Chapter IV Spectral Sequences:

The theory of spectral sequences was introduced by J. Leray [1]. Leray obtained spectral sequences from differential filtered modules (see below). A more general procedure of obtaining spectral sequences was introduced by W. S. Massey in his theory of exact couples [2]. Yet another way of obtaining spectral sequences was introduced by S. Eilenberg, and is expounded in his forthcoming book with H. Cartan [3]. This method has the advantage that there is both an inductive and a direct definition of the term E^r in the spectral sequence, and consequently will be followed here.

Notation and Conventions: Let $\tilde{Z}$ be the set $Z \cup \{-\infty, \infty\}$. Order $\tilde{Z}$ by $-\infty < r < \infty$ for $r \in Z$.

Definition 4.1: Let $\mathcal{A}$ be the category such that

1) objects of $\mathcal{A}$ are pairs (p,q) of elements of $\tilde{Z}$ such that $p \geq q$, and

2) a map in $\mathcal{A}$ is an assignment to an object (p,q) in $\mathcal{A}$ another object (p',q') in $\mathcal{A}$ such that $p' \geq p$, $q' \geq q$.

If $\alpha : (p,q) \longrightarrow (p',q')$ and $\beta : (p',q') \longrightarrow (p'',q'')$ are maps in $\mathcal{A}$ we say that (α, β) is a couple if $q = q'$, $p' = p''$, and $q'' = p$ (see [4], p. 114). In other words there is a correspondence between couples and triples (p,q,r) of elements of $\tilde{Z}$ such that $p \geq q \geq r$, the correspondence being that which assigns to the triple (p,q,r) the couple (α,β), where $\alpha : (q,r) \longrightarrow (p,r)$, and $\beta:(p,r) \longrightarrow (p,q)$.

Notation: Let Λ be a commutative ring with unit. Denote by $\mathcal{G}_\Lambda$ the category of Λ-modules and Λ-homomorphisms, and by $\mathcal{G}_\Lambda{}'$ the category of graded Λ-modules and graded Λ-homomorphisms.

Definition 4.2: A covariant ∂-functor on the category with couples $\mathcal{A}$ consists of a covariant functor $H:\mathcal{A} \longrightarrow \mathcal{G}_\Lambda$ together with a homomorphism $\partial_{(\alpha,\beta)} : H(C) \longrightarrow H(A)$ for each couple (α,β) in $\mathcal{A}$, $\alpha: A \longrightarrow B$, $\beta:B \longrightarrow C$, satisfying the following condition:

1) if

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} \\ A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C_1 \end{array}$$

is a commutative diagram in $\mathcal{A}$, where (α,β) and (α_1, β_1) are couples, then

$$\begin{array}{ccc} H(C) & \xrightarrow{\partial(\alpha,\beta)} & H(A) \\ \downarrow{\scriptstyle H(\gamma_3)} & & \downarrow{\scriptstyle H(\gamma_1)} \\ H(C_1) & \xrightarrow{\partial(\alpha_1,\beta_1)} & H(A_1) \end{array}$$

is a commutative diagram.

2) For every couple (α,β) in $\mathcal{A}$, $\alpha:A \longrightarrow B$, $\beta:B \longrightarrow C$,

the sequence

$$\cdots \longrightarrow H(A) \xrightarrow{H(\alpha)} H(B) \xrightarrow{H(\beta)} H(C) \xrightarrow{\partial(\alpha,\beta)} H(A) \longrightarrow \cdots$$

is exact.

If $H: \mathcal{A} \longrightarrow \mathcal{G}'_\Lambda$ satisfies 1) and 2) above, and if in addition $\partial_{(\alpha,\beta)}: H_{n+1}(C) \longrightarrow H_n(A)$ for every couple $(\alpha,\beta): A \longrightarrow B \longrightarrow C$, then H will be said to be a <u>graded covariant ∂-functor</u> on $\mathcal{A}$ ([4], p. 115).

<u>Definition 4.3:</u> Let M be a differential Λ-module. A <u>filtration</u> on M is a set of submodules $\{F_pM\}_{p \in \widetilde{Z}}$ such that

1) $F_pM \subset F_{p+1}M$,

2) $dF_pM \subset F_pM$,

3) $F_{-\infty}M = 0$

4) $F_{\infty}M = M$

If M is a graded differential Λ-module, the filtration will be assumed to be compatible with the gradation, i.e. $F_pM = \Sigma_n (F_pM) \cap M_n$ for all $p \in \widetilde{Z}$.

The module M together with its differential operator and filtration is called a <u>differential filtered Λ-module</u>, and if it is graded it is called a <u>differential graded filtered Λ-module.</u>

<u>Definition 4.4:</u> Let $\{M, F_pM\}$ be a differential filtered Λ-module. If (p,q) is an object of $\mathcal{A}$, let $H(p,q) = H(F_pM/F_qM)$, and if $\alpha: (p,q) \longrightarrow (p',q')$ is a

map, let $H(\alpha):H(p,q) \longrightarrow H(p',q')$ be the natural map. If $\alpha:(p,r) \longrightarrow (p,r)$, $\beta:(p,r) \longrightarrow (p,q)$ is a couple in $\mathcal{a}$, then there is an exact sequence

$$0 \longrightarrow F_qM/F_rM \longrightarrow F_pM/F_rM \longrightarrow F_pM/F_qM \longrightarrow 0$$

and a resulting exact sequence

$$\cdots \longrightarrow H(q,r) \longrightarrow H(p,r) \longrightarrow H(p,q) \xrightarrow{\partial} H(q,r) \longrightarrow \cdots .$$

Let $\partial_{(\alpha,\beta)}:H(p,q) \longrightarrow H(q,r)$ be the homomorphism denoted by ∂ in this exact sequence (Henceforth $\partial_{(\alpha,\beta)}$ will be denoted merely by ∂.)

It is evident that the functor H just defined and the homomorphisms $\partial:H(p,q) \longrightarrow H(q,r)$ form a <u>covariant</u> <u>∂-functor</u> on $\mathcal{a}$, and that this functor is graded if M is graded. This covariant ∂-functor is said to be the one associated with the differential filtered Λ-module $\{M,F_pM\}$.

<u>Definition 4.5:</u> If $H:\mathcal{a} \longrightarrow \mathcal{g}$ is a covariant ∂-functor, define

$$Z_p^r = \text{Image } H(p,p-r) \longrightarrow H(p,p-1)$$

$$B_p^r = \text{Image } \partial:H(p+r-1,p) \longrightarrow H(p,p-1)$$

for $r,p \in \tilde{Z}, r \geq 2$. If H is graded, define

$$Z_{p,q}^r = \text{Image } H_{p+q}(p,p-r) \longrightarrow H_{p+q}(p,p-1)$$

$$B_{p,q}^r = \text{Image } H_{p+q+1}(p+r-1,p) \longrightarrow H_{p+q}(p,p-1)$$

<u>Lemma:</u> $\cdots Z_p^r \supset Z_p^{r+1} \supset \cdots \supset Z_p^\infty \supset B_p^\infty \supset \cdots \supset B_p^{r+1} \supset B_p^r \supset \cdots$, and $\cdots Z_{p,q}^r \supset Z_{p,q}^{r+1} \supset \cdots \supset Z_{p,q}^\infty \supset B_{p,q}^\infty \supset \cdots \supset B_{p,q}^{r+1} \supset B_{p,q}^r \supset \cdots$

The proof of this lemma is straightforward, and will be omitted.

<u>Definition 4.6:</u> If $H: \mathcal{A} \longrightarrow \mathcal{G}_\Lambda$ is a covariant ∂-functor, define $E^r_p = Z^r_p/B^r_p$ for $r, p \in \tilde{Z}$, $r \geq 2$. Define $E^1_p = H(p, p-1)$, and set $E^r = \Sigma_p E^r_p$. If H is graded, set $E^r_{p,q} = Z^r_{p,q}/B^r_{p,q}$, $E^r_p = \Sigma_q E^r_{p,q}$, $E^r = \Sigma_{p,q} E^r_{p,q}$. $\{E^r\}_{r \geq 2}$ is <u>the spectral sequence of H</u>. If H is the covariant ∂-functor associated with a differential filtered Λ-module $\{M, F_p M\}$, the spectral sequence will sometimes be denoted by $\{E^r(M)\}$. Further in this case $E^0_p(M) = F_p M/F_{p-1}M$, and $E^0(M) = \Sigma_p E^0_p(M)$.

We now have spectral sequences defined, but we have not as yet proved two of their basic properties. First, E^{r+1} should be the homology of E^r with respect to some differential operator. Second, if M is a filtered Λ-module, $E^\infty(M)$ should approximate $H(M)$ in a certain sense. We now proceed to define $d^r: E^r \longrightarrow E^r$ so that E^{r+1} will be isomorphic to $H(E^r)$.

