there is a sequence $\{V^r\}$ with $V^r \in \mathcal{V}^r$ open in F_{n+r}, and $V^r \cap F_{n+r-1} = V^{r-1}$. So let $\tilde{V} = (\bigcup_r V^r) \cap U_V$, where $U_V \in \mathcal{U}$, which contains V. Since $\tilde{V} \cap F_{n+r} = V^r \cap U_V \cap F_{n+r}$ and is hence open in F_{n+r}, $\tilde{V}$ is open in X since X has the weak topology on the union. Let $\tilde{\mathcal{V}} = \{ \tilde{V} : V \in \mathcal{V} \}$.

To prove that $\mathcal{V}$ is locally finite, we use the test collections. If $x \in F_{n+r}$, then let $x \in W^0 \in \mathcal{W}^r_r$. There is a sequence $\{W^i\}$ with $W^i \in \mathcal{W}^{r+i}_r$, and $W^i \cap F_{n+r+i-1} = W^{i-1}$. Let $\hat{W} = \bigcup_i W^i$, and as above, $\hat{W}$ is open in X, and is thus an open neighborhood of x, and if $\hat{W} \cap \hat{V} \neq \emptyset$, then for some i, sufficiently large, $W^i \cap V^{r+i} \neq \emptyset$, which by construction implies that $W^0 \cap V^r \neq \emptyset$, and this is only the case for a finite number of V's.

This completes the proof of the first keystone of of the result we want. For the second, we consider $f : A \dashrightarrow Y$ with A a closed subspace of X, and we form the adjunction space $X \cup_f Y$, by the obvious quotient map $\phi : X \cup Y \dashrightarrow X \cup_f Y$.

5. Theorem - If X and Y are paracompact, then $X \cup_f Y$ is paracompact. Y is completely regular or paracompact.

Proof: complete regularity : For a point x in $X \cup_f Y$ and a disjoint closed set B, we have two cases: i) $x \in \phi(X-A)$. In this case we find a continuous function $g : X \dashrightarrow I$, which is 1 at x and is 0 on $A \cup (\phi^{-1}(B) \cap X)$, by complete regularity of X. The map $X \cup Y \dashrightarrow I$, which is g on X and 0 on Y is obviously consistent with the identification of ϕ and hence defines the required map. ii) $x \in \phi(Y)$. Define $g : Y \dashrightarrow I$ which is 1 at x and 0 on $\phi^{-1}(B) \cap Y$. On the closed subset $A \cup (\phi^{-1}(B) \cap X)$ we define the continuous function to I, which is gf on A and 0 on $\phi^{-1}(B) \cap X$. By the Tietze extension theorem this extends to a continuous function $g' : X \dashrightarrow I$ and the map $X \cup Y \dashrightarrow I$ which is g' on X and g on Y is obviously consistent with the identification of ϕ and hence defines the required map.

This proves that $X \cup_f Y$ is completely regular and, a fortiori, regular.

Now given an open cover $\mathcal{U}$ of $X \cup_f Y$. Identify Y with the closed subset of $X \cup_f Y$, $\phi(Y)$. $\mathcal{U} \cap Y$ has an open (in Y), locally finite refinnment $\mathcal{V}$, with each element of $\mathcal{V}$ an F_σ, since Y is paracompact. $f^{-1}\mathcal{V} = \phi^{-1}\mathcal{V}$, is a (relatively) open, locally finite cover of A, by F_σ's. Let $\mathcal{W}$ be an open, locally finite cover of Y, by F_σ's such that each member of $\mathcal{W}$ meets only a finite number of elements of $\mathcal{V}$. By lemma 3, we can find locally finite collections of open sets $\mathcal{V}'$ and $\mathcal{W}'$, of X, such that $\mathcal{V}' \cap A = f^{-1}\mathcal{V}$ and $\mathcal{W}' \cap A = f^{-1}\mathcal{W}$ and an element of $\mathcal{W}'$ meets an element of $\mathcal{V}'$ iff the corresponding elements of $\mathcal{V}$ and $\mathcal{W}$ meet, and we may obviously assume that $\mathcal{V}'$ refines $\phi^{-1}\mathcal{U}$. For $V \in \mathcal{V}$, or $W \in \mathcal{W}$, assume that $V' \in \mathcal{V}'$ or $W' \in \mathcal{W}'$ such that $V' \cap A = f^{-1}(V)$ and $W' \cap A = f^{-1}(W)$, and then let $\hat{V} = \phi(V' \cup \hat{V}) \cap U_V$ for $V \subset U_V$ and $\hat{W} = \phi(W' \cup \hat{W})$. $\{\hat{V} : V \in \mathcal{V}\}$ is an open collection in $X \cup_f Y$ and we see that it is locally finite by testing with the $\hat{W}$'s, finally, this collection refines $\mathcal{U}$, and covers $\phi(Y)$. $B = X - \cup \phi^{-1}\mathcal{X}$ is a closed subset of X disjoint from A. By paracompactness of X, we can find a locally finite collection $\mathcal{Y}$ of open sets of X, which i) covers B, ii) are each contained in X-A, iii) refines $\phi^{-1}\mathcal{U}$. $\mathcal{Y} \cup \mathcal{X}$ is the required locally finite refinement of $\mathcal{U}$.

8. Corollary– Every CW complex is paracompact in the weak topology.

Proof: Let K be a CW complex. K^0 is discrete and hence paracompact. K^n is an adjunction space of K^{n-1} with a disjoint union of n-cells and hence K^n [illegible]

It is possible to strengthen this corollary somewhat:

7. Theorem - Every subspace of a CW complex is paracompact.

Proof: It is easily seen that it suffices to prove this for open subspaces. If U is an open subspace of K, then since K is perfectly normal (I. theorem 10), U is an F_σ, and is hence an union of an ascending sequence $\{F_n\}$ of closed subsets of K, and since K is normal we can assume that $F_n \subset \operatorname{Int} F_{n+1}$. It follows that U has the weak topology on the union and hence U is paracompact by theorem 4.

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1. K. Morita - "On Spaces Having the Weak Topology II" Proc. Japan Acad. 30 - 1954.

Notes to John Milnor's "On Spaces Having the Homotopy Type of a CW Complex" :

1. The class $\mathcal{W}_0$.

Prep. 1. – If $f : K \xrightarrow{\sim} K$, and L is the smallest subcomplex of K containing $f(A)$, then L is finite and dominates A.

Prep. 2. – That L is countable follows from the closure finiteness of K, together with the lemma that a Lindelöf subspace of a CW complex K meets only countably many open cells. The proof of this is analogous to the proof of the lemma in the compact – finite case.

2. The class $\mathcal{W}^n$.

Theorem 2. (c) <--> (d) : To prove that the open cover U is locally finite, let $V = \{x : \xi_\beta > \frac{1}{4} \mathrm{Max}_\gamma\, \xi_\gamma\}$.
If $x \in K_s$, then $\{y : \mathrm{Max}|\xi_\gamma(x) - \xi_\gamma(y)| < \mathrm{Min}(\frac{1}{8}\mathrm{Max}_\gamma \xi_\gamma(x), \frac{1}{8}\mathrm{Max}_\gamma \xi_\gamma(y))$ is a neighborhood of x and if it intersects U , then $x \in V$.
Since $\{V_\gamma\}$ is easily seen to be point finite the result follows.

Lemma 2. To prove that $\sum_\gamma \mathrm{Min}(\xi_\gamma, \eta_\gamma)$ is a continuous function, let (x_n, y_n) be a net converging to (x,y) and let J be the finite set of vertices at which each x or y has a non-zero barycentric coordinate. Since $\sum_J \mathrm{M}(\xi_\gamma, \eta_\gamma)$ is continuous we need only check that $\sum_{J'} \mathrm{Min}(\xi_\gamma^n, \eta_\gamma^n)$ converges to $\sum_{J'} \mathrm{Min}(\xi_\gamma, \eta_\gamma) = 0$. But we have $0 \leq \sum_{J'} \mathrm{Min}(\xi_\gamma^n, \eta_\gamma^n) \leq$ $\sum_{J'} \xi_\gamma^n + \sum_{J'} \eta_\gamma^n \leq d(x, x_n) + d(y, y_n)$ where d is the "sum of absolute values" metric. The right goes to zero and hence the term we want goes to zero.

Combining results of this paper with results of chapter 1 of the notes, we can prove the following theorem:

Theorem: Let $f : X \to Y$ be a continuous map, with X and Y in $\mathcal{W}$, then the triad $(M_f; X\times 0, Y) \in \mathcal{W}^3$.

Proof: By lemma 26, this triad is dominated by a triad $(M_g; P\times 0, Q)$ where P and Q are CW-complexes and g is a continuous map $g : P \to Q$. g is homotopic to a cellular map g_1 (by a homotopy g_t, with $g_0 = g$, let us say) and $(M_{g_1}; P\times 0, Q)$ is a CW triad by theorem 16 of the notes so the result reduces, by theorem 1 of the paper to a proof that if $g_0 \sim g_1$, then $(M_{g_0}; P\times 0, Q)$ is homotopy equivalent to $(M_{g_1}; P\times 0, Q)$.

Let $H : M_{g_0} \to M_{g_1}$ be defined as :

$H(q) =$	q		$q \in Q$
$H(p,t) =$	$(p, 2t)$	$0 \leq t \leq \frac{1}{2}$	$p \in P$
$H(p,t) =$	$g_{2-2t}(p)$	$\frac{1}{2} \leq t \leq 1$	$p \in P$

Let $K : M_{g_1} \to M_{g_0}$ be defined as :

$K(q) =$	q		$q \in Q$
$K(p,t) =$	$(p, 2t)$	$0 \leq t \leq \frac{1}{2}$	$p \in P$
$K(p,t) =$	$g_{2t-1}(p)$	$\frac{1}{2} \leq t \leq 1$	$p \in P$

These are easily seen to be homotopy inverses and the homotopies can be kept fixed on $P\times 0$ and Q. Thus giving the required result.

This generalizes to n-ads directly. That is, if $f : \mathcal{A} \to \mathcal{B}$ is a map of n-ads, then we define the mapping cylinder 3n-ad to be $(M; M_1, \ldots, M_{n-1}, A, B, A_1, \ldots, A_{n-1}, B_1, \ldots B_{n-1})$, where M_i is the mapping cylinder of $f : A_i \to B_i$. Since Lemma 26 following generalization:

Theorem: Let $f : \not{A} \dashrightarrow \not{B}$ be a map of n-ads. If $\not{A}$ and $\not{B}$ are in $\mathcal{W}^n$, then $\not{M}$ is in $\mathcal{W}^{3n}$.

Theorem: If $(A;A_1,\ldots,A_{n-1},B)$ is in $\mathcal{W}^{n+1}$, then $(A/B;A_1/B\cap A_1,\ldots,A_{n-1}/B\cap A_{n-1})$ is in $\mathcal{W}^n$.

Proof: If the n+1-ad has the homotopy type of $(K;K_1,\ldots,K_{n-1},L)$ then the n-ad has the homotopy type of $(K/L;K_1/L\cap K_1,\ldots,K_{n-1}/L\cap K_{n-1})$.

Chap. II : Homology of CW Complexes

In this chapter we shall be considering homology and cohomology theories on several categories. These will all be categories of pairs of spaces (X,A) with $A \subset X$, satisfying certain weak restrictions which are required for homology theory (see Eilenberg and Steenrod pages 4 and 5). In particular, each such category is closed under the application of the restriction functor $R : R(X,A) = (A,\emptyset)$ and if $f : (X,A) \longrightarrow (X',A')$ then $Rf = f|A$. A homology theory on such a category, $\mathcal{C}$, is:

a) For each integer q, a functor $H_q : \mathcal{C} \longrightarrow \mathfrak{G}$, where $\mathfrak{G}$ is the category of abelian groups.

b) For each integer q, a natural transformation $\partial_q : H_q \longrightarrow H_{q-1} \cdot R$, with the index of ∂_q usually dropped.

These functors and natural transformations satisfy the following axioms:

i-(Exactness) For each pair (X,A) in $\mathcal{C}$, the sequence:

$$\cdots H_q(A,\emptyset) \xrightarrow{i_*} H_q(X,\emptyset) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} H_{q-1}(A,\emptyset) \cdots$$

is exact, where the maps $i : (A,\emptyset) \longrightarrow (X,\emptyset)$ and $j : (X,\emptyset) \longrightarrow (X,A)$ are inclusion maps and we write i_* for $H_q i$, etc.

ii-(Homotopy) If $F : (X,A)\times I \longrightarrow (Y,B)$, F_0 and F_1 are all elements of $\mathcal{C}$, then $F_{0*} = F_{1*}$.

iii-(Excision) If U is an open subset of X with $\bar{U} \subset \operatorname{Int} A$, and if $(X-U,A-U) \longrightarrow (X,A)$ is admissible, then it induces isomorphisms of the homology groups.

iv-(Dimension) If P is a space in $\mathcal{C}$ consisting of a single point, then $H_q(P,\emptyset) = 0$, if $q \neq 0$.

We will, in general, write $H_q(A)$ for $H_q(A,\emptyset)$.

For cohomology on $\mathcal{C}$, we require:

a) For each integer q, a contravariant functor $H^q : \mathcal{C} \longrightarrow \mathcal{G}$.

b) For each integer q, a natural transformation $\partial : H^q \longrightarrow H^{q+1}\cdot R$.

Satisfying the duals of the four axioms of homology.

In generalized or extraordinary homology and cohomology theories the dimension axiom is dropped. Consequently, it will be of interest to note at what stages the dimension axiom is required in the proofs of this chapter. Finally, if we speak of a homology or cohomology theory on a class of topological spaces, we shall mean a theory on the category of pairs of such spaces with the morphisms all continuous maps of such pairs.

First we consider a homology, or cohomology, theory on open subsets of CW complexes.

1. Theorem- If K_1 and K_2 are subcomplexes of a CW complex K, then the inclusion map, $i : (K_1, K_1 \cap K_2) \longrightarrow (K_1 \cup K_2, K_2)$ induces isomorphisms in each dimension. Consequently, $(K; K_1, K_2)$ is a proper triad.

Proof: Without loss of generality we may assume that $K = K_1 \cup K_2$. K is then a neighborhood of K_1. Let V be an open neighborhood of K_1, as defined in the proof of theorem 11, and let F_t be the homotopy constructed there. i can be factored: $(K_1, K_1 \cap K_2) \xrightarrow{i_1} (V, V \cap K_2) \xrightarrow{i_2} (K, K_2)$. i_2 induces isomorphisms by excision. On the other hand K_1 is a strong deformation retract of V, and we note that the deformation F_t was constructed in [illegible] then so is each point $F_t(x)$. Thus, $K_1 \cap K_2$ is a strong deformation retract of $V \cap K_2$, and in fact, the pair $(K_1, K_1 \cap K_2)$ is a strong deformation retract of $(V, V \cap K_2)$ and hence i_1 induces isomorphisms by the homotopy axiom.

2. Theorem– Let K be a CW complex with subcomplexes $K_1, \ldots, K_r, L$ such that $K = L \cup \bigcup_{i=1}^{r} K_i$, and $K_i \cap K_j \subset L$ for $i \neq j$. Let $L_i = K_i \cap L$, and let $k_i : (K_i, L_i) \subset (K, L)$ be the inclusions. Then:

a) Homology case: The homomorphisms k_{i*} form an injective representation of $H_*(K,L)$ as a direct sum. (Where H_* represents the graded system of homology groups.)

b) Cohomology case: The homomorphisms k_i^* form a projective representation of $H^*(K,L)$ as a direct sum. (Where H^* represents the graded system of cohomology groups.)

Proof: $r=1$: Since $K = L \cup K_1$, this follows from theorem 1.

$r=2$: By theorem 1, the triad $(K; K_1 \cup L, K_2 \cup L)$ is proper. Hence, the maps $k_i^! : (K_i \cup L, L) \subset (K, L)$ induce the proper direct sum diagrams by Eilenberg and Steenrod theorems I.14.2 and I.14.2.c. Furthermore, $k_i = k_i^!$ preceded by the inclusion $(K_i, L_i) \subset (K_i \cup L, L)$ which induces isomorphisms by another application of theorem 1.

Assuming the result for r-1, we prove it for r : Let $K' = \bigcup_{i=1}^{r-1} K_i$. By inductive hypothesis, the maps $k_i^! : (K_i, L_i) \subset (K' \cup L, L)$ induce a direct sum, but by the case for $r=2$, k_r and the inclusion $(K' \cup L, L) \subset (K, L)$ induce a direct sum.

Note that the special case of this theorem where $L = \emptyset$ is true without the restriction to CW complexes, as Eilenberg and Steenrod theorems I.13.2 and I.13.2c. Many homology and cohomology theories, such as singular and Cech satisfy the following generalization of this theorem to the infinite case, which we will codify as a fifth axiom:

v-(Direct Sum) If $X = \bigcup_{i \in I} X_i$ (disjoint union, with the weak topology, over some index set I) and the inclusions $k_i : X_i \subset X$ are all admissable then the homomorphisms k_{i*} form an injective representation of $H_*(X)$ as a direct sum. (Dually for cohomology.)

We will now consider the rather general case of a homology or cohomology theory on CW pairs, that is, on pairs (K,L) where K is a CW complex and L is a subcomplex. Furthermore, we will require that in addition to the Eilenberg and Steenrod axioms, this additional axiom. We will eventually show, following Milnor (1), that singular homology and cohomology are characterized on CW pairs by these axioms. This is the smallest category, we will consider, and the proof will go through for the category of spaces having the homotopy type of CW complexes. Thus, we will not be explicit until we need to be and prove results simultaneously for the category of CW pairs, category of CW complexes, ie. pairs (K,L) of CW complexes where $L \subset K$ but need not be a subcomplex, and spaces having the homotopy type of CW complexes.

First, we will consider a property which we shall show is closely related to the direct sum axiom. Let a CW complex, K, be the union of an increasing sequence of subcomplexes: $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots$ Applying H_* to this sequence we get a direct system of groups, and applying H^* we get an inverse system. The inclusion map $k_i : K_i \subseteq K$ induces maps $k_{i*} : H_*(K_i) \longrightarrow H_*(K)$ and $k_i^* : H^*(K) \longrightarrow H^*(K_i)$. We now give definitions of direct and inverse limits of sequences which while slightly different are equivalent to the specialization to sequences of the usual definitions.

Definition- Given a direct sequence of groups $G_1 \xrightarrow{p} G_2 \xrightarrow{p} \ldots$ the direct limit is defined as the cokernel of the map $\sum_i G_i \xrightarrow{d} \sum_i G_i$, which maps g into g-pg, ie. $d(g_1, g_2, \ldots) = (g_1, g_2 - pg_1, g_3 - pg_2, \ldots)$. We will write this as G_∞ or $\underrightarrow{\mathrm{Lim}}\, G_i$.

Definition- Given an inverse sequence of groups $G_1 \xleftarrow{p} G_2 \xleftarrow{p} G_3 \ldots$ the inverse limit is defined as the kernel of the map $\prod_i G_i \longrightarrow \prod_i G_i$, which maps g into g-pg, ie. $d(g_1, g_2, \ldots) = (g_1 - pg_2, g_2 - pg_3, \ldots)$. We shall write this as G^∞ or $\underleftarrow{\mathrm{Lim}}\, G_i$. We will also require notation for the cokernel of this map which we shall write as $L'(G_i)$, the first derived functor of the inverse limit functor.

We note that the use of the word "functor", above, is justified. Direct and Inverse limits (as well as L') are functors since maps of direct (or inverse) sequences induce maps of the direct sum (product) which commute with the map d and hence induce limit maps on the kernel or cokernel of d.

Also we note that in the topological case above we get limit maps $k_\infty : \overrightarrow{\mathrm{Lim}}\, H_*(K_i) \longrightarrow H_*(K)$ and $k^\infty : H^*(K) \longrightarrow \overleftarrow{\mathrm{Lim}}\, H^*(K_i)$.

3. Theorem (Milnor)- In the above situation:

a) Homology case- $k_\infty : \overrightarrow{\mathrm{Lim}}\, H_n(K_i) \longrightarrow H_n(K)$ is an isomorphism.

b) Cohomology case- $k^\infty : H^n(K) \longrightarrow \overleftarrow{\mathrm{Lim}}\, H^n(K_i)$ is an epimorphism with kernel naturally isomorphic to $L'(H^{n-1}(K_i))$.

Note: In addition to the Eilenberg-Steenrod axioms this proof requires axiom v, though we only use it for the case when the index set I is countable. We do not use axiom iv.

Proof: Let L denote the CW complex:

$K_1 x[0,1] \cup K_2 x[1,2] \cup K_3 x[2,3] \cup \ldots$ and let L_1 be the union of all of the $K_i x[i-1,i]$ with i odd. Similarly, let L_2 be the union of all of the $K_i\, x[i-1,i]$ with i even. Each of L_1 and L_2 are CW complexes by I. theorem 14, and the fact that the weak disjoint union of CW complexes is obviously a CW complex. L is then a CW complex by I. theorem 16 and L_1 and L_2 are subcomplexes. The projection map L --> K induces isomorphisms of homotopy groups in all dimensions. This is because the restriction to a map $K_1 x[0,1] \cup \ldots \cup K_n x[n-1,n] \longrightarrow K_n$ is a homotopy equivalence and L is the weak union of the left hand terms, while K is the weak union of the right-hand terms, and because the homotopy groups are continuous under weak direct limit. Then by Whitehead's theorem (I.27), applied to each component of L, we have that the projection is a homotopy equivalence. Note that each component of L is of the form $L \cap Cx[0,\infty)$ for C a component of K.