<u>Lemma:</u> If $p \geq q \geq r \geq s$ then

$$H(p,q) \xrightarrow{\partial} H(q,r) \xrightarrow{\partial} H(r,s), \text{ and } \partial\partial = 0.$$

<u>Proof:</u> This follows immediately from the commutativity of the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H(p,r) \arrow[r] & H(p,q) \arrow[r, "\partial"] & H(q,r) \arrow[r] \arrow[d, "\partial"] & H(p,r) \arrow[r] \arrow[dl, "\partial"] & \cdots \\
 & & & H(r,s) & &
\end{tikzcd}$$

and the fact that the horizontal sequence is exact.

Definition 4.7: Notice that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H(p-1,p-r) \arrow[r] \arrow[dr, "\partial"] \arrow[drr, bend right=40, "\partial"'] & H(p,p-r) \arrow[r, "j"] \arrow[d, "\partial"] \arrow[dr, "\partial"] & H(p,p-1) \arrow[r] & \cdots \\
 & & H(p-r,p-2r) \arrow[r] & H(p-r,p-r-1) &
\end{tikzcd}$$

is commutative. Consequently there is a natural map $\partial^r : Z_p^r \longrightarrow E_{p-r}^r$ such that $\partial^r(z)$ is the equivalence class of $\partial j^{-1} z \in H(p-r,p-r-1)$. Further it follows from the commutativity of the diagram

$$\begin{tikzcd}
H(p+r-1,p) \arrow[rr, "\partial"] \arrow[dr, "\partial"'] & & H(p,p-1) \\
 & H(p,p-r) \arrow[ur] \arrow[d, "\partial"] & \\
 & H(p-r,p-r-1) &
\end{tikzcd}$$

and the fact that $\partial\partial = 0$, that $\partial^r(B_p^r) = 0$. Define $d^r : E_p^r \longrightarrow E_{p-r}^r$ to be the homomorphism induced by $\partial^r : Z_p^r \longrightarrow E_{p-r}^r$. Further denote by d^r the induced endomorphism of E^r.

Proposition 4.8: $d^r \circ d^r = 0$, and $H(E^r)$ is naturally isomorphic with E^{r+1}.

Proof: The fact that $d^r \circ d^r = 0$ follows from the diagram

$$\begin{tikzcd}
H(p,p-r) \arrow[rr] \arrow[d, "\partial"] \arrow[drr, "\partial"] & & H(p,p-1) \\
H(p-r,p-2r) \arrow[rr] \arrow[d, "\partial"] \arrow[drr, "\partial"] & & H(p-r,p-r-1) \\
H(p-2r,p-3r) \arrow[rr] & & H(p-2r,p-2r-1)\ ,
\end{tikzcd}$$

and the fact that $\partial\partial = 0$.

Further it follows from the diagram

$$\begin{tikzcd}
H(p,p-r-1) \arrow[r] \arrow[d] & H(p,p-r) \arrow[r] \arrow[d, "\partial"] & H(p,p-1) \\
0 = H(p-r-1,p-r-1) \arrow[r] & H(p-r.p-r-1) &
\end{tikzcd}$$

that the sequence

$$0 \longrightarrow Z_p^{r+1}/B_p^r \longrightarrow E_p^r \xrightarrow{d^r} E_{p-r}^r$$

is exact, or that the sequence

$$Z_p^{r+1} \longrightarrow H(E^r) \longrightarrow 0$$

is exact. From the diagram

$$\begin{tikzcd}
H(p+r,p) \arrow[rr] \arrow[d, "\partial"] \arrow[drr, "\partial"] & & H(p+r,p+r-1) \\
H(p,p-r) \arrow[rr] \arrow[drr, "\partial"] & & H(p,p-1) \\
& & H(p-r,p-r-1)
\end{tikzcd}$$

and the fact that $\partial\partial = 0$, it follows that B_p^{r+1} = kernel $Z_p^{r+1} \longrightarrow H(E^r)$, or that $E_p^{r+1} \xrightarrow{\approx} H(E^r)$

as was to be proved.

Notice that if $H: \mathcal{A} \longrightarrow \mathcal{G}'_\Lambda$, then

$d^r: E^r_{p-q} \longrightarrow E^r_{p-r,q+r-1}$

<u>Definition 4.9:</u> We define a filtration on $H(\infty, -\infty)$ by setting $F_p(H(\infty, -\infty)) = \text{Image } H(p, -\infty) \longrightarrow H(\infty, -\infty)$

<u>Proposition 4.10:</u> $E^0_p(H(\infty, -\infty))$ is naturally isomorphic to E^∞_p for all $p \in \tilde{Z}$.

<u>Proof:</u> Recall that $Z^\infty_p = \text{Image } H(p, -\infty) \longrightarrow H(p, p-1)$, $B^\infty_p = \text{Image } H(\infty, p) \xrightarrow{\partial} H(p, p-1)$. Further the sequence

$\cdots \longrightarrow H(p-1, -\infty) \longrightarrow H(p, -\infty) \longrightarrow H(p, p-1) \longrightarrow \cdots$

is exact, and Image $H(p-1, -\infty) \longrightarrow H(\infty, -\infty) = F_{p-1} H(\infty, -\infty)$. Therefore there is a natural map $E^\infty_p \longrightarrow E^0_p H(\infty, -\infty)$, and this map is clearly an epimorphism. However, it follows from a similar argument that it is a monomorphism, and the result follows.

<u>Proposition 4.11:</u> Suppose that $H: \mathcal{A} \longrightarrow \mathcal{G}'_\Lambda$ is a graded covariant ∂-functor, and

1) $H(p,q) = 0$ if $p < 0$, and

2) $H_n(p,q) = 0$ if $n \leq q$, then

$E^r_{p,q}$ is naturally isomorphic with $E^\infty_{p,q}$ for $r > \sup\{p, q+1\}$.

<u>Proof</u>: Suppose that $r > \sup\{p, q+1\}$. The horizontal sequence

$$\begin{tikzcd}
 & & H(p,p-1) & & \\
\cdots \arrow[r] & H(p-r,-\infty) \arrow[r] & H(p,-\infty) \arrow[r] \arrow[u] & H(p,p-r) \arrow[r] \arrow[ul] & \cdots
\end{tikzcd}$$

is exact, and $H(p-r,-\infty) = 0$. Therefore, $H(p,-\infty) \approx H(p,p-r)$, and $Z_p^r = Z_p^\infty$.

Further, the horizontal sequence in the diagram

$$\begin{tikzcd}
 & & H_{p+q}(p,p-1) & & \\
\cdots \arrow[r] & H_{p+q+2}(\infty,p+r-1) \arrow[r] & H_{p+q+1}(p+r-1,p) \arrow[r] \arrow[u, "\partial"] & H_{p+q+1}(\infty,p) \arrow[r] \arrow[ul, "\partial"'] & H_{p+q+1}(\infty,p+r-1)
\end{tikzcd}$$

is exact, $H_n(\infty, p+r-1) = 0$ for $n = p+q+2, p+q+1$, and hence $B^r_{p,q} = B^\infty_{p,q}$. Then the proof is complete.

<u>Definition:</u> If $H: \mathcal{A} \longrightarrow \mathcal{G}'_\Lambda$ is a graded covariant ∂ - functor, then H is <u>regular</u> if $H(p,q) = 0$ if $p < 0$, and $H_n(p,q) = 0$ if $n \leq q$; in other words if the hypotheses of the preceding proposition are fulfilled.

If $\{M, F_p M\}$ is a differential graded filtered Λ - module, then $\{F_p M\}$ is a <u>regular</u> filtration if

1) $F_p M = 0$ for $p < 0$, and
2) $M_p \subset F_p M$.

Notice that this definition assures that the covariant

∂- functor associated with $\{M, F_p M\}$ is regular. Almost all of the filtration in which we shall be interested have this property.

We are now in a position to prove the exact sequence theorem of Serre [5], which will be used extensively later in the notes.

Theorem 4.12: Suppose that $H: \mathcal{a} \longrightarrow \mathcal{G}'_\wedge$ is a regular covariant ∂- functor, and that i, j, r are positive integers with $i < j, r \geq 2$. Suppose further that if $i \leq n \leq j$ then

1) (a_n, b_n) and (c_n, d_n) are pairs of integers such that $n = a_n + b_n = c_n + d_n$, and $a_n < c_n$,
2) $E^r_{p,q} = 0$ if $p+q = n-1$, $p \leq a_n - r$, and
3) $E^r_{p,q} = 0$ if $p+q = n$, $(p,q) \notin \{(a_n,b_n),(c_n,d_n)\}$
4) $E^r_{p,q} = 0$ if $p+q = n+1$, $p \geq c_n + r$.

Under these hypotheses there is an exact sequence

$$E^r_{a_j,b_j} \longrightarrow H_j(\infty, -\infty) \longrightarrow E^r_{c_j,d_j} \longrightarrow E^r_{a_{j-1},b_{j-1}} \longrightarrow \cdots$$
$$\cdots \longrightarrow E^r_{a_i,b_i} \longrightarrow H_i(\infty, -\infty) \longrightarrow E^r_{c_i,d_i} .$$

Proof: It follows immediately from the hypotheses of the theorem that $E^\infty_{p,q} = 0$ if $p+q = n$, $(p,q) \notin \{(a_n,b_n), (c_n,d_n)\}$, where $i \leq n \leq j$. From this fact and proposition 4.10, with gradation considered, it follows that there is an exact sequence

$$0 \longrightarrow E^{\infty}_{a_n,b_n} \longrightarrow H_n(\infty, -\infty) \longrightarrow E^{\infty}_{c_n,d_n} \longrightarrow 0.$$

However it follows from 2) above that if $n > 1$, then either

a) $r \leq s = c_n - a_{n-1}$ and $E^{\infty}_{c_n,d_n}$ is the kernel of

$$d^s : E^s_{c_n,d_n} \longrightarrow E^s_{a_{n-1},b_{n-1}}, \text{ or}$$

b) $r > c_n - a_{n-1}$ and $E^{\infty}_{c_n,d_n} = E^r_{c_n,d_n}$.

Consequently there is an exact sequence

$$0 \longrightarrow E^{\infty}_{a_n,b_n} \longrightarrow H_n(\infty, -\infty) \longrightarrow E^s_{c_n,d_n} \longrightarrow E^s_{a_{n-1},b_{n-1}} .$$

However $E^s_{c_n,d_n} = E^r_{c_n,d_n}$, $E^s_{a_{n-1},b_{n-1}} = E^r_{a_{n-1},b_{n-1}}$, and $E^{\infty}_{a_{n-1},b_{n-1}}$ is the cokernel of $d^s : E^s_{c_n,d_n} \longrightarrow E^s_{a_{n-1},b_{n-1}}$ in case a, or $E^s_{a_{n-1},b_{n-1}}$ in case b. This follows from 2) and 4) in the hypotheses of the theorem. These facts combine to imply that there is an exact sequence

$$0 \longrightarrow E^{\infty}_{a_n,b_n} \longrightarrow H_n(\infty, -\infty) \longrightarrow E^r_{c_n,b_n} \longrightarrow$$

$$E^r_{a_{n-1},b_{n-1}} \longrightarrow H_{n-1}(\infty, -\infty) \longrightarrow E^{\infty}_{c_{n-1},d_{n-1}} .$$

To complete the proof it is necessary only to continue in this manner.

<u>Definiton 4.13:</u> If M, M' are filtered Λ- modules, then $f : M \longrightarrow M'$ is <u>filtration preserving</u>, or is a <u>map of filtered modules</u> if $f(F_p M) \subset F_p M'$, for $p \in \widetilde{Z}$.

If $f, g : M \longrightarrow M'$ are maps of differential filtered modules, a <u>homotopy of degree s</u> between f, g is a Λ- homomorphism $D : M \longrightarrow M'$ such that

1) $dD + Dd = f - g$, and

2) $D(F_pM) \subset F_{p+s}M'$.

If M and M' are graded, it will be assumed that $D(M_n) \subset M'_{n+1}$.

Proposition 4.14: 1) If $f:M \longrightarrow M'$ is a map of differential filtered Λ- modules, then f induces

$$f^r:E^r(M) \longrightarrow E^r(M')$$

a map of differential Λ-modules for $r \geq 0$, and further if M and M' are graded, then $f^r(E^r_{p,q}(M)) \subset E^r_{p,q}(M')$).

2) If $f,g:M \longrightarrow M'$ are maps of differential filtered Λ- modules which are homotopic by a homotopy of degree s, then $f^r = g^r$ for $r > s$.

Proof: The first part of the proposition is obvious, and its proof will be omitted.

To prove the second part, it suffices to show that if D is a homotopy of degree s between f and g, then D induces a homotopy D^s between f^s and g^s. If $x \in F_pM$ represents $[x] \in E^s_p(M)$, define $D^s[x] = [Dx] \in E^s_{p+s}(M')$. It will be left to the reader to verify that the definition is independent of the choice of representatives, and that $d^sD^s + D^sd^s = f^s - g^s$.

The preceding definition and proposition could have been extended to include maps of covariant ∂- functors on $\mathcal{A}$. However, to avoid complications we now abandon covariant

∂- functors, and for the remainder of this chapter consider only spectral sequences which arise from filtered modules.

Before proceeding to the proof of some comparison theorems, we first study coefficient sequences.

<u>Definition 4.15</u>: If N is a differential graded Λ- module, and $\mathcal{G}$ is a Λ - module, then $\mathcal{G}\otimes_\Lambda N$ is the differential graded Λ-module such that $(\mathcal{G}\otimes_\Lambda N)_q = \mathcal{G}\otimes_\Lambda N_q$, and $d(a\otimes b) = a\otimes db$ for $a \in G$, $b\in N$. The homology of $\mathcal{G}\otimes_\Lambda N$ is denoted by $H(N;\mathcal{G})$.

If $\mathcal{G}$ is graded, then $\mathcal{G}_p\otimes_\Lambda N_q$ is the submodule of gradation (p,q) of the bigraded differential module $\mathcal{G}\otimes_\Lambda N$, and $d(a\otimes b) = (-1)^p a\otimes db$ if $a\in\mathcal{G}_p, b\in N_q$. Thus $\mathcal{G}\otimes_\Lambda N = \Sigma_p\, \mathcal{G}_p\otimes_\Lambda N$. The elements of total degree (or gradation) n in $\mathcal{G}\otimes_\Lambda N$ are those of $\Sigma_{p+q=n}\,\mathcal{G}_p\otimes N_q$.

<u>Definition 4.16:</u> Let $f: M \longrightarrow M'$ be a map of differential graded modules. The <u>mapping cylinder of f</u> is the differential graded Λ- module M'' such that

1) $M''_q = M_{q-1} + M_q + M'_q$, and

2) $d(a,b,c) = (-da, db-a, dc+f(a))$.

Let $i: M \longrightarrow M''$ be the map defined by $i(b) = (0,b,0)$ $j: M'' \longrightarrow M'$ the map defined by $j(a,b,c) = f(b) + c$, and $\lambda: M' \longrightarrow M''$ by $\lambda(c) = (0,0,c)$. Let $D: M'' \longrightarrow M''$ be defined by $D(a,c,c) = (b,0,0)$.

If M,M' are filtered and f is filtration preserving, define $F_pM'' = F_{p-1}M + F_pM + F_pM'$.

<u>Proposition 4.17:</u> Under the conditions of the preceding definition we have

1) $f = ji$,

2) $j\lambda$ is the identity,

3) $dD + Dd = \lambda j$ - identity, and

4) if f is filtration preserving, D is a homotopy of degreee 1.

<u>Corollary:</u> $j_x : H(M'') \longrightarrow H(M')$, and if f is a map of filtered modules, then $j^2 : E^2(M'') \xrightarrow{\approx} E^2(M')$.

<u>Definition 4.18</u> Let N be a differential graded Λ- module, $f: \mathcal{G} \longrightarrow \mathcal{G}'$ a map of Λ- modules. Then $f \otimes 1 : \mathcal{G} \otimes N \longrightarrow \mathcal{G}' \otimes N$. Let M be the mapping cylinder of $f \otimes 1$. Then M is the <u>mapping cylinder for N of the coefficient homomorphism</u> f. Define

$$F_pM = \mathcal{G} \otimes \Sigma_{q \leq p} N_p + \mathcal{G} \otimes \Sigma_{q < p} N_p + \mathcal{G}' \otimes \Sigma_{q \leq p} N_p$$

Further let A be the kernel of f, and C the cokernel.

Note that the filtration $\{F_pM\}$ induces a filtration $\{F_pM'\}$ on $M' = M/\mathcal{G} \otimes N$.

<u>Proposition 4.19:</u> If in addition to the hypotheses of the preceding definition N is a free Λ - module, then there is an exact sequence

$$\cdots \longrightarrow H_{p-1}(N;A) \longrightarrow H_p(M') \longrightarrow H_p(N;C) \longrightarrow H_{p-2}(N;A) \longrightarrow \cdots$$

Proof: We have $E^0_{p,q}(M') = 0$ if $q \neq 0,1$, and $E^0_{p,1} = \mathcal{G} \otimes N_p$, $E^0_{p,0} = \mathcal{G}' \otimes N_p$. By an easy calculation $E^1_{p,1} = A \otimes N_p$, $E^1_{p,0} = C \otimes N_p$. The proposition now follows from Theorem 4.12.

Collary: If in addition $H_0(N) \simeq \Lambda + H'_0(N)$, then

1) $H_0(M') = 0$ implies $C = 0$, and
2) $H_0(M') = H_1(M') = 0$ implies $A = C = 0$, and $f: G \xrightarrow{\approx} G'$.