a) Let $j_i : K_i \longrightarrow K_i \times [i]$ in the obvious fashion. We consider the triad $(L;L_1,L_2)$. This triad is proper. While we cannot apply theorem 1 (our homology theory need not be defined on all the open subsets of CW complexes), we can apply its method to each of the excisions in question, and in each case we note that the neighborhood V can be chosen to be a subcomplex of L so that the method of proof of theorem 1, goes through. In fact each set $K_i \times [i-1,i]$ can be thickened by adding on $K_{i-1} \times [i-3/2,i-1]$. Each of the subcomplexes L_1, L_2 and $L_1 \cap L_2$ can be represented as an injective direct sum by the j_{i*}'s by an application of the direct sum axiom and in the cases of L_1 and L_2, the homotopy axiom also, eg. $H_*(K_1) \oplus H_*(K_3) \oplus H_*(K_5) \oplus \ldots \approx H_*(L_1)$ by $\sum_{i\,odd} j_{i*}$. Using these identifications, we compute $\psi : H_*(L_1 \cap L_2) \longrightarrow H_*(L_1) \oplus H_*(L_2)$, of the Mayer-Vietoris sequence. If $h \in H_*(K_i)$ then $\psi(h) = h - ph$ for i odd and $-h+ph$ for i even, where $p : H_*(K_i) \longrightarrow H_*(K_{i+1})$ is induced by the inclusion map, ie. $\psi(h_1,h_2,h_3,\ldots) = (h_1, ph_2+h_3,\ldots) \oplus (-ph_1-h_2, -ph_3-h_4,\ldots)$. It is convenient to precede ψ by the automorphism α of $H_*(L_1 \cap L_2)$ which multiplies each h_i by $(-1)^{i+1}$. We shuffle the terms on the right side of the equation to obtain: $\psi\alpha(h_1,h_2,h_3,\ldots) = (h_1, h_2-ph_1, h_3-ph_2,\ldots)$. From this expression it is obvious that ψ has kernel zero and that the following commutes:

$$\begin{array}{ccc} \sum H_*(K_i) & \xrightarrow{d} & \sum H_*(K_i) \\ \downarrow \approx & & \downarrow \approx \\ H_*(L_1 \cap L_2) & \xrightarrow{\psi\alpha} & H_*(L_1) \oplus H_*(L_2). \end{array}$$

Hence, we have an isomorphism of the cokernel of d, $\overrightarrow{\mathrm{Lim}}\, H_*(K_i)$ with the cokernel of $\psi\alpha$ = cokernel of ψ which is, by the Mayer-Vietoris sequence, since ψ has kernel zero, $H_*(L)$. Furthermore, this isomorphism followed by the isomorphism of $H_*(L)$ and $H_*(K)$ induced by projection is precisely the map k_∞, this since j_i followed by this projection is the inclusion k_i.

b) As in a) we can calculate the Mayer-Victoris map $\psi: H^*(L_1) \oplus H^*(L_2) \longrightarrow H^*(L_1 \cap L_2)$, $\psi((h_1,h_3,\ldots) \oplus (h_2,h_4,\ldots))$ $= (h_1-ph_2,-h_2+ph_2,h_3-ph_4,\ldots)$ where p is the map induced by inclusion $p: H^*(K_{i+1}) \longrightarrow H^*(K_i)$. So if α is the automorphism of $H^*(L_1 \cap L_2)$ which multiplies the i^{th} place by $(-1)^{i+1}$, we have, as in a), the commutative diagram:

$$\begin{array}{ccc}
\prod H^*(K_i) & \xrightarrow{\ d\ } & \prod H^*(K_i) \\
\approx\uparrow & & \uparrow\approx \\
H^*(L_1) \oplus H^*(L_2) & \xrightarrow{\psi\alpha} & H^*(L_1 \cap L_2)
\end{array}$$

This time, however, the Mayer-Vietoris sequence doesn't break up into short exact sequences since need not be onto. However, by exactness we do get a map from $H^*(L)$ into the kernel of d, which is $\overleftarrow{\mathrm{Lim}}\, H^*(K_i)$, and which, preceded by the map induced by the projection of $L \longrightarrow K$, is just k^∞. This map is onto as shown in the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & \varprojlim H^*(K_i) \arrow[r] & \prod H^*(K_i) \arrow[r, "d"] & \prod H^*(K_i) \\
 & 0 \arrow[r] & \phi(H^*(L)) \arrow[u, "\approx"'] \arrow[r, "incl"] & H^*(L_1) \oplus H^*(L_2) \arrow[u, "\approx"] \arrow[r, "\alpha\psi"] & H^*(L_1 \cap L_2) \arrow[u] \\
H^*(L) \arrow[uurr, "k^\infty"] \arrow[urr, "\phi"'] & & & &
\end{tikzcd}$$

Since ϕ is onto its image, k^∞ is the composite of an onto map and an isomorphism. We also get that the kernel of k^∞ is precisely the kernel of ϕ. Since $\alpha^2 =$ identity, the Mayer-Vietoris sequence remains exact when we replace ψ by $\alpha\psi$ and Δ by $\Delta\alpha$. Hence, the kernel of ϕ is the image of $\Delta\alpha$ which equals the image of Δ, Which is isomorphic to $L'(H^*(K_i))$, with a lowering of the index by one, as can be seen from the diagram:

$$\begin{tikzcd}
\prod H^{n-1}(K_i) \arrow[r, "d"] & \prod H^{n-1}(K_i) \arrow[r] & L'(H^{n-1}(K_i)) \arrow[r] & 0 \\
H^{n-1}(L_1) \oplus H^{n-1}(L_2) \arrow[u, "\approx"] \arrow[r, "\alpha\psi"] & H^{n-1}(L_1 \cap L_2) \arrow[u, "\approx"] \arrow[r, "\Delta\alpha"] & \operatorname{Im}\Delta = \text{kernel } k^\infty_{(n)} \arrow[u, "\approx"'] \arrow[r] & 0.
\end{tikzcd}$$

This proves b).

Note that theorem 3 implies axiom v, for the special case of a countable index set.

The previous theorem can be proved for singular homology and cohomology without assuming that K and the K_i's are CW complexes. We only require that the union have the weak topology on the sequence. This can either be proved directly by considering singular simplices or using the previous method replacing that application of theorem I.14, by the Whitehead theorem that a map which induces isomorphisms of homotopy must induce isomorphisms of homology.

The paradigm case of a complex as an increasing union of a sequence of subcomplexes is, of course, the representation of K as the union of skeletons K^n. If L is a subcomplex of K, let $\overline{K}^n = K^n \cup L$.

4. Lemma- Let $i_\sigma : (e^n_\sigma, S^{n-1}_\sigma) \dashrightarrow (\overline{K}^n, \overline{K}^{n-1})$, $\sigma \in I^n_K - I^n_L$ be the set of attaching maps of the n-cells to $\overline{K}^{n-1}$.

a) Homology case- $i_{\sigma *} : H_*(e^n_\sigma, S^{n-1}_\sigma) \dashrightarrow H_*(\overline{K}^n, \overline{K}^{n-1})$ is an injective representation of a direct sum.

b) Cohomology case- $i_\sigma^* : H^*(\overline{K}^n, \overline{K}^{n-1}) \dashrightarrow H^*(e^n_\sigma, S^{n-1}_\sigma)$ is a projective representation of a direct product.

Proof: Let (E,S) be the disjoint union of the $(e^n_\sigma, S^{n-1}_\sigma)$ with the weak topology. We will assume that each e^n_σ is a unit disc with center the origin in some n-dimensional real vector space and will use vector notation throughout. Define $i : (E,S) \dashrightarrow (\overline{K}^n, \overline{K}^{n-1})$ so that $i|e^n_\sigma = i_\sigma$. By the additivity axiom, it suffices to show that i induces isomorphisms.

Let $J = \overline{K}^{n-1} \cup i(\{x \in E : x = ty \text{ with } y \in S \text{ and } t \in [\tfrac{1}{2}, 1]\})$, and let $U = \overline{K}^{n-1} \cup i(\{x \in E : x = ty \text{ with } y \in S \text{ and } t \in (2/3, 1]\})$. J is a closed subset of $\overline{K}^n$ and U is an open subset of $\overline{K}^n$. Furthermore, $\overline{K}^{n-1}$ is obviously a deformation retract of J and we have the diagram:

$$\begin{tikzcd}
(E,S) \arrow[r, dashed, "i"] \arrow[dr, "i'"] \arrow[ddr, "i'"] \arrow[dddr, "i'"'] & (\overline{K}^n, \overline{K}^{n-1}) \arrow[d, phantom, "\cap\ \text{induces} \cong \text{ by homotopy axiom}"] \\
& (\overline{K}^n, J) \arrow[d, phantom, "\cup\ \text{induces} \cong \text{ by excision axiom}"] \\
& (\overline{K}^n - U, J - U) \arrow[d, phantom, "\cup\ \text{induces} \cong \text{ by homotopy axiom}"] \\
& i'(E,S)
\end{tikzcd}$$

Where $i' : (E,S) \dashrightarrow (\overline{K}^n, J)$ by $x \dashrightarrow i(\tfrac{1}{2}x)$. i' and i are homotopic as maps into $(\overline{K}^n, J)$ and i' is a homeomorphism onto its image. Finally, the image of i' is a deformation retract of the

Since the diagram commutes, the fact that $i' : (E,S) \dashrightarrow \operatorname{Im} i'$ is a homeomorphism and hence induces isomorphisms and the fact that i' is homotopic to i and hence induce the same map it follows that i induces isomorphisms.

5. Corollary- If $\overline{K}^n$ is the n-skeleton of (K,L) a CW complex pair,

a) Homology case- then $H_m(\overline{K}^n,\overline{K}^{n-1}) = 0$, for $m \neq n$

b) Cohomology case- then $H^m(\overline{K}^n,\overline{K}^{n-1}) = 0$, for $m \neq n$.

Proof: This follows from Eilenberg and Steenrod I. theorem 16.4, which proves that $H_m(e^n,S^{n-1}) = 0$ for $m \neq n$, and theorem 4, or I. theorem 16.4c and theorem 4 for the cohomology case. These calculations require the dimension axiom and this is the first time that we have required it. Since the rest of the characterization requires this corollary, we can no longer dispense with the dimension axiom.

We can now define the homological and cohomological chain complexes associated with the CW pair (K,L).

Definition- For a CW pair (K,L) :

a) Homology case- We define the chain complex $C_*(K,L)$ as follows: $C_n(K,L) = H_n(\overline{K}^n,\overline{K}^{n-1})$, with the boundary $\partial : C_n(K,L) = H_n(\overline{K}^n,\overline{K}^{n-1}) \dashrightarrow C_{n-1}(K,L) = H_{n-1}(\overline{K}^{n-1},\overline{K}^{n-2})$ as the boundary of the triple $(\overline{K}^n,\overline{K}^{n-1},\overline{K}^{n-2})$.

b) Cohomology case- We define the cochain complex $C^*(K,L)$ as follows: $C^n(K,L) = H^n(\overline{K}^n,\overline{K}^{n-1})$ with the coboundary $\delta: C^n(K,L) = H^n(\overline{K}^n,\overline{K}^{n-1}) \dashrightarrow C^{n+1}(K,L) = H^{n+1}(\overline{K}^{n+1},\overline{K}^n)$ as the coboundary of the triple $(\overline{K}^{n+1},\overline{K}^n,\overline{K}^{n-1})$.

That these actually are chain and cochain complexes requires the proof that $\partial\partial = 0$ and $\delta\delta = 0$. But $\partial\partial$ can be factored as: $H_n(\overline{K}^n,\overline{K}^{n-1}) \xrightarrow{\overline{\partial}} H_{n-1}(\overline{K}^{n-1}) \longrightarrow H_{n-1}(\overline{K}^{n-1},\overline{K}^{n-2}) \xrightarrow{\overline{\partial}} H_{n-2}(\overline{K}^{n-2}) \longrightarrow H_{n-2}(\overline{K}^{n-2},\overline{K}^{n-3})$, which contains two consecutive maps of the exact sequence of the pair $(\overline{K}^{n-1},\overline{K}^{n-2})$ whose composition is thus 0. $\delta\delta = 0$ follows similarly.

Our immediate goal is the proof of the theorem that homology of the pair (K,L), ie. $H_*(K,L)$, is precisely the homology of the chain complex, ie. $H(C_*(K,L))$ and similarly for cohomology. To this end, a preliminary lemma, and corollaries.