Proof: The last term of the exact sequence of 4.19 are

$$\cdots \longrightarrow H_0(N;A) \longrightarrow H_1(M') \longrightarrow H_1(N;C)$$
$$\longrightarrow H_{-1}(N;A) \longrightarrow H_0(M') \xrightarrow{\approx} H_0(N;C)$$

Therefore if $H_0(M') = 0$, we have $H_0(N;C) = 0$, and since $H_0(N) = \Lambda + H'_0(N)$ it follows that $C = 0$. Now if $C = 0$, $H_0(N;A) \xrightarrow{\approx} H_1(M')$ and the result follows.

It is not difficult to prove that if Λ is a principal ideal domain, then the exact sequence of 4.19 reduces to

$$0 \longrightarrow H_q(N;A) \longrightarrow H_{q+1}(M') \longrightarrow H_{q+1}(N;C) \longrightarrow 0$$

Further, even in the general case, there is an exact sequence

$$\cdots \longrightarrow H_q(N;\mathcal{G}) \longrightarrow H_q(N;\mathcal{G}') \longrightarrow H_q(M') \longrightarrow H_{q-1}(N;\mathcal{G}) \longrightarrow \cdots$$

since M' is the relative mapping cylinder of $\mathcal{G} \otimes N \longrightarrow \mathcal{G}' \otimes N$. If $A = 0$, then $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}' \longrightarrow C \longrightarrow 0$, and $H_q(M') \simeq H_q(N;C)$. Thus the preceding exact sequence reduces to the usual one coming from the exact sequence of coefficients $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}' \longrightarrow C \longrightarrow 0$. Similarly if $C = 0$, then $H_{q-1}(N;A) \approx H_q(M')$, and our exact sequence reduces to the usual one corresponding to the exact sequence of coefficients $0 \to \mathcal{a} \to g \to g' \to 0$.

Proposition 4.20: Let $f: M \to M'$ be a map of differential filtered Λ- modules, and let M'' denote the relative mapping cylinder of f. Then there is an exact sequence

$$\cdots \to E^2_p(M) \to E^2_p(M') \to E^2_p(M'') \to E^2_{p-1}(M) \to \cdots,$$

and further if f is a map of graded Λ- modules, there are exact sequences

$$\cdots \to E^2_{p,q}(M) \to E^2_{p,q}(M') \to E^2_{p,q}(M'') \to E^2_{p-1,q}(M) \to \cdots$$

for each q.

Proof: Let $M^{\#}$ be the mapping cylinder of f. Then there is an exact sequence

$$0 \to M \xrightarrow{i} M^{\#} \xrightarrow{j} M'' \to 0 .$$

Further there is a map $\lambda : M^{\#} \to M$ such that λ^i is the identity defined by $\lambda(a,b,c) = b$. The map λ is only a map of Λ- modules, and is not compatible with d. However it induces a map $\lambda^0 : E^0(M^{\#}) \to E^0(M)$, and for this map we have $\lambda^0 d^0 = d^0 \lambda^0$. It now follows easily that there is an exact sequence

$$0 \to E'(M) \xrightarrow{i} E'(M^{\#}) \to E'(M'') \to 0 .$$

On passing to homology this gives rise to an exact sequence

$$\cdots \to E^2_p(M) \to E^2_p(M^{\#}) \to E^2_p(M'') \to E^2_{p-1}(M) \to \cdots$$

Now noting that $E^2(M^{\#})$ is naturally isomorphic with

$E^2(M')$ by 4.17 and 4.14, the result follows.

We now wish to prove a comparison theorem for spectral sequences of differential graded Λ- modules. Since the hypotheses of this theorem are somewhat complicated, they will be listed first in a section of their own preceding the theorem.

<u>Hypotheses of the theorem:</u> Let $g: M \longrightarrow M'$ be a map of differential graded filtered Λ- modules, $h: U \longrightarrow U'$ a map of graded Λ- modules, $\bar{g}: N \longrightarrow N'$ a map of differential graded Λ- modules, and suppose that N, N' are free Λ- modules. Finally, suppose there is given a commutative diagram

$$\begin{array}{ccc} E^1(M) & \xrightarrow{g^1} & E^1(M') \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\psi'} \\ U \otimes_\Lambda N & \xrightarrow{h \times \bar{g}} & U' \otimes_\Lambda N' \end{array}$$

where $\psi(E^1_{p,q}(M)) \subset U_q \otimes N_p$, $\psi'(E^1_{p,q}(M')) \subset U'_q \otimes N'_p$, such that ψ and ψ' are maps of differential Λ- modules, and induce isomorphisms

$\psi_*: E^2(M) \xrightarrow{\approx} H(N;U)$ and $\psi'_*: E^2(M') \xrightarrow{\approx} H(N';U')$.

Under all the preceding hypotheses, one has the following two theorems:

<u>Theorem A:</u> If $g_*: H(M) \longrightarrow H(M')$ is an isomorphism, $h: U \longrightarrow U'$ is an isomorphism, and if $U_0 \simeq \Lambda + U'_0$, then $\bar{g}_*: H(N) \longrightarrow H(N')$ is an isomorphism.

Theorem B: If $g_*: H(M) \longrightarrow H(M')$ is an isomorphism, $\bar{g}_*: H(N) \longrightarrow H(N')$ is an isomorphism, and $H_0(N) \simeq \Lambda + H_0'(N)$, then $h: U \longrightarrow U'$ is an isomorphism.

Proof of Theorem A: We may as well assume that h is the identity map. Let $M^{\#}$ be the mapping cylinder of g, M'' the relative mapping cylinder. Further let $N^{\#}$ be the mapping cylinder of $\bar{g}$, N'' the relative mapping cylinder. Since $E_p'(M^{\#}) = E_{p-1}'(M) + E_p'(M) + E_p'(M')$, we now have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E'(M) \arrow[r] \arrow[d, "\psi"] & E'(M^{\#}) \arrow[r] \arrow[d, "\psi^{\#}"] & E'(M'') \arrow[r] \arrow[d, "\psi''"] & 0 \\
0 \arrow[r] & U \otimes N \arrow[r] & U \otimes N^{\#} \arrow[r] & U \otimes N'' \arrow[r] & 0
\end{tikzcd}$$

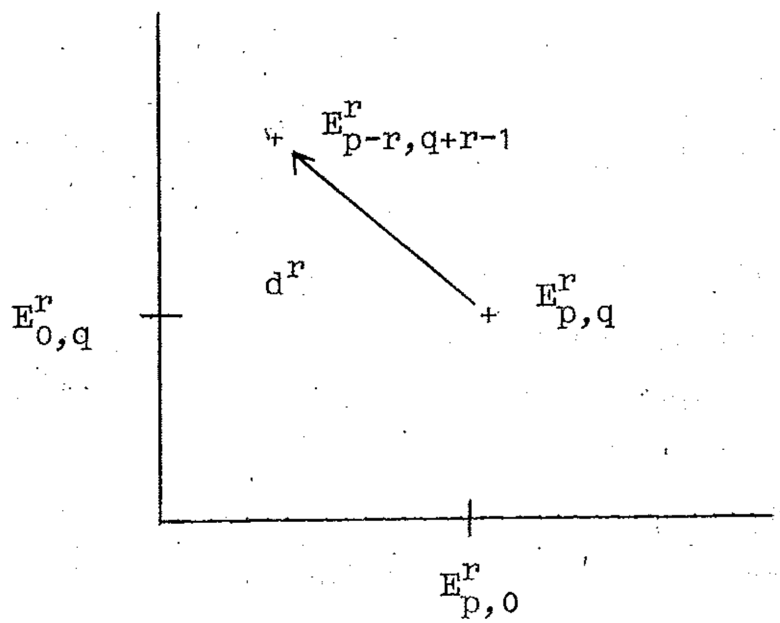
of differential modules such that the horizontal lines are exact. Passing to homology, we have the commutative diagram

$$\begin{tikzcd}
\arrow[r] & E^2_{p,q}(M) \arrow[r] \arrow[d, "\approx"] & E^2_{p,q}(M^{\#}) \arrow[r] \arrow[d, "\approx"] & E^2_{p,q}(M'') \arrow[r] \arrow[d] & E^2_{p-1,q}(M) \arrow[r] \arrow[d, "\approx"] & \cdots \\
\arrow[r] & H_p(N;U_q) \arrow[r] & H_p(N^{\#};U_q) \arrow[r] & H_p(N'';U_q) \arrow[r] & H_{p-1}(N;U_q) \arrow[r] & \cdots
\end{tikzcd}$$

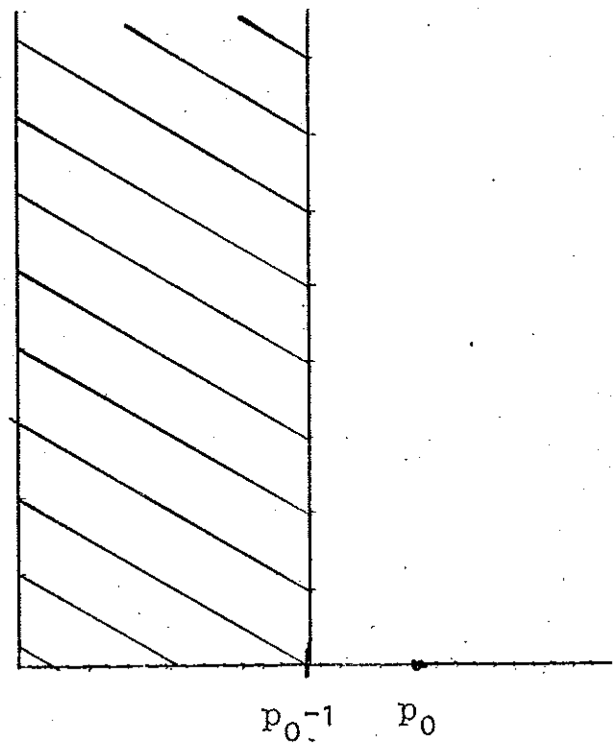
with exact horizontal lines. Therefore, by the 5-lemma, we have $E^2_{p,q}(M'') \simeq H_p(N'';U_q)$. Now since $g_*: H(M) \xrightarrow{\approx} H(M')$ we have $H(M'') = 0$, and hence $E^{\infty}_{p,q}(M'') = 0$ for all p,q. Assume that $H_p(N'') = 0$ for $p < p_0$. This means that

$H_p(N''; U_q) = 0$ for $p < p_0$, or that $E^2_{p,q}(M'') = 0$ for $p < p_0$. However $d^r : E^2_{p_0,0} \longrightarrow E^2_{p_0-r,r-1}$ and therefore we have $E^2_{p_0,0} = E^\infty_{p_0,0} = 0$, or $H_{p_0}(N;U_0) = 0$. Now since $U_0 = \Lambda + U'_0$ this means that $H_{p_0}(N'') = 0$, and proceeding inductively we have $H_p(N'') = 0$ for all p. Then because N'' was the relative mapping cylinder of $\bar{g} : N \longrightarrow N'$, $\bar{g}_* : H(N) \longrightarrow H(N')$ is an isomorphism.

The basis for the preceding argument may be found by making a diagram for $E^r(M'')$ by plotting $E^r_{p,q}$ at the point (p,q) in the first quadrant of the plane.



Now in this diagram d^r is represented by an arrow going up and to the left. In the preceding argument the assertion that $H_0(N'') = 0$ for $p < p_0$ meant that $E^2_{p,q}(M'') = 0$ for $p < p_0$, or that only 0 groups appear in the shaded portion of the diagram

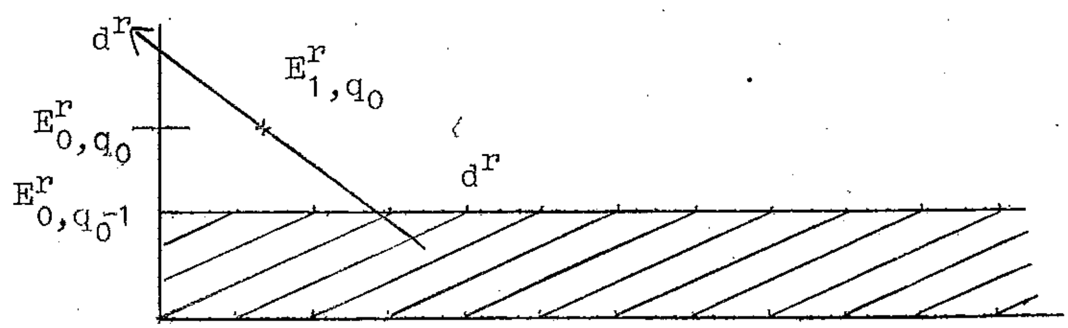


Consider now $E^r_{p_0,0}$: it sits on the horizontal axis, and therefore contains no boundaries. Further since d^r slopes up and to the left, it is mapped into zero. In other words we have the well-known principle that a spectral sequence with $E^\infty = 0$ identically has no corners.

<u>Proof of Theorem B:</u> In this case we may assume that $\bar{g}$ is the identity. Let $M^{\#}$ be the mapping cylinder of g, M'' the relative mapping cylinder, and let N'' be the mapping cylinder of $h \otimes 1 : U \otimes N \longrightarrow U' \otimes N$. We then show as before that $E^2(M'') \simeq H(N'')$, and recall that N'' is just the mapping cylinder associated with a coefficient homomorphism which we have already studied (4.19 and the corollary to 4.19).

Let A_q be the kernel of $h: U_q \longrightarrow U'_q$, and let C_q be the cokernel. Now since $g_* : H(M) \longrightarrow H(M')$ is an isomorphism, $H(M'') = 0$, and $E^{\infty}_{p,q}(M'') = 0$. Therefore $E^2_{0,0}(M'') = 0$; but $E^2_{0,0}(M'') = H_0(N; C_0)$. Therefore $C_0 = 0$. Now we also have $E^2_{1,0}(M'') = H_0(N; A_0) = 0$ from the corollary to 4.19. Therefore $A_0 = 0$. Suppose now that $A_q = C_q = 0$ for $q < q_0$. Then $H_{p,q}(N'') = 0$ for $q < q_0$, or $E^2_{p,q}(M'') = 0$ for $q < q_0$. This means that $E^r_{p,q}(M'') = 0$ for $q < q_0$, $r \geq 2$. Consider E^r_{0,q_0}. It consists entirely of d^r cycles for $r \geq 2$, and since $d^r : E^r_{r,q_0+1-r} \longrightarrow E^r_{0,q_0}$, it contains no boundaries. Therefore $E^2_{0,q_0} = E^{\infty}_{0,q_0} = 0$. However, $E^2_{0,q_0} = H_0(N; C_{q_0})$ and this means that $C_{q_0} = 0$. Now consider E^r_{1,q_0}. Again it consists entirely of d^r cycles for $r \geq 2$, and contains no boundaries. Therefore $E^2_{1,q_0} = E^{\infty}_{1,q_0} = 0$; but $E^2_{1,q_0} = H_0(N; A_{q_0})$, and therefore $A_{q_0} = 0$. Proceeding by induction we have $A_q = C_q = 0$ for all q, so that $h : U_q \longrightarrow U'_q$ is an isomorphism for all q. Thus the proof is complete.

The idea of the preceding proof is again that there can be no "corners" in a spectral sequence with $E^{\infty} = 0$. For $E^2_{p,q} = 0$ for $q < q_0$ means there are only 0-groups in the shaded region



A version of theorem A involving only spectral sequences was proved by Borel, and by Serre, but is unpublished. However, theorem A as it stands will suffice for what we need here. For completeness we now state a well known theorem of Leray.

Theorem C: If $h: U \longrightarrow U'$ is an isomorphism, and $\overline{g}_*: H(N) \longrightarrow H(N')$ is an isomorphism, then $g_*: H(M) \longrightarrow H(M')$ is an isomorphism.

This theorem may be proved by the usual procedure of observing that since $g^{\infty}: E^{\infty}(M) \longrightarrow E^{\infty}(M')$ is an isomorphism, $g_*^0: E^0(H(M)) \longrightarrow E^0(H(M'))$ is also an isomorphism.

Chapter V

DGA Algebras and the Construction of Cartan

We shall now prepare to make Cartan's calculation of $H_q(X)$, where X is an Eilenberg-MacLane space; i.e. $\pi_q(X) = 0$ for $q \neq n$, $\pi_n(X) = \pi$. A number of preliminary notions are necessary before we can actually do this, and we shall present these in a manner similar to that of [1]. In the course of this work we shall obtain a special case of a theorem of Borel [2] which is useful in the study of the topology of Lie groups.

Conventions: In this chapter Λ will denote a fixed commutative ring with unit. If N and N' are graded Λ- modules, $N = \Sigma_{n \geq 0} N_n$, $N' = \Sigma_{n \geq 0} N'_n$, then $N \otimes_\Lambda N'$ is the graded Λ-module such that $(N \otimes_\Lambda N')_n = \Sigma_{r+s=n} N_r \otimes_\Lambda N'_s$. If N, N' are differential graded Λ- modules, then $N \otimes_\Lambda N'$ is a differential graded Λ- module with

$$d(x \otimes y) = dx \otimes y + (-1)^r x \otimes dy$$

for $x \in N_r$, $y \in N'$.

Definitions: A graded Λ-algebra is a pair (A, ϕ) where A is a graded Λ-module, and $\phi : A \otimes_\Lambda A \longrightarrow A$ is a homomorphism of graded Λ- modules such that if we denote $\phi(x \otimes y)$ by $x \cdot y$, then $(x \cdot y) z = (x \cdot y) \cdot z$.

If, in addition to the preceding, A is a differential graded Λ- module, and ϕ is a homomorphism of differential graded Λ- modules, then (A,ϕ) is a <u>differential graded Λ - algebra</u>.

Usually either a graded Λ - algebra or a differential graded Λ-algebra will be denoted merely by the symbol for its underlying module .