6. Lemma- For a CW pair (K,L),

a) the homomorphism $H_q(\overline{K}^n,L) \longrightarrow H_q(\overline{K}^{n+1},L)$ induced by inclusion is an isomorphism for $q \neq p, p+1$ and is onto if $q = p$, and is mono if $q = p+1$.

b) the homomorphism $H^q(\overline{K}^{n+1},L) \longrightarrow H^q(\overline{K}^n,L)$ induced by inclusion is an isomorphism for $q \neq n, n+1$ and is onto if $q = n+1$, and is mono if $q = n$.

Proof: Consider the sequence of the triple $(\overline{K}^{n+1},\overline{K}^n,L)$ with the result from corollary 5.

7. Corollary- For a CW pair (K,L)

a) $H_n(\overline{K}^{n-1},L) = 0$.

b) $H^n(\overline{K}^{n-1},L) = 0$.

Proof: From lemma 6, it follows by induction that $H_n(\overline{K}^{n-1},L) \approx H_n(\overline{K}^{n-r},L)$ for $r = 2,\dots,n+1$ and $H_n(\overline{K}^{-1},L) = H_n(\overline{L},L) = 0$. Similarly for cohomology.

8. Lemma- For a CW pair (K,L)

a) the homomorphism $H_n(\overline{K}^m,L) \to H_n(K,L)$ induced by inclusion is an isomorphism for $m > n$.

b) the homomorphism $H^n(K,L) \to H^n(\overline{K}^m,L)$ induced by inclusion is an isomorphism for $m > n$.

Proof: First, we prove that $H_n(\overline{K}^m) \to H_n(K)$ induced by inclusion, is an isomorphism. By the exact sequence of the pair $(\overline{K}^m,\overline{K}^{m+1})$ and corollary 5, $H_n(\overline{K}^m) \to H_n(\overline{K}^{m+1})$ is an isomorphism. By the result on direct limits of sequences, theorem 3, and a general result on direct limits (Eilenberg and Steenrod VIII theorem 4.13), the inclusion $H_n(\overline{K}^m) \to H_n(K)$ is an isomorphism.

Now the result follows from the "five lemma" applied to the sequence of the pair $(\overline{K}^m,L)$; and the pair (K,L) :

$$\begin{array}{ccccccccc}
H_n(L) & \to & H_n(\overline{K}^m) & \to & H_n(\overline{K}^m,L) & \to & H_{n-1}(L) & \to & H_{n-1}(\overline{K}^m) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_n(L) & \to & H_n(K) & \to & H_n(K,L) & \to & H_{n-1}(L) & \to & H_{n-1}(K).
\end{array}$$

Dually for cohomology.

We can now state and prove the following:

9. Theorem- For a CW pair (K,L)

a) Homology case- $H_*(K,L)$ is naturally isomorphic to $H(C_*(K,L))$.

b) Cohomology case- $H^*(K,L)$ is naturally isomorphic to $H(C^*(K,L))$.

Proof: Given all this preliminary work, the proof is a matter of looking at a diagram.

a)

$$\begin{tikzcd}
 & H_{n+1}(\overline{K}^{n+1},\overline{K}^{n}) \arrow[d, "\partial"] \arrow[dr, "\partial"] & & H_{n-1}(\overline{K}^{n-2},L) = 0 \arrow[d] \\
0 = H_n(\overline{K}^{n-1},L) \arrow[r] & H_n(\overline{K}^{n},L) \arrow[r, "j_{3*}"] \arrow[d, "j_{2*}"] & H_n(\overline{K}^{n},\overline{K}^{n-1}) \arrow[r, "\partial"] \arrow[dr, "\partial"] & H_{n-1}(\overline{K}^{n-1},L) \arrow[d] \\
H_n(K,L) & H_n(\overline{K}^{n+1},L) \arrow[l, "\approx"', "j_{1*}"] \arrow[d] & & H_{n-1}(\overline{K}^{n-1},\overline{K}^{n-2}) \\
 & 0 & &
\end{tikzcd}$$

This diagram has exact rows and columns. If $Z_n(K,L) \xrightarrow{\eta} C_n(K,L)$ is the inclusion map of the cycles into the chains and $Z_n(K,L) \xrightarrow{\nu} H_n(C_*(K,L))$ is the projection of the cycles onto the homology group, then the required map is defined by the switchback $\nu\eta^{-1} j_{3*} j_{2*}^{-1} j_{1*}^{-1}$. We leave to the reader the diagram chases that prove this map to be well defined and an isomorphism.

b)

$$\begin{tikzcd}
 & & 0 & \\
H^{n-1}(\overline{K}^{n-1},\overline{K}^{n-2}) \arrow[d] \arrow[dr, "\delta"] & & H^n(\overline{K}^{n+1},L) \arrow[d, "j_2^*"'] & H^n(K,L) \arrow[l, "\approx"', "j_1^*"] \\
H^{n-1}(\overline{K}^{n-1},L) \arrow[r, "\delta"] \arrow[d] & H^n(\overline{K}^{n},\overline{K}^{n-1}) \arrow[r, "j_3^*"] \arrow[dr, "\delta"] & H^n(\overline{K}^{n},L) \arrow[r] \arrow[d, "\delta"] & H^n(\overline{K}^{n-1},L) = 0 \\
H^{n-1}(\overline{K}^{n-2},L) = 0 & & H^{n+1}(\overline{K}^{n+1},\overline{K}^{n}) &
\end{tikzcd}$$

If $\eta : Z^n(K,L) \to C^n(K,L)$ and $\nu : Z^n(K,L) \to H_n(C^*(K,L))$ are the inclusion and projection of the cocycles respectively, then the switchback of the isomorphism is $\nu\eta^{-1} j_3^{*-1} j_2^* j_1^*$. We again leave the details to the reader.

The isomorphisms of a) and b) are obviously natural under cellular maps, $f : (K,L) \to (K',L')$ since such a map "projects" either of the above diagrams for (K,L) onto the corresponding one for (K',L') (or vice-versa in the contravariant case). For non-cellular maps f, we define the induced map on be a cellular approximation of f, ie. let g_t: $L \dashrightarrow L'$ be a homotopy with $g_0 = f|L$ and g_1 cellular. Extend this to a homotopy h_t with $h_0 = f$ and h_t: $K \dashrightarrow K'$, then $h_1|L$ is cellular and hence h_1 is homotopic (rel L) to g which is cellular, and hence f is homotopic by the resultant of these two homotopies to g. Unfortunately, the map of chains or cochains induced by the cellular approximation g, of f, is not well-defined as a function of f. The map $0 \dashrightarrow \frac{1}{2}$ of $0 \dashrightarrow I$, has two cellular approximations $0 \dashrightarrow 0$ and $0 \dashrightarrow 1$ and these induce different maps on the chain complexes. However, any two such induced maps give the same maps on homology namely the image under our isomorphism of f_* or f^*, by the homotopy axiom.

Now consider a CW pair (K,L). It is easy to see, using lemma 4, that the inclusion maps $(L^n, L^{n-1}) \dashrightarrow (K^n, K^{n-1}) \dashrightarrow (\overline{K}^n, \overline{K}^{n-1})$ yield a short exact sequence of chains:

$$0 \dashrightarrow C_*(L) \dashrightarrow C_*(K) \dashrightarrow C_*(K,L) \dashrightarrow 0$$

and cochains: $0 \dashrightarrow C^*(K,L) \dashrightarrow C^*(K) \dashrightarrow C^*(L) \dashrightarrow 0$.
The exact homology (or cohomology) sequence of the short exact sequence of complexes is term-by-term isomorphic to the exact homology sequence (or cohomology sequence) of the pair. Furthermore, by naturality the isomorphisms commute with the mappings induced by the inclusions. Thus, the question of the isomorphism of the two sequences is reduced to the question of commutativity of the isomorphism of theorem 9 a) (resp. 9 b)) with the boundary operators (or coboundary operators), of the two sequences. Before we can prove the commutativity, we shall require a preliminary lemma.

10. Lemma- For a CW pair (K,L)

a) the homomorphism $H_n(K^n,L^{n-1}) \to H_n(\overline{K}^n,L)$ induced by inclusion, is onto.

b) the homomorphism $H^n(\overline{K}^n,L) \to H^n(K^n,L^{n-1})$ induced by inclusion, is one-to-one.

Proof: a) The homomorphism factors into $H_n(K^n,L^{n-1}) \xrightarrow{(1)} H_n(K^n,L^n) \xrightarrow{(2)} H_n(\overline{K}^n,L^n) \xrightarrow{(3)} H_n(\overline{K}^n,L)$. We will show that each of these three maps are onto.

(1) Sequence of the triple (K^n,L^n,L^{n-1}) :

$H_n(K^n,L^{n-1}) \to H_n(K^n,L^n) \xrightarrow{\partial} H_{n-1}(L^n,L^{n-1}) = 0.$

(2) Sequence of the triple $(\overline{K}^n,K^n,L^n)$:

$H_n(K^n,L^n) \to H_n(\overline{K}^n,L^n) \to H_n(\overline{K}^n,K^n) \approx$ (lemma 8, on $(\overline{K}^n,K^n)$, noting that $(\overline{K}^n)^{n+1} = K^{n+1}$) $H_n(K^{n+1},K^n) = 0.$

(3) Sequence of the triple $(\overline{K}^n,L,L^n)$:

$H_n(\overline{K}^n,L^n) \to H_n(\overline{K}^n,L) \xrightarrow{\partial} H_{n-1}(L,L^n) \approx$ (Lemma 8 on (L,L^n))

$H_{n-1}(L^{n+1},L^n) = 0.$

b) The homomorphism factors into

$H^n(\overline{K}^n,L) \to H^n(\overline{K}^n,L^n) \to H^n(K^n,L^n) \to H^n(K^n,L^{n-1})$. It is proved as in (a) that each of these three maps has kernel zero, by considering the sequence of the relevant triple.

11. Theorem- For a CW pair (K,L), the isomorphism of theorem 9 a (resp. 9 b) commutes with boundary (resp. coboundary) opprators.

Proof: The proofs are a demonstration that the larger rectangle in each of the following diagrams commutes.

a)

$$\begin{tikzcd}
 & & Z_n(K,L) \arrow[d, "\eta"'] \\
H_n(K,L) \arrow[ddd, "\partial"'] & H_n(\overline{K}^n,L) \arrow[l] \arrow[r] & H_n(\overline{K}^n,\overline{K}^{n-1}) \\
 & H_n(K^n,L^{n-1}) \arrow[u, "e"] \arrow[r] \arrow[dd, "\partial"'] & H_n(K^n,K^{n-1}) \arrow[u] \arrow[d, "\partial"] \\
 & & H_{n-1}(K^{n-1},K^{n-2}) \\
H_{n-1}(L) & H_{n-1}(L^{n-1}) \arrow[l] \arrow[r] & H_{n-1}(L^{n-1},L^{n-2}) \arrow[u] \\
 & & Z_{n-1}(L) \arrow[u, "\eta"']
\end{tikzcd}$$

e is an epimorphism by lemma 10.

b)

$$\begin{tikzcd}
 & & Z^n(K,L) \arrow[d, "\eta"'] \\
H^n(K,L) \arrow[r] & H^n(\overline{K}^n,L) \arrow[d, "m"'] & H^n(\overline{K}^n,\overline{K}^{n-1}) \arrow[l] \arrow[d] \\
 & H^n(K^n,L^{n-1}) & H^n(K^n,K^{n-1}) \arrow[l] \\
 & & H^{n-1}(K^{n-1},K^{n-2}) \arrow[u, "\delta"'] \arrow[d] \\
H^{n-1}(L) \arrow[uuu, "-\delta"] \arrow[r] & H^{n-1}(L^{n-1}) \arrow[uu, "\delta"] & H^{n-1}(L^{n-1},L^{n-2}) \arrow[l] \\
 & & Z^{n-1}(L) \arrow[u]
\end{tikzcd}$$

m is a monomorphism by lemma 10.