The graded Λ- algebra A has a <u>unit</u> if there exists an element $1 \in A_0$ such that $1 \cdot x = x \cdot 1 = x$ for $x \in A$, and it is <u>anti-commutative</u> if $x \cdot y = (-1)^{rs} y \cdot x$ for $x \in A_r, y \in A_s$.

The ring Λ itself will be considered as either

1) a Λ- module,
2) a graded Λ- module N such that $N_n = 0$ for $n > 0$, and $N_0 = \Lambda$
3) a differential graded Λ- module with $d = 0$,
4) a graded Λ- algebra, or
5) a differential graded Λ- algebra.

If A, A' are (differential) graded Λ-algebras, then $A \otimes_\Lambda A'$ is the (differential) graded Λ- algebra such that $(x \otimes y)\cdot(x' \otimes y') = (-1)^{rs} xx' \otimes yy'$ for $x' \in A_r, y \in A'_s$.

Notice that if A is a graded Λ-algebra, then the multiplication $\phi : A \otimes_\Lambda A \longrightarrow A$ is a homomorphism of graded Λ- algebras if and only if A is anti-commutative.

<u>Definitions:</u> An <u>augmentation</u> of a (differential) graded Λ- module N is a homomorphism $\varepsilon : N \longrightarrow \Lambda$ of (differential)

graded Λ-modules. A <u>DGA-module</u> is a differential graded Λ-module N together with an augmentation $\varepsilon : N \longrightarrow \Lambda$.

If N, N' are DGA-modules, then $N \otimes N'$ is a DGA-module with $\varepsilon(n \otimes n') = \varepsilon(n)\varepsilon(n')$.

An <u>augmentation</u> of a (differential) graded Λ-algebra A is a homomorphism $\varepsilon : A \longrightarrow \Lambda$ of (differential) graded Λ algebras with unit. Note that this implies that ε is an epimorphism. A <u>DGA-algebra</u> is a differential graded Λ-algebra together with an augmentation $\varepsilon : A \longrightarrow \Lambda$.

<u>Example 1:</u> Let X be a semi-simplicial complex. Then $C(X)_N \otimes \Lambda$ is in a natural way a DGA-module. It already has a differential operator and a gradation, so it suffices to define an augmentation. This is done by setting $\varepsilon = 0$ on positive dimensional elements, and $\varepsilon(x \otimes \lambda) = \lambda$ for $x \in X_0$, $\lambda \in \Lambda$.

<u>Example 2:</u> It was pointed out in Chapter III that if X, X', X'' are semi-simplicial complexes, then the diagram

$$\begin{tikzcd}
(C(X)_N \otimes C(X')_N) \otimes C(X'')_N \arrow[r, "\nabla \otimes 1"] \arrow[dd, "\approx"] & C(X \times X')_N \otimes C(X'')_N \arrow[dr, "\nabla"] & \\
 & & C(X \times X' \times X'')_N \\
C(X)_N \otimes (C(X')_N \otimes C(X'')_N) \arrow[r, "1 \otimes \nabla"] & C(X)_N \otimes C(X' \times X'')_N \arrow[ur, "\nabla"] &
\end{tikzcd}$$

is commutative.

This means that if Γ is a monoid complex, and a multiplication is defined in $C(\Gamma)_N$ by the diagram

$$C(\Gamma)_N \otimes C(\Gamma)_N \xrightarrow{\nabla} C(\Gamma \times \Gamma)_N \longrightarrow C(\Gamma)_N$$

where $C(\Gamma \times \Gamma)_N \longrightarrow C(\Gamma)_N$ is the homomorphism induced by the multiplication in Γ, then $C(\Gamma)_N$ is a differential graded algebra over the ring of integers. Further it is not difficult to see that the unit of Γ_0 gives rise to a unit in the algebra $C(\Gamma)_N$. Consequently $C(\Gamma)_N \otimes \Lambda$ is in a natural way a DGA-algebra. Finally if Γ is commutative we have a commutative diagram

$$\begin{tikzcd}
C(\Gamma)_N \otimes C(\Gamma)_N \arrow[r, "\nabla"] \arrow[d, "T"] & C(\Gamma \times \Gamma)_N \arrow[d, "T'"] \arrow[dr] & \\
& & C(\Gamma)_N \\
C(\Gamma)_N \otimes C(\Gamma)_N \arrow[r, "\nabla"] & C(\Gamma \times \Gamma)_N \arrow[ur] &
\end{tikzcd}$$

where $T(x \otimes y) = (-1)^{rs}\, y \otimes x$ for y of dim s, x of dim r, and T' is the map induced by the map of $\Gamma \times \Gamma$ into itself which interchanges factors. Therefore, if Γ is commutative, then $C(\Gamma)_N$ is an anti-commutative DGA-algebra.

Example 3: If A is a DGA-algebra, then $H_*(A) = \Sigma H_n(A)$ is a DGA-algebra with d identically zero.

Definition: If A is a DGA-algebra, then a graded augmented (left) A-module is a graded augmented module M and a homomorphism $\phi : A \otimes_\Lambda M \longrightarrow M$ of graded augmented modules such that if we write $\phi(a \otimes m) = a \cdot m$ for $a \in A, m \in M$, then $a \cdot (a' \cdot m) = (a \cdot a') \cdot m$ for $a, a' \in A$, and $1 \cdot m = m$.

M is a DGA-module over A if, in addition to the preceding, ϕ is a homomorphism of DGA-modules.

Definition: If A, A' are DGA-algebras and $f: A \longrightarrow A'$ is a DGA homomorphism, M a DGA-module on A, and M' a DGA-module on A', then $g: M \longrightarrow M'$ is a DGA-homomorphism compatible with f if the diagram

$$\begin{array}{ccc} A \otimes_{\Lambda} M & \xrightarrow{\quad f \otimes g \quad} & A' \otimes_{\Lambda} M' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad g \quad} & M' \end{array}$$

is a commutative diagram of maps of DGA-modules.

Definition: If A is a DGA-algebra, then a construction on A consists of

1) a filtered DGA-module M on A such that if $m \in F_p M$, $a \in A$, then $a \cdot m \in F_p M$
2) a DGA-module N,
3) a homomorphism of DGA modules $p: M \longrightarrow N$ which is compatible with $\varepsilon: A \longrightarrow \Lambda$, and
4) a homomorphism of graded augmented left A-modules $\nabla: A \otimes_{\Lambda} N \longrightarrow M$ subject to the following conditions:

a) $p \nabla (1 \otimes n) = n$,

b) $p F_r M \subset \Sigma_{q \leq r} N_r$,

c) if $F_r(A \otimes_{\Lambda} N) = \Sigma_{q \leq r} A \otimes_{\Lambda} N_r$, then $\nabla(F_r(A \otimes_{\Lambda} N)) \subset F_r M$, and

d) $\nabla^0 : E^0(A \otimes_\Lambda N) \longrightarrow E^0(M)$ is a homomorphism of DGA-modules such that $\nabla^1 : E^1(A \otimes_\Lambda N) \xrightarrow{\approx} E'(M)$.

A construction on A will be denoted by (A, N, M).

Definition: A construction (A,N,M) is free if $\nabla : A \otimes N \longrightarrow M$ is an isomorphism of filtered Λ-modules, and N is a free Λ-module. In this case we will frequently identify $A \otimes N$ and M as Λ-modules. Note, however, that the differential operator in M is not necessarily the natural one of $A \otimes N$; in fact it is usually twisted.

Definition: A DGA module M is acyclic if $\varepsilon : M \longrightarrow \Lambda$ induces an isomorphism $\varepsilon_\lambda : H(M) \longrightarrow \Lambda$, or in other words if $\cdots \longrightarrow M_n \xrightarrow{d} M_{n-1} \longrightarrow \cdots \xrightarrow{d} M_0 \xrightarrow{\varepsilon} \Lambda$ is an exact sequence.

A construction (A,N,M) is acyclic if M is acyclic.

Theorem 1: Let (A,N,M) be a free construction, (A',N',M') an acyclic construction, and $f : A \longrightarrow A'$ a DGA homomorphism. Under these conditions there exists a DGA homomorphism $g : M \longrightarrow M'$ which is compatible with f. If g' is another such homomorphism, then there is a homotopy $D : M \longrightarrow M'$ such that

$dD + Dd = g - g'$, and $D\, a \cdot m = (-1)^r f(a) Dm$ for $a \in A_r$. Further if the filtration on M is regular then g is filtration preserving, and D is a homotopy of degree 1.

Proof: Let C_1 be a basis for N_1 over Λ. For this proof, identify $x \in C_1$ with $1 \otimes x \in M$. Now if $x \in C_0$, define $g(x)$ to be any element of M_0' such that $\varepsilon g(x) = \varepsilon(x)$. If $y \in A \otimes N_0$ then y may be written uniquely as $\sum a_j \otimes x_j$ where $x_j \in C_0$ and $g(y)$ is defined to be $\sum f(a_j)\, g(x_j)$.

For $x \in C_1$, we have $dx \in A \otimes N_0$ and $\varepsilon(dx) = 0$. Therefore $g(dx)$ is defined and $\varepsilon g(dx) = 0$. Define $g(x)$ to be some element of M_1' such that $dg(x) = g(dx)$. Now if $y \in A \otimes N_1$, $y = \sum a_j \otimes x_j$ where $x_j \in C_1$ and we define $g(y)$ to be $\sum f(a_j)\, g(x_j)$.

Suppose now that g is defined on $A \otimes \sum_{q<r} N_q = F_{r-1} M$. For $x \in C_r$; we have $dx \in F_{r-1}M$, $g(dx)$ is defined and $dg(dx) = 0$. Therefore we may define $g(x)$ to be any element of M'_r such that $dg(x) = g(dx)$. Consequently the existence of g is proved.

Let g' be another map compatible with f. Then for $x \in C_0$, $\varepsilon g(x) = \varepsilon(x) = \varepsilon g'(x)$, and $\varepsilon(g(x) - g'(x)) = 0$. Define Dx to be any element of M_1' such that $dDx = g(x) - g'(x)$. Now extend D to F_0M by defining $D\, a \otimes x = (-1)^r f(a) Dx$ for $x \in A_r$.

Suppose that D is defined on $F_{r-1}M$. Then for $x \in C_r$, we have $dx \in F_{r-1}M$, $g(x) - g'(x) - Ddx$ is a cycle belonging to M'_r, and we define Dx to be any element of M'_{r+1} such that $dDx = g(x) - g'(x) - Ddx$.

Notice that $g(1 \otimes N_r) \subset F_rM'$ if M' has a regular filtration (i.e. $M'_r \subset F_rM'$), and then $g(A \otimes N_r) \subset F_rM'$, since for $x \in A'$, $m \in F_rM'$ we have $x\text{-}m \in F_rM'$. The same reasoning shows that $DF_rM \subset F_{r+1}M'$, or that D is of degree 1.

<u>Definitons</u>: If (A,N,M) and (A',N',M') are constructions, a map of the first into the second consists of a DGA homomorphism $f: A \longrightarrow A'$ together with a filtration preserving DGA homomorphism $g: M \longrightarrow M'$ which is compatible with f. Under the preceding conditions the map of constructions will be said to be compatible with f. Further, since g is filtration preserving, g induces $g^r: E^r(M) \longrightarrow E^r(M')$. Now consider Λ as an $H(A)$ module by defining $x.a = x \cdot \varepsilon(a)$ for $x \in \Lambda$, $a \in H(A)$. Similarly consider Λ as an $H(A')$ module. Then $N = \Lambda \otimes_{H(A)} E^1(M)$, and $N' = \Lambda \otimes_{H(A')} E^1(M')$, and there is a DGA homomorphism $\overline{g}: N \longrightarrow N'$ induced by g^1, or by g.

<u>Theorem 1:</u> Let (A,N,M) be a free construction, (A',N',M') an acyclic construction with a regular filtration, and $f: A \longrightarrow A'$ a DGA homomorphism. Then there is a map of (A,N,M) into (A',N',M') compatible with f. Further the induced homomorphism $\overline{g}_*: H(N) \longrightarrow H(N')$ is independent of

the choice of such a map.

<u>Proof:</u> The first part of this theorem is just a restatement of Theorem 1. To prove the last part suppose $g, g' : M \longrightarrow M'$ are compatible with f. Let D be a homotopy between g and g' satisfying the conditions of Theorem 1, and define $\overline{D} : N \longrightarrow N'$ by $\overline{D}x = pDx$ for $x \in C_r$ where C_r is a basis for N_r as in Theorem 1, and $p : M' \longrightarrow N'$ is the projection map of the construction (A', N', M'). One verifies easily that $d\overline{D} + \overline{D}d = \overline{g} - \overline{g}'$.

<u>Theorem 2:</u> Suppose that (A,N,M) and (A',N',M') are constructions, $f : A \longrightarrow A'$ and $g : M \longrightarrow M'$ are DGA homomorphisms which determine a map of constructions, and N, N' are free Λ-modules. Under these conditions if $f_x : H(A) \longrightarrow H(A')$ is an isomorphism and $g_x : H(M) \longrightarrow H(M')$ is an isomorphism, then $\overline{g}_x : H(N) \longrightarrow H(N')$ is also an isomorphism.

The preceding theorem is almost a special case of Theorem A of chapter 4. The difference is that we have not assumed that the isomorphism $H(A) \otimes N \longrightarrow E^1(M)$ is compatible with differential operators. This, however, is the case if $H_0(A) = \Lambda$. With $H_0(A) = \Lambda$, the map $p : E^1_{q,0} \longrightarrow N_q$ is an isomorphism, and therefore the differential operator d' is of the correct form on $\sum_q E^1_{q,0}$. Now as a left $H(A)$ module, $E^1(M) = H(A) \otimes N$, and $d'(x \otimes y) = (-1)^{\dim x} x \cdot d'(1 \otimes y) = (-1)^{\dim x} x(1 \otimes dy) = (-1)^{\dim x} x \otimes dy$, and we see that in this case the differential operator is just the usual one in $H(A) \otimes N$.

We will now indicate the changes necessary in the proof of Theorem A to prove the above theorem without the assumption that $H_0(A) = \Lambda$. Let M'' be the relative mapping cylinder of $g: M \longrightarrow M'$, and N'' the relative mapping cylinder of $\bar{g}: N \longrightarrow N'$. It is easily seen that $E^1(M'') = H(A) \otimes N''$, and to use the same proof as before we need to know that $E^1_{q,0}(M'') = 0$ for $q \leq p$ implies that $E^1_{p,q}(M'') = 0$ for all q.

Let $N^{\#} = H_0(A) \otimes N''$. We have a differential operator in $N^{\#}$ induced by d^1. Further $E^1(M'') = H(A) \otimes_{H_0(A)} N^{\#}$. Let G be any right $H_0(A)$ module, and define $H(N^{\#}; G)$ to be $H(G \otimes_{H_0(A)} N^{\#})$. Now $E^2_{p,q}(M'') = H_p(N^{\#}; H_q(A))$, and $N'' = \Lambda \otimes_{H_0(A)} N^{\#}$. Therefore to prove the theorem it suffices to show that $H_q(N^{\#}; H_0(A)) = 0$ for $q \leq p$ implies that $H_q(N^{\#}; G) = 0$ for $q \leq p$ for any right $H_0(A)$ module G. However, the fact that $H_q(N^{\#}; H_0(A)) = 0$ for $q \leq p$ implies that $H_q(N^{\#}; F) = 0$ for $q \leq p$ where F is any free $H_0(A)$ module. Suppose now that $0 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 0$ is an exact sequence of right $H_0(A)$ modules. Then since $N^{\#}$ is a free $H_0(A)$ module (this follows since N'' is a free Λ-module), the sequence

$$0 \longrightarrow R \otimes_{H_0(A)} N^{\#} \longrightarrow F \otimes_{H_0(A)} N^{\#} \longrightarrow G \otimes_{H_0(A)} N^{\#} \longrightarrow 0$$

is exact, and there is a resulting exact sequence

$$\cdots \longrightarrow H_q(N;R) \longrightarrow H_q(N;F) \longrightarrow H_q(N;G) \longrightarrow H_{q-1}(N;R) \longrightarrow \cdots \quad ;$$

Consequently for F free we have $H_q(N;G) \simeq H_{q-1}(N;R)$ for $q \leq p$, and by induction this implies the desired result.

Having given some properties of constructions, we shall now show how they arise. We shall first prove that any twisted Cartesian product (Γ,B,E) (Definition 2.13) gives rise to a construction, provided is a monoid complex. To do this some preliminary definitions are needed.

Definitions: If (Γ,B,E) is a twisted Cartesian product, let $\nabla':C(\Gamma)_N \otimes C(B)_N \longrightarrow C(E)_N$ be the composition of the natural map $\nabla:C(\Gamma)_N \otimes C(B)_N \longrightarrow C(\Gamma \times B)_N$ of the Eilenberg-Zilber Theorem (Chapter 3, p. 17) and the identification of $C(E)_N$ and $C(\Gamma \times B)_N$ as groups. We shall say that a simplex $\sigma \in E$ is of filtration p if its projection lies in the p-skeleton of B, i.e. may be written as $s_{i_1} \cdots s_{i_r}\tau$ where $\tau \in B$ is a simplex of dimension less than or equal to p. Define $F_pC(E)_N$ to be the subgroup generated by simplexes of filtration p. Further when Γ is a monoid complex consider $C(E)_N$ as a left $C(\Gamma)_N$ module by using the diagram

$$C(\Gamma)_N \otimes C(E)_N \xrightarrow{\nabla} C(\Gamma \times E)_N \longrightarrow C(E)_N$$

all maps being the natural ones.

Proposition: If Γ is a monoid complex, and (Γ,B,E) is a twisted Cartesian product, then $(C(\Gamma)_N,C(B)_N,C(E)_N)$ is a construction with a regular filtration.