The proofs follow from diagram chases of the two diagrams using the fact that each rectangle involving the term introduced in the center of the diagram (ie. $H_n(K^n,L^{n-1})$ in (a) and $H^n(K^n,L^{n-1})$ in (b)) commutes, and the properties of e amd m mentioned above. The actual chase is left to the reader, who can find it in Eilenberg and Steenrod pages 98-100.

To review what we have done: given an additive homology theory on the category of CW pairs, we have constructed a chain complex C_* for each pair and proved that H_* as a homology functor is equivalent to HC_*. Furthermore, given any two such homology theories H_* and $\overline{H}_*$, we note that by lemma 4, the groups of the corresponding chain complexes C_* and $\overline{C}_*$ are identical. If the boundary operators could be proved to agree in the two chain complexes then the composition of the two equivalences would prove that H_* is equivalent to $\overline{H}_*$. We show below that in the subcategory of (infinite) simplicial pairs, the boundary operators do, indeed agree, as they are both the boundary operators of the simplicial groups. All of this dualizes for cohomology.

Let $s^n = v_0 \dots v_n$ be fixed as the standard n-simplex. A simplicial complex K, with a fixed ordering of the vertices, is a CW complex with maps i_σ for $\sigma = A_0 \dots A_n$, an n-simplex, defined as the simplicial map which takes $v_k \longrightarrow A_k$. We will use the notation and theorems of Eilenberg and Steenrod chapter III, sections 3 and 4 and theorem 6.4, which deal with homology and cohomology theories on a simplex and its subcomplexes.

a) Homology case- By lemma 4, we knew that $\{ i_{\sigma *}: H_n(s, \partial s) \longrightarrow H_n(\overline{K}^n, \overline{K}^{n-1})$, $\sigma \in I^n_K - I^n_L \}$ represents the latter group as a direct sum. We define for such a $\sigma = A_0 \dots A_n$, $gA_0 \dots A_n \in C_n(K,L) = i_{\sigma *}(gs)$. If $\sigma \in I^n_L$ we define this symbol as 0 and this still equals $i_{\sigma *}(gs)$ since i_σ then factors through (K,K). We must prove that

$gA_0 \dots A_n = \sum (-1)^k gA_0 \dots \hat{A}_k \dots A_n$. We use the diagram:

$$\begin{array}{ccc} H_n(s, \partial s) & \xrightarrow{i_{\sigma *}} & H_n(\overline{K}^n, \overline{K}^{n-1}) \\ \partial \downarrow & & \partial \downarrow \\ H_{n-1}(\partial s, \partial s^{n-2}) & \xrightarrow{j_*} & H_{n-1}(\overline{K}^{n-1}, \overline{K}^{n-2}) \\ \uparrow i_{k*} & \nearrow i_{\partial_k \sigma *} & \\ H_{n-1}(s', \partial s') & & \end{array} \qquad s = s^n,\ s' = s^{n-1}.$$

By Eilenberg and Steenrod, theorem 6.4 for the case of a simplex mod boundary, we have $\partial(gs) = \sum(-1)^k i_{k*}(gs')$. Hence, $\partial\, i_{\sigma *}(gs) = j_* \partial(gs) = \sum(-1)^k j_* i_{k*}(gs') = \sum (-1)^k\, i_{\partial_k \sigma *}(gs')$, which is the above required result.

b) Cohomology case- By lemma 4, we know that $\{ i_\sigma^* : H^n(\overline{K}^n, \overline{K}^{n-1}) \longrightarrow H^n(s, \partial s)\ ,\ \sigma \in I_K^n - I_L^n \}$ represents the former group as a direct product. We define for such a $\sigma = A_0 \dots A_n$, and an element $c \in H^n(K, L)$, $c(A_0 \dots A_n) = g$ where $i_\sigma^*(c) = gs$, or in slightly different notation, $c(A_0 \dots A_n) = i_\sigma^*(c)(s)$. We must then prove that $(\delta c)(A_0 \dots A_{n+1}) = \sum(-1)^k c(A_0 \dots \hat{A}_k \dots A_{n+1})$. We use the diagram:

$$\begin{array}{ccc} H^{n+1}(s, \partial s) & \xleftarrow{i_\sigma^*} & H^{n+1}(\overline{K}^{n+1}, \overline{K}^n) \\ \delta \uparrow & & \delta \uparrow \\ H^n(\partial s, \partial s^{n-1}) & \xleftarrow{j^*} & H^n(\overline{K}^n, \overline{K}^{n-1}) \\ \downarrow i_k^* & \swarrow i_{\partial_k \sigma}^* & \\ H^{n-1}(s', \partial s') & & \end{array} \qquad s = s^{n+1},\ s' = s^n$$

By Eilenberg and Steenrod, theorem 6.4c for $\tilde{c} \in H^n(\partial s, \partial s^{n-1})$ $(\delta \tilde{c})(s) = \sum (-1)^k i_k^*(\tilde{c})(s')$. So we have for $c \in H^n(\overline{K}^n, \overline{K}^{n-1})$ $(i_\sigma^*(\delta c))(s) = (\delta(j^*(c)))(s) = \sum(-1)^k i_k^* j^*(c)(s') = \sum(-1)^k i^*_{\partial_k \sigma}(s')$.

Since singular homology and cohomology are additive, the above proves that the simplicial homology and cohomology groups, being naturally equivalent to singular, are topological invariants. Moreover, we have the following:

12. Theorem- On the category of simplicial pairs,

a) Homology case- any additive homology theory is naturally equivalent to singular homology theory.

b) Cohomology case- any additive cohomology is naturally equivalent to singular cohomology.

Let $\mathcal{W}$ be a full subcategory of pairs of topological spaces, containing all simplicial pairs and such that if (X,A) is in $\mathcal{W}$, then X and A have the homotopy type of CW complexes.

13. Theorem- On the category $\mathcal{W}$, any additive homology theory is naturally equivalent to singular homology theory and any additive cohomology theory is naturally equivalent to singular cohomology theory.

Proof: J.H.C. Whitehead (2) constructs a functor S'' from the category of topological spaces to the full subactegoyy of (possibly infinite) simplicial complexes ($S''(X)$ is the second derivation of the realization of the singular complex $S(X)$). If $A \subset X$, then $S''(A)$ is a subcomplex of $S''(X)$ and if $f : (X,A) \to (X',A')$ is a map of pairs then $S''f|S''(A) = S''f|A$, and so $S''f$ is a map of pairs $(S''X, S''A) \to (S''X', S''A')$. There is also defined a natural transformation ω from S'' to the identity functor, ie. $\omega : S''(X) \to X$ which induces then by Whitehead's theorem (I. theorem 27), ω is a homotopy equivalence.

Let H_* and H^* be respectively the singular homology and singular cohomology functors.

Let $H_\#$ be another additive homology theory. Since $\omega : S''X \to X$ and $\omega : S''A \to A$ are homotopy equivalences, the "five lemma" implies that ω_* and $\omega_\#$ are isomorphisms for all pairs (X,A) in $\mathcal{W}$. Let $h : H_* \to H_\#$ be the natural equivalence of H_* with $H_\#$ on simplicial pairs, which we know exists by theorem 12. The required equivalence is $\omega_\# h\, \omega_*^{-1}$, ie. $H_*(X,A) \xleftarrow{\omega_*} H_*(S''X, S''A) \xrightarrow{h} H_\#(S''X, S''A) \xrightarrow{\omega_\#} H_\#(X,A)$. Naturality and commutativity with the boundary operators easily follows from the naturality of and the corresponding property of h.

Similarly, if $H^\#$ is another additive cohomology theory and $h : H^* \to H^\#$ is a natural equivalence on simplicial pairs, then the required equivalence is given by $\omega^{\#-1} h\, \omega^*$, ie.
$H^*(X,A) \xrightarrow{\omega^*} H^*(S''X, S''A) \xrightarrow{h} H^*(S''X, S''A) \xleftarrow{\omega^\#} H^\#(X,A)$.

This completes the proof of Milnor's characterization of singular homology and cohomology theory. However, the expression of the chain complexes C_* and C^* given by lemma 4, and the isomorphism of theorem 9, are themselves useful tools and we shall spend the rest of the chapter examining them, for singular homology and cohomology. We note that since the singular theories are defined on all topological spaces, theorems 1 and 2 hold for singular. So for the remainder of the chapter, H_* and H^* will be singular homology and cohomology with C_* and C^* the corresponding chain complexes.

We will now begin to consider different coefficient groups. So we will write $C_*(K,L;G)$ for the chain groups of $H_*(K,L;G)$ and similarly for cohomology. $C_*(K,L)$ and $C^*(K,L)$ will be reserved for G = the integers, Z.

We will begin by investigating the relation between the chain complexes C^* and C_* and the singular complex S. It is desirable that the isomorphism of HC_* with H_* be realized by an actual chain map of S to C_*. We can almost get this.

Given a CW complex K, consider its singular complex, $S(K)$. $S_n(K)$ is the free group on continuous mappings $T : s^n \longrightarrow K$. Now $(s^n, \partial s^n)$ is a simplicial pair and is hence a CW pair. So we can consider $S^c(K)$ a subcomplex of $S(K)$, where $S^c_n(K)$ is the free group on those maps $T : s^n \longrightarrow K$, which are cellular. If T is cellular then $T^{(i)}$ is, and so this actually is a subcomplex of $S(K)$. We assert that $S^c(K)$ is an admissable subcomplex of $S(K)$, in the sense of Eilenberg and Zilber (3), The two conditions that must be verified are: (i) For x_0 a fixed base point of K, the constant simplices $s^n \longrightarrow x_0$ are in $S^c(K)$, and (ii) If $T \in S_n(K)$ and $T^{(i)} \in S^c(K)$, for each i, then there exists $T' \in S^c_n(K)$, such that T is homotopic to T' (rel ∂s^n). Choosing the base point x_0 to be some point of K^0, (i) is true and (ii) follows from the cellular approximation theorem. It follows (see Eilenberg and Zilber (3)) that the inclusion map $S^c(X) \subset S(X)$ is a (chain) homotopy equivalence.

If L is a subcomplex of K, then $S^c(L) = S(L) \cap S^c(K)$ and hence $S^c(K)$ is relatively admissable, and hence the inclusion of pairs $(S^c(K), S^c(L)) \subset (S(K), S(L))$ is a (chain) homotopy equivalence. It follows from this that the map induced by inclusions : $S^c(K)/S^c(L) \to S(K)/S(L)$ is also a homotopy equivalence. Since We also have $(S^c(K) \otimes G, S^c(L) \otimes G) \subset (S(K) \otimes G, S(L) \otimes G)$ is a homotopy equivalence, we get that $(S^c(K)/S^c(L)) \otimes G \to (S(K)/S(L)) \otimes G$ is a homotopy equivalence, and $\mathrm{Hom}(S(K)/S(L), G) \to \mathrm{Hom}(S^c(K)/S^c(L), G)$ is a homotopy equivalence.

14. Theorem- There exist natural maps

$\alpha : (S^c(K)/S^c(L)) \otimes G \to C_*(K,L;G)$ and

$\beta : C^*(K,L;G) \to \mathrm{Hom}(S^c(K)/S^c(L);G)$ which induce the isomorphisms of theorem 9.

Proof: $\alpha((\mathrm{cls}\ T) \otimes g) = T_*(gs)$, where $T : (s, \partial s) \to (\overline{K}^n, \overline{K}^{n-1})$ $(s = s^n)$, since T is cellular. We note that $(T, g) \to T_*(gs)$ is bilinear and hence is a a well-defined map of the tensor product $S^c(K) \otimes G$. Furthermore, if $T \in S(L)$, then T factors through (L,L) and hence T_* has image zero in $H_n(\overline{K}^n, \overline{K}^{n-1})$. α is thus well defined.

$\beta(c)(\mathrm{cls}\ T)$ (where $c \in H^n(\overline{K}^n, \overline{K}^{n-1})$) $= T^*(c)(s)$, where $T : (s, \partial s) \to (\overline{K}^n, \overline{K}^{n-1})$ $(s = s^n)$. As above if $T \in S(L)$, then $T^* = 0$ and so β is well-defined.

To prove that α induces the main isomorphism:

We have the diagram:

$$
\begin{array}{ccccc}
H^c_n(K,L) & \xleftarrow{i^c_*} & H^c_n(\overline{K}^n, L) & \xrightarrow{j^c_*} & H^c_n(\overline{K}^n, \overline{K}^{n-1}) \\
\downarrow k & & \downarrow k' & & \downarrow k'' \\
H_n(K,L) & \xleftarrow{i_*} & H_n(\overline{K}^n, L) & \xrightarrow{j_*} & H_n(\overline{K}^n, \overline{K}^{n-1})
\end{array}
$$

We will prove that α induces $k'' j^c_* i^{c-1}_*$ which equals $j_* i_*^{-1} k$

since k, k' and k" are isomorphisms.

Let $\Sigma_i(\text{cls } T_i)\otimes g_i$ represent a homology class of $H^c_n(K,L;G)$. $k''\, j^c_*\, i^{c-1}_*(\{\Sigma_i(\text{cls } T_i)\otimes g_i\})$ is represented, as a class in $H_n(\overline{K}^n,\overline{K}^{n-1})$ by $\Sigma_i(\text{cls}' T_i)\otimes g_i$. Since $T_{i*}(\{\text{cls } 1_s\otimes g_i\}) = \{\text{cls}' T_i\otimes g_i\}$, we will be finished if we can prove that cls $1_s\otimes g$ represents the homology class, gs, of $H_n(s,\partial s)$. If $n = 0$, this is by definition of the identification of G as the coefficient group by $g \dashrightarrow \{1_{s^0}\otimes g\}$. To complete an inductive proof, it suffices to show that the incidence isomorphism $[s^n:s^{n-1}]$ takes $\{\text{cls } 1_{s^n}\otimes g\}$ into $\{\text{cls } 1_{s^{n-1}}\otimes g\}$. The indidence isomorphism is the boundary of the triad, $(s^n;s^{n-1},c^{n-1})$ (c^{n-1} is the faces of s^n other than $s^{n-1} = \partial_0 s^n$):

$$H_n(s^n,\partial s^n) \xrightarrow{\partial} H_{n-1}(\partial s^n) \xrightarrow{\theta} H_{n-1}(\partial s^n, c^{n-1}) \longleftarrow H_{n-1}(s^{n-1},\partial s^{n-1}).$$

$$\{\text{cls } 1_{s^n}\otimes g\} \longrightarrow \{\Sigma(-1)^k\, 1_{\partial_k s^n}\otimes g\} \longrightarrow \{\Sigma(-1)^k\, \text{cls}'\, 1_{\partial_k s^n}\otimes g\} =$$

$\{\text{cls}' 1_{s^{n-1}}\otimes g\} \longleftarrow \{\text{cls } 1_{s^{n-1}}\otimes g\}$ (Notation following Eilenberg and Steenrod pages 78-79).

To prove that ρ induces the main isomorphism:

We have the diagram:

$$\begin{array}{ccccc}
H^n(K,L) & \xrightarrow{i^*} & H^n(\overline{K}^n,L) & \xleftarrow{j^*} & H^n(\overline{K}^n,\overline{K}^{n-1}) \\
k\downarrow & & k'\downarrow & & k''\downarrow \\
H^n_c(K,L) & \xrightarrow{i^*_c} & H^n_c(\overline{K}^n,L) & \xleftarrow{j^*_c} & H^n_c(\overline{K}^n,\overline{K}^{n-1})
\end{array}$$

We will show that ρ induces $i^{*-1}_c j^*_c\, k'' | Z^n(K,L)$ which $= k\, i^{*-1} j^* | Z^n(K,L)$, since k, k' and k" are isomorphisms.

Let $c : S_n(\overline{K}^n)/S_n(\overline{K}^{n-1}) \dashrightarrow G$ represent an element of $Z^n(K,L)$, that is, if δ_n is the coboundary operator $C^n(K,L) \dashrightarrow C^{n+1}(K,L)$, then $\delta_n\{c\} = 0$. To interpret what this means about the homomorphism c, we look at the short exact sequence of complexes:

$0 \longrightarrow \mathrm{Hom}(S(\overline{K}^{n+1})/S(\overline{K}^{n}),G) \longrightarrow \mathrm{Hom}(S(\overline{K}^{n+1})/S(\overline{K}^{n-1}),G) \longrightarrow$

$\mathrm{Hom}(S(\overline{K}^{n})/S(\overline{K}^{n-1}),G) \longrightarrow 0.$

δ_n is the connecting homomorphism of the cohomology sequence of this short exact sequence. To say that $\delta_n\{c\} = 0$ is to say that c can be pulled back to a c' in $\mathrm{Hom}(S_n(\overline{K}^{n+1})/S_n(\overline{K}^{n-1}),G)$ such that $\delta c'$ is a coboundary when pulled back to $\mathrm{Hom}(S(\overline{K}^{n+1})/S(\overline{K}^{n}),G)$, ie. is δd, and replacing d' by c'-d if necessary, we can assume that $\delta c' = 0$. Thus, to say that $\delta\{c\} = 0$ for $c : S_n(\overline{K}^{n})/S_n(\overline{K}^{n-1}) \longrightarrow G$, is to say that there exists a $c' : S_n(\overline{K}^{n+1})/S_n(\overline{K}^{n-1}) \longrightarrow G$ extending c, and such that $c'\partial = 0$. We obviously have that $j_c^* k''\{c\}$ is represented by the homomorphism $: S_n^c(\overline{K}^{n})/S_n^c(K) \longrightarrow S^c(\overline{K}^{n})/S^c(\overline{K}^{n-1}) \xrightarrow{c} G$. By the above conditions, c' is a cocycle of $\mathrm{Hom}(S_n^c(K)/S_n^c(L),G)$ and hence represents $i_c^{*-1} j_c^* k'' c$. For cls T in $S_n^c(K)/S_n^c(L)$, $c'(\mathrm{cls}\ T) = c(\mathrm{cls}\ T) = T^*(c)(\mathrm{cls}\ 1_{s^n})$. As in the homology case we reduce to a question in $H^n(s^n, \partial s^n)$.

We must prove that if $\tilde{c}$ represents an element of $H^n(s^n, \partial s^n)$, then $\{\tilde{c}\}(s) = \tilde{c}(1_s)$. By definition, $\{\tilde{c}\}(s) = g$, where $\{\tilde{c}\} = gs$. So it suffices to prove that $\langle gs, \{\mathrm{cls}\ 1_s\}\rangle = g$ where $\langle\ ,\ \rangle$ is the Kronecker index, which commutes with boundary operators and hence with the incidence isomorphisms, so the above follows by induction, and the fact (proved in the last section) that $\{\mathrm{cls}\ 1_s\}$ represents the homology class gs.

We note that α and β are certainly natural under cellular maps.

We next turn to the universal coefficient theorems.

15. Theorem- There exist natural isomorphisms $\eta : C_*(K,L)\otimes G \longrightarrow C_*(K,L:G)$ and $\zeta : C^*(K,L;G) \longrightarrow \mathrm{Hom}(C_*(K,L),G)$ which commute with α and β. These are the maps given by the universal coefficient theorems.

Proof: By "given by the universal coefficient theorems" we mean that $\eta : H_n(\overline{K}^n,\overline{K}^{n-1}) \otimes G \longrightarrow H_n(\overline{K}^n,\overline{K}^{n-1};G)$ and $\zeta : H^n(\overline{K}^n,\overline{K}^{n-1};G) \longrightarrow \mathrm{Hom}(H_n(\overline{K}^n,\overline{K}^{n-1}),G)$ are the universal coefficient maps for each n. These are certainly natural and commute with boundari and when we prove commutativity with α and β, we will have commutativity on homology, with the main isomorphism, by theorem 14. We must show that the following commute:

$$\begin{tikzcd}
S^c(K)/S^c(L)\otimes G \arrow[d, "\alpha\otimes 1_G"] \arrow[dr, "\alpha"] & \\
C_*(K,L)\otimes G \arrow[r, "\eta"] & C_*(K,L;G)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(S^c(K)/S^c(L),G) & \\
\mathrm{Hom}(C_*(K,L),G) \arrow[u, "\mathrm{Hom}(\alpha,G)"] & C^*(K,L;G) \arrow[l, "\zeta"'] \arrow[ul, "\beta"']
\end{tikzcd}$$

The first requires, for a pair cls $T\otimes g$, that $(T_*\otimes 1_G)(\{\text{cls } 1_s\}\otimes g) = T_*(\{\text{cls } 1_s\otimes g\})$. But since $\{\text{cls } 1_s\otimes g\} = \eta(\{\text{cls } 1_s\}\otimes g)$, the former equality holds by naturality of the universal coefficient theorem map η.

The second requires that for $c \in C^*(K,L;G)$ and cls $T \in S^e(K)/S^e(L)$, that $\mathrm{Hom}(\alpha, G)\,\zeta(c)\,(T) = \varphi(c)(T)$:
The term on the left is $\zeta(c)(T_* \text{ cls } 1_s) = \mathrm{Hom}(T_*,G)\,\zeta(c)(\{\text{cls } 1_s\})$.
The term on the right is $\zeta(T^*c)(\{\text{cls } 1_s\}) = \zeta\, T^*(c)(\{\text{cls } 1_s\})$.
These two are equal by naturality of the universal coefficient theorem map ζ. (Note the appearance of ζ, in the term on the right. $\varphi(c)(T) = T^*(c)(s)$, which we have shown is the same as taking a cocycle to represent $T^*(c)$ and evaluating at cls1_s, which is the definition of $\zeta(T^*(c))(\{\text{cls } 1_s\})$, also.)

Now we return to theorem 2 and give the promised generalization.

16. Theorem— Let K be a CW complex with subcomplexes $L, K_1, K_2, \ldots$ such that $K = L \cup \bigcup_{i=1}^{\infty} K_i$, and $K_i \cap K_j \subset L$ for $i \neq j$. Let $L_i = K_i \cap L$, and let $k_i : (K_i, L_i) \subset (K,L)$ be the inclusions. Then: the homomorphisms $k_{i*} : H_*(K_i, L_i; G) \longrightarrow H_*(K, L; G)$ form an injective representation of $H_*(K,L;G)$ as a direct sum.

Proof: Let $M_j = L \cup \bigcup_{i=1}^{j} K_i$, then the inclusions $(K_i, L_i) \subset (M_j, L)$ $(i \leq j)$ induce a direct sum representation of $H_*(M_j, L)$. Now $\underrightarrow{\mathrm{Lim}}\, H_*(M_j)$ is canonically isomorphic to $H_*(K)$ and the direct limit of the constant sequence $H_*(L)$ (for each j), is canonically isomorphic to $H_*(L)$ and hence, since direct limits preserve exactness, $\underrightarrow{\mathrm{Lim}}\, H_*(M_j, L) \approx H_*(K,L)$, but the term on the left is isomorphic to $\mathrm{Lim} \sum_{i \leq j} H_*(K_i, L_i)$ by theorem 2 applied to (M_j, L), and this is isomorphic to $\sum_i H_*(K_i, L_i)$.

A direct proof of the cohomology version of this result would involve a reproving of Milnor's limit theorem for the relative case. Instead, we can obtain the cohomology version directly from theorem 16. For if we have a situation as in the hypothesis of theorem 16, we can define $\overline{K_i^m} = K_i^m \cup L_i$ and theorem 16 implies that $H_m(\overline{K_i^m}, \overline{K_i^{m-1}}) \dashrightarrow H_m(\overline{K^m}, \overline{K^{m-1}})$ is a direct sum decomposition of the latter and hence we have that $C_*(K_i,L_i) \dashrightarrow C_*(K,L)$ is a direct sum decomposition of $C_*(K,L)$ as a complex. Hence, by the identification ζ of theorem 15, we get that $C^*(K,L) \dashrightarrow C^*(K_i,L_i)$ is a projective representation of $C^*(K,L)$ as a direct product. Such a representation is preserved when homology is taken and hence we get the dual result:

17.Corollary- Let K be a CW complex with subcomplexes $L, K_1, K_2, \ldots$ such that $K = L \cup \bigcup_{i=1}^{\infty} K_i$, and $K_i \cap K_j \subset L$ for $i \neq j$. Let $L_i = K_i \cap L$, and let $k_i : (K_i,L_i) \subset (K,L)$ be the inclusion map. Then: the homomorphisms $k_i^* : H^*(K,L;G) \dashrightarrow H^*(K_i,L_i;G)$ form a projective representation of $H^*(K,L;G)$ as a direct product.

We also formalize the under lying result about the chain an cohhain complexes:

18.Corollary- Let (K,L), (K_1,L_1), (K_2,L_2),... be as in the hypothesis of theorem 16. Then:

a) Homology case- $C_*(K_i,L_i) \dashrightarrow C_*(K,L)$ is an injective representation of $C_*(K,L)$ as a direct sum.

b) Cohomology case- $C^*(K,L) \dashrightarrow C^*(K_i,L_i)$ is a projective representation of $C^*(K,L)$ as a direct product.

We now turn to an examination of the homology of product complexes. We will require in our constructions, the notions associated with that of simplicial object, and simplicial modules, ie. semi-simplicial complexes, and the Alexander-Whitney map, as constructed in the proof of the Eilenberg-Zibber theorem, for all of which see MacLane pages 233-245.

Consider pairs of spaces (X_o,U_o) and (X_1,U_1). The singular complexes $S(X_o,U_o)$ and $S(X_1,U_1)$ are simplicial modules, where $S_n(X_o,U_o)$ is $S_n(X_o)/S_n(U_o)$. We can take their products as simplicial modules and we assert that there is a natural mapping of $S(X_o,U_o)xS(X_1,U_1)$ into the singular complex of the product $S((X_o,U_o)x(X_1,U_1)) = S(X_oxX_1, U_oxX_1\cup X_oxU_1)$.

$(S(X_o,U_o)xS(X_1,U_1))_n = S_n(X_o,U_o)\otimes S_n(X_1,U_1) =$

$S_n(X_o)/S_n(U_o)\otimes S_n(X_1)/S_n(U_1) \approx$ (since the singular complexes are free) $S_n(X_o)\otimes S_n(X_1)/S_n(U_o)\otimes S_n(X_1) + S_n(X_o)\otimes S_n(U_1)$.

Thus, we use the following exact sequence:

$$0 \longrightarrow H(U_oxU_1) \longrightarrow S(X_oxU_1)\oplus S(U_oxX_1) \longrightarrow S(X_oxX_1) \longrightarrow S(X_o,U_o)xS(X_1,U_1) \longrightarrow 0.$$

We identify $S(X_oxU_1)\oplus S(N_oxX_1)/S(U_oxU_1)$ and the image of $S(X_oxU_1)\oplus S(U_oxX_1)$ in $S(X_oxX_1)$ and we call this complex $S(X_oxU_1)+S(U_oxX_1)$. The inclusion of this complex into $S(X_oxU_1\cup U_oxX_1)$ and the identity on $S(X_oxX_1)$ induce a map k:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S(X_oxU_1)+S(U_oxX_1) & \longrightarrow & S(X_oxX_1) & \longrightarrow & S(X_o,U_o)xS(X_1,U_1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & S(X_oxU_1\cup U_oxX_1) & \longrightarrow & S(X_oxX_1) & \longrightarrow & S((X_o,U_o)x(X_1,U_1)) & \longrightarrow & 0
\end{array}$$

k is obviously natural with respect to mappings of products of pairs.

We consider three short exact sequences of complexes:

$$0 \to S(X_0,U_0)xS(U_1,A_1) \to S(X_0,U_0)xS(X_1,A_1) \to S(X_0,U_0)xS(X_1,U_1) \to 0$$

$$0 \to S(U_0,A_0)xS(X_1,U_1) \to S(X_0,A_0)xS(X_1,U_1) \to S(X_0,U_0)xS(X_1,U_1) \to 0$$

$$0 \to S(X_0xU_1 \cup U_0xX_1, A_0xX_1 \cup U_0xU_1 \cup X_0xA_1) \to S(X_0xX_1, X_0xX_1 \cup U_0xU_1 \cup X_0xA_1)$$
$$\to S(X_0xX_1, X_0xU_1 \cup U_0xX_1) \to 0$$

(For triples (X_i,U_i,A_i) $(i=0,1)$).

The corresponding exact homology sequences have boundary operators ∂_0, ∂_1 and ∂.

We assert that the following diagram commutes, giving a relation between the three boundary operators:

$$\begin{tikzcd}
H_*(S(X_0,U_0)xS(X_1,U_1)) \arrow[r, dashed, "k_*"] \arrow[dd, "\partial_0+\partial_1"'] & H_*((X_0,U_0)x(X_1,U_1)) \arrow[d, "\partial"] \\
 & H_*(X_0xU_1 \cup U_0xX_1, A_0xX_1 \cup U_0xU_1 \cup X_0xA_1) \\
\begin{array}{c} H_*(S(X_0,U_0)xS(U_1,A_1)) \\ \oplus \\ H_*(S(U_0,A_0)xS(X_1,U_1)) \end{array} \arrow[r, dashed, "k_*"] & \begin{array}{c} H_*((X_0,U_0)x(U_1,A_1)) \\ \oplus \\ H_*((U_0,A_0)x(X_1,U_1)) \end{array} \arrow[u, "i_{0*}+i_{1*}"']
\end{tikzcd}$$

Our method of proof is to amalgamate the first two sequences together and map the result into the third dequence.

If we consider the 3x3 diagram of which the first two sequences are the right and bottom rows, we get the short exact sequence:

$$0 \to (S(U_0,A_0)xS(X_1,A_1) \oplus S(X_0,A_0)xS(U_1,A_1))/S(U_0,A_0)xS(U_1,A_1)$$
$$\to S(X_0,A_0)xS(X_1,A_1) \to S(X_0,U_0)xS(X_1,U_1) \to 0$$

However, the first group is isomorphic, by a switchback, to $S(X_o,U_o) x S(U_1,A_1) + S(U_o,A_o) x S(X_1,U_1)$ and thus we have our "amalgamated" sequence by substituting this and replacing the first inclusion by an obvious (that is, obvious from the 3x3 diagram, the drawing of which is left to the reader), switchback.

The map of the amalgamated sequence onto the third sequence is given by an intermediate (but not exact) sequence:

$$0 \to S(X_o,U_o) x S(U_1,A_1) \oplus S(U_o,A_o) x S(X_1,U_1) \to S(X_o,A_o) x S(X_1,A_1) \to S(X_o,U_o) x S(X_1,U_1) \to 0$$

$$\Big\downarrow k$$

$$S((X_o,U_o) x (U_1,A_1)) \oplus S((U_o,A_o) x (X_1,U_1)) \to S((X_o,A_o) x (X_1,A_1)) \to S((X_o,U_o) x (X_1,U_1))$$

$$\Big\downarrow \text{incl.}$$

$$0 \to S(X_o x U_1 \cup U_o x X_1, A_o x X_1 \cup U_o x U_1 \cup X_o x A_1) \to S(X_o x X_1, A_o x X_1 \cup U_o x U_1 \cup X_o x A_1) \to S(X_o x X_1, X_o x U_1 \cup U_o x X_1) \to 0$$

The triplet of maps represented by k, commute because k is natural. The triplet represented by incl. are all inclusions and hence commute with the inclusions which define the middle and lower sequences.

This proves our result since the boundary of the first sequence is easily proven to be $\partial_o + \partial_1$. Furthermore, k followed by the inclusions induce k_* and $(i_{o*} + i_{1*})k_*$ on the two end complexes.

Consider the situation when U_i is open in X_i $(i=0,1)$. $F = \{X_0 x U_1, U_0 x X_1\}$ is an open cover of $X_0 x U_1 \cup U_0 x X_1$ and, in the notation of Eilenberg and Steenrod VII. theorem 8.2, S

$$S(X_0 x U_1)+S(U_0 x X_1) = S(X_0 x U_1 \cup U_0 x X_1, F) \subset S(X_0 x U_1 \cup U_0 x X_1),$$

and the inclusion is a chain equivalence by the theorem just mentioned. It follows that in this case k is a chain equivalence.

Assume now that U_i is a deformation retract of V_i in X_i, with V_i open $(i=0,1)$, and that $X_0 x U_1 \cup U_0 x X_1$ is a deformation retract of $X_0 x V_1 \cup V_0 x X_1$, then in the commutative diagram:

$$\begin{array}{ccc} S(X_0,U_0) x S(X_1,U_1) & \xrightarrow{k} & S((X_0,U_0)x(X_1,U_1)) \\ \downarrow & & \downarrow \\ S(X_0,V_0) x S(X_1,V_1) & \xrightarrow{k} & S((X_0,V_0)x(X_1,V_1)) \end{array}$$

the bottom map and the two inclusions induce isomorphisms of homology and cohomology and hence so does the top map.

The other homological map which we will require is the Alexander-Whitney map $f: S(X_0,U_0) x S(X_1,U_1) \to S(X_0,U_0) \otimes S(X_1,U_1)$ which is a natural map which induces isomorphisms of homology and cohomology groups (it is, in fact, a chain equivalence).

Using these two homomorphisms and the Künneth formula we will define the homology product of pairs of spaces: $p: H_*(X_0,U_0) \otimes H_*(X_1,U_1) \to H_*((X_0,U_0)x(X_1,U_1))$. It is defined by setting $p = k_* f_*^{-1} \xi$, where ξ is the map of the Künneth formula $\xi: H_*(X_0,U_0) \otimes H_*(X_1,U_1) \to H_*(S(X_0,U_0) \otimes S(X_1,U_1))$. The homology product is natural since [illegible] over the index set of such pairs $(j,1)$ is [illegible] in the [illegible].

k, f and ξ are each natural.

For triples (X_i,U_i,A_i) $(i=0,1)$ we assert that the following:

$$\begin{tikzcd}
H_*(X_0,U_0)\otimes H_*(X_1,U_1) \arrow[r, "p"] \arrow[dd, "\partial\otimes 1 + 1\otimes\partial"'] & H_*((X_0,U_0)\times(X_1,U_1)) \arrow[d, "\partial"] \\
 & H_*(X_0\times U_1\cup U_0\times X_1,\ A_0\times X_1\cup U_0\times U_1\cup X_0\times A_1) \\
\begin{matrix} H_*(X_0,U_0)\otimes H_*(U_1,A_1) \\ \oplus \\ H_*(U_0,A_0)\otimes H_*(X_1,U_1) \end{matrix} \arrow[r, "p"] & \begin{matrix} H_*((X_0,U_0)\times(U_1,A_1)) \\ \oplus \\ H_*((U_0,A_1)\times(X_1,U_1)) \end{matrix} \arrow[u, "i_{0*}+i_{1*}"']
\end{tikzcd}$$

commutes.

For we know that f is a mapping of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & S(X_0,U_0)\times S(U_1,A_1) \arrow[r] \arrow[d, "f"] & S(X_0,U_0)\times S(X_1,A_1) \arrow[r] \arrow[d, "f"] & S(X_0,U_0)\times S(X_1,U_1) \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & S(X_0,U_0)\otimes S(U_1,A_1) \arrow[r] & S(X_0,U_0)\otimes S(X_1,A_1) \arrow[r] & S(X_0,U_0)\otimes S(X_1,U_1) \arrow[r] & 0
\end{tikzcd}$$

Hence, f induces a mapping of the homology sequences and thus commutes with the boundary operator. ξ, as the Künneth formula map, is natural with respect to connecting homomorphisms and hence we have the following commutative diagram:

$$\begin{tikzcd}
H_*(X_0,U_0)\otimes H_*(X_1,U_1) \arrow[r, "\xi"] \arrow[d, "1\otimes\partial"] & H_*(S(X_0,U_0)\otimes S(X_1,U_1)) \arrow[d, "\partial_0"] & H_*(S(X_0,U_0)\times S(X_1,U_1)) \arrow[l, "f_*"'] \arrow[d, "\partial_0"] \\
H_*(X_0,U_0)\otimes H_*(U_1,A_1) \arrow[r] & H_*(S(X_0,U_0)\otimes S(U_1,A_1)) & H_*(S(X_0,U_0)\times S(U_1,A_1)) \arrow[l, "f_*"']
\end{tikzcd}$$

And similarly for ∂_1. Combining this diagram with the previously constructed diagram for k_* gives the required result.

Now given two CW pairs (K_0,L_0) and (K_1,L_1), we will show that $C_*(K_0,L_0)\otimes S_*(K_1,L_1)$ is isomorphic as a chain complex to $C_*((K_0,L_0)\times_k(K_1,L_1))$.

We note first, that the singular complexes of $K\times K'$ and $K\times_k K'$ are identical and hence in the homology product formulae

As a preliminary, we consider the homology product on $H_*(\overline{K}_o^i,\overline{K}_o^{i-1}) \otimes H_*(\overline{K}_1^j,\overline{K}_1^{j-1})$, with $i+j = n$.

First, we note that Σ is an isomorphism since the Tor term in the Kunneth formula is zero. f_* we know to be an isomorphism, and for k_*, we let $V_o = \{\overline{K}_o^i - i_\sigma(c_\sigma) : \sigma \in I^i_{K_o} - I^i_{L_o}\}$, c_σ = center e_σ, and similarly, define V_1. V_o and V_1 are open in $\overline{K}_o^i$ and $\overline{K}_1^j$ respectively and it is easily seen that $\overline{K}_o^{i-1}$ and $\overline{K}_1^{j-1}$ are strong deformation retracts of V_o and V_1, respectively, and that $\overline{K}_o^{i-1} x \overline{K}_1^j \cup \overline{K}_o^i x \overline{K}_1^{j-1}$ is a strong deformation retract of $V_o x \overline{K}_1^j \cup \overline{K}_o^i x V_1$. Hence, k_* is an isomorphism and it follows that the homology product on $H_*(\overline{K}_o^i,\overline{K}_o^{i-1}) \otimes H_*(\overline{K}_1^j,\overline{K}_1^{j-1})$ is an isomorphism.

$$(H_*(\overline{K}_o^i,\overline{K}_o^{i-1}) \otimes H_*(\overline{K}_1^j,\overline{K}_1^{j-1}))_n = \sum_{m+q=n} H_m(\overline{K}_o^i,\overline{K}_o^{i-1}) \otimes H_q(\overline{K}_1^j,\overline{K}_1^{j-1})$$
$$= \text{(by corollary 5) } H_i(\overline{K}_o^i,\overline{K}_o^{i-1}) \otimes H_j(\overline{K}_1^j,\overline{K}_1^{j-1}).$$

We now define the map $\gamma : C_*(K_o,L_o) \otimes C_*(K_1,L_1) \longrightarrow C_*((K_o,L_o) x_k (K_1,L_1))$, on the n^{th} group as follows:

$$\begin{array}{ll}
\sum_{i+j=n} H_i(\overline{K}_o^i,\overline{K}_o^{i-1}) \otimes H_j(\overline{K}_1^j,\overline{K}_1^{j-1}) & \\
\quad\quad \big\downarrow \Sigma p_{ij} & p_{ij} - \text{ homology product on the } (i,j) \text{ pair} \\
\sum_{i+j=n} H_n((\overline{K}_o^i,\overline{K}_o^{i-1}) x_k (\overline{K}_1^j,\overline{K}_1^{j-1})) & \\
\quad\quad \big\downarrow \Sigma m_{ij*} & m_{ij} - \text{ inclusion of the } (i,j) \text{ pair} \\
H_n((\overline{K_o x_k K_1})^n, (\overline{K_o x_k K_1})^{n-1}) &
\end{array}$$

Since each p_{ij} is an isomorphism, Σp_{ij} is, and Σm_{ij*} is an isomorphism, by theorem 2.

To prove that γ is a chain map, we observe that the following diagram commutes:

$$\begin{tikzcd}
H_n((\bar K_0^i,\bar K_0^{i-1})x_k(\bar K_1^j,\bar K_1^{j-1})) \arrow[r, "m_{ij*}"] \arrow[d, "\partial"] & H_n((\overline{K_0x_kK_1})^n,(\overline{K_0x_kK_1})^{n-1}) \arrow[d, "\partial_n"] \\
H_{n-1}(\bar K_0^i x_k \bar K_1^{j-1}\cup \bar K_0^{i-1}x_k\bar K_1^j,\ \bar K_0^{i-2}x_k\bar K_1^j\cup \bar K_0^{i-1}x_k\bar K_1^{j-1}\cup \bar K_0^i x_k \bar K_1^{j-2}) \arrow[r, "m_{ij*}"] & H_{n-1}((\overline{K_0x_kK_1})^{n-1},(\overline{K_0x_kK_1})^{n-2}) \\
H_{n-1}((\bar K_0^i,\bar K_0^{i-1})x_k(\bar K_1^{j-1},\bar K_1^{j-2})) \oplus H_{n-1}((\bar K_0^{i-1},\bar K_0^{i-2})x_k(\bar K_1^j,\bar K_1^{j-1})) \arrow[u, "m_{i,j-1*}+m_{i-1,j*}"] \arrow[ur, "m_{i,j-1*}+m_{i-1,j*}"'] &
\end{tikzcd}$$

Patching this diagram together with the diagram for p_{ij} on the triples $(\bar K_0^i,\bar K_0^{i-1},\bar K_0^{i-2})$ and $(\bar K_1^j,\bar K_1^{j-1},\bar K_1^{j-2})$, and summing over pairs (i,j) with $i+j = n$, gives the diagram which proves that $\gamma\partial_n = \partial_n\gamma$, ie. that γ commutes with the boundary operator of chains and is hence a chain map.

19. Theorem- For CW pairs (K_0,L_0) and (K_1,L_1), there are natural isomorphisms:

$$C_*(K_0,L_0;G_0)\otimes C_*(K_1,L_1;G_1) \approx C_*((K_0,L_0)x_k(K_1,L_1);G_0\otimes G_1)$$
$$C^*(K_0,L_0;G_0)\otimes C^*(K_1,L_1;G_1) \approx C^*((K_0,L_0)x_k(K_1,L_1);G_0\otimes G_1).$$

Proof: These isomorphisms could be constructed directly as γ was constructed above, but instead we will use γ and the isomorphisms η and ζ of theorem 15, to prove the results:

For homology: $C_*(K_0,L_0;G_0)\otimes C_*(K_1,L_1;G_1) \xleftarrow{\eta\otimes\eta}$
$C_*(K_0,L_0)\otimes G_0\otimes C_*(K_1,L_1)\otimes G_1 \xrightarrow{a} C_*(K_0,L_0)\otimes C_*(K_1,L_1)\otimes G_0\otimes G_1$
$\xrightarrow{\gamma\otimes G_0\otimes G_1} C_*((K_0,L_0)x_k(K_1,L_1))\otimes G_0\otimes G_1 \xrightarrow{\eta} C_*((K_0,L_0)x_k(K_1,L_1);G_0\otimes G_1)$

For cohomology: $C^*(K_0,L_0;G)\otimes C^*(K_1,L_1;G) \xrightarrow{\zeta_0\otimes\zeta_1}$
$\mathrm{Hom}(C_*(K_0,L_0),G_0)\otimes \mathrm{Hom}(C_*(K_1,L_1),G_1) \xrightarrow{b}$

$$\mathrm{Hom}(C_*(K_o,L_o)\otimes C_*(K_1,L_1),G_o\otimes G_1) \xleftarrow{\mathrm{Hom}(\gamma,G_o\otimes G_1)}$$
$$\mathrm{Hom}(C_*((K_o,L_o)x_k(K_1,L_1)),G_o\otimes G_1) \xleftarrow{\zeta} C^*((K_o,L_o)x_k(K_1,L_1);G_o\otimes G_1)$$

Where a is the "middle four interchange" and b is the "Hom-$\otimes$ interchange" (see MacLane pages 194-195). This first is always an isomorphism and the second is an isomorphism in this case because $C_*(K_o,L_o)$ and $C_*(K_1,L_1)$ are both complexes of free abelian groups.

We shall, in the following chapters, use the isomorphisms of theorem 17 as identifications. We also use as an identification, the influence of γ "on the cells". That is, if we, suggestively, if ambiguously, let σ also represent $i_{\sigma*}$(a generator of $H_n(e^n,S^{n-1})$) in $H_n(\overline{K}^n_o,\overline{K}^{n-1}_o)$, and similarly τ for $H_n(\overline{K}^n_1,\overline{K}_1^{\,n-1})$, then it is easily verified that $\gamma(\sigma\otimes\tau) = \sigma x\tau$.

The case in which we are most interested will be, of course, where $(K_1,L_1) = (I,\emptyset)$ or (I,S^o). In these cases, x_k is just x.

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