Proof: All statements which need to be verified follow at once except the assertion that $\nabla: C(\Gamma)_N \otimes C(B)_N \longrightarrow C(E)_N$ commutes with d^0 and induces an isomorphism $\nabla': H(C(\Gamma)_N) \otimes C(B)_N \longrightarrow E^1(C(E)_N)$. We shall prove this by showing that $E^0(C(E)_N) = E^0(C(\Gamma \times B)_N)$, that this identification is compatible with d^0, and that the proposition is true for a Cartesian product.

First identify E and $\Gamma \times B$ as sets. Then we have to consider $\partial_i(\sigma \times \tau)$ where $\sigma \times \tau$ is of filtration p. If $i > 0$ it does not matter whether we mean the i-th face operator in E or $\Gamma \times B$ by ∂_i. If $i = 0$ we still have the relation $\partial_0(\sigma \times \tau) = \partial_0 \sigma \cdot \partial_0(1 \times \tau)$. The fact that $\sigma \times \tau$ is of filtration p means that $\tau = s_{i_0} \cdots s_{i_r} \tau'$ where $\tau' \in B$ has dimension less than or equal to p. If τ' has dimension less than p, then $\sigma \times \tau$ represents the zero element in $C(E)_N$. Therefore assume that dimension $(\tau') = p$. Now $\partial_0(1 \times \tau) = \partial_0(1 \times s_{i_0} \cdots s_{i_r} \tau') = \partial_0 s_{i_0} \cdots s_{i_r}(1 \times \tau')$. Assuming, as we may, that $i_0 > \cdots > i_r$, it follows that the element $\partial_0(1 \times \tau) = s_{i_0 - 1} \cdots s_{i_r - 1} \partial_0(1 \times \tau')$ is of filtration $(p-1)$ unless $i_r = 0$. In this case $\partial_0(1 \times \tau) = s_{i_0 - 1} \cdots s_{i_{r-1} - 1}(1 \times \tau')$, and this formula is independent of whether we mean the 0'th face operator of E or $\Gamma \times B$ by ∂_0. Thus, we have shown that $E^0(C(E)_N) = E^0(C(\Gamma \times B)_N)$. It therefore remains to show that $\nabla': H(C(\Gamma)_N) \otimes C(B)_N \longrightarrow E^1(C(\Gamma \times B)_N)$ is an isomorphism. To show this, recall that we have defined a map

$f: C(\Gamma \times B)_N \longrightarrow C(\Gamma)_N \otimes C(B)_N$ (Chapter 3) such that $f\nabla$ is the identity[1] and ∇f is homotopic to the identity. Since f is filtration preserving, $f'\nabla'$ is the identity, and to prove the proposition we need only show that $\nabla' f'$ is the identity. For this it suffices to know that the homotopy of f with the identity is of degree 0. However, this is indeed the case, for the homotopy is natural.

The following comments may help to clarify the last assertion. The fact that the homotopy is natural means that if $f: X \longrightarrow X'$ and $g: Y \longrightarrow Y'$ are maps of semi-simplicial complexes, then the homotopy commutes with the induced map of $C(X \times Y)_N \longrightarrow C(X' \times Y')_N$. However, any simplex of a Cartesian product $X \times Y$ is the image of a simplex of $\Delta_p \times \Delta_q$ for some p and q, and every simplex of Δ_p or Δ_q can be obtained by applying face and degeneracy operations to the basic simplex. Therefore the fact that the homotopy is natural means that it may be expressed by using face and degeneracy operations. However, from the very definition of the filtration on the chains of a Cartesian product or a twisted Cartesian product product it is evident that the filtration can not be raised by applying face and degeneracy operations.

<u>Definition</u>: A construction (A,N,M) satisfies the <u>condition B'</u> if

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1 In Chapter 3 it only stated that $f\nabla$ is homotopic to the identity. However, one verifies easily that it is actually equal to the identity.

1) $\varepsilon : N_0 \xrightarrow{\approx} \Lambda$, and

2) $x \in Z_q(M)$ and $\varepsilon(x) = 0$ imply that there exists a unique $y \in \nabla(1 \otimes N_{q+1})$ such that $dy = x$.

The construction satisfies the <u>condition B</u> if it satisfies the condition B' and is free.

<u>Theorem 3:</u> If (A.N,M) is a free construction, (A',N',M') is a construction satisfying the condition B', and $f:A \longrightarrow A'$ is a DGA homomorphism, then there is a unique map of (A,N,M) into (A',N',M') such that $\nabla(1 \otimes N)$ maps into $\nabla(1 \otimes N')$.

One we note that the condition B' implies that the construction is acyclic, the proof of this theorem is entirely similar to the proof of Theorem 1, except that at each stage where a choice had to be made in the proof of the earlier theorem, there is now available a unique element of $\nabla(1 \otimes N')$ satisfying the required conditions.

<u>Theorem 4</u>: If A is a DGA algebra, and kernel $\varepsilon : A \longrightarrow \Lambda$ is a free Λ-module, there exists a construction (A,N,M) satisfying the condition B. Further if (A,N',M') is another such construction, then there is a unique isomorphism of (A,N,M) with (A,N',M') which maps $\nabla(1 \otimes N)$ into $\nabla(1 \otimes N')$.

The uniqueness is clear from the preceding theorem. It remains to prove existence. This will be done in two different ways. The first way is perhaps more intuitive, but is valid only if Λ is a principal ideal domain.

First proof of existence: We assume now that Λ is a principal ideal domain. Recall that over a principal ideal domain any submodule of a free module is free. Therefore $\hat{A}$ = kernel $\varepsilon : A \longrightarrow \Lambda$ is automatically free. Proceeding with the construction, let $N_0 = \Lambda$, $M_0 = A_0 \otimes N_0 = A_0$, let $N_1 = \hat{A}_0$, $M_1 = A_1 \otimes N_0 + A_0 \otimes N_1$, and define $d : 1 \otimes N_1 \longrightarrow A_0$ to be the natural map.

Suppose that N_q and M_q are defined for $q \leq r$ so as to satisfy the condition B. We have $M_q = \sum_{i+j=q} A_i \otimes N_j$. Define N_{r+1} = kernel $d : M_r \longrightarrow M_{r-1}$, and $M_{r+1} = \sum_{i+j=r+1} A_i \otimes N_j$. Further define $d : 1 \otimes N_{r+1} \longrightarrow M_r$ to be the natural map. It is now evident that (A,N,M) is a construction satisfying the condition B.

Second proof of existence: Again let $\hat{A}$ denote kernel $\varepsilon : A \longrightarrow \Lambda$. Define $\bar{B}^0(A) = \Lambda$, and for $n > 0$, $\bar{B}^n(A)$ to be the tensor product of $\hat{A}$ with itself n-times, and denote an element of $\bar{B}^n(A)$ by $[a_1, \ldots, a_n]$. Define a new gradation in $\bar{B}^n(A)$ by setting dimension $[a_1, \ldots, a_n] = n + \sum \alpha_i$ where α_i = dimension a_i. Define $\bar{B}(A)$ to be $\sum \bar{B}^n(A)$, and $B(A)$ to be $A \otimes \bar{B}(A)$. The object now is to place a differential operator in $B(A)$ so that $(A, \bar{B}(A), B(A))$ is a construction satisfying the condition B.

Denote $A \otimes \bar{B}^n(A)$ by $B^n(A)$ and denote an element of this module by $a[a_1,\ldots,a_n]$. Define $s: B(A) \longrightarrow B(A)$ by setting $s(a[a_1,\ldots,a_n]) = [a-\varepsilon(a), a_1,\ldots,a_n]$. We want s to be a contracting homotopy for $B(A)$, i.e. we want the relation $ds + sd = 1 - \varepsilon$ to hold, where 1 is the identity map. Since $B(A)$ is to be a left A-module we shall have the relation $d(a\cdot x) = (da)\cdot x + (-1)^{\alpha} a\cdot dx$, where α = dimension a. Therefore it suffices to define d on $\bar{B}(A)$. On $\bar{B}_0(A)$, d is zero. On $\bar{B}_1(A)$ define $d[a_1] = a_1 \in A \otimes \bar{B}_0(A)$. Assume that d is defined on $\bar{B}^r(A)$ for $r \leq n$, such that $d: \bar{B}^r(A) \longrightarrow B^r(A)$. A typical element of $\bar{B}^{n+1}(A)$ may be written as $[a_1,\ldots,a_{n+1}] = s\,a_1[a_2,\ldots,a_{n+1}]$. Define $d[a_1,\ldots,a_{n+1}] = a_1[a_2,\ldots,a_{n+1}] - sda_1[a_2,\ldots,a_{n+1}]$. Then $dd[a_1,\ldots,a_{n+1}] = da_1[a_2,\ldots,a_{n+1}] - dsda_1[a_2,\ldots,a_{n+1}]$, and assuming by induction that $ds + sd = 1 - \varepsilon$ this last expression is zero. Consequently d is defined, and $d^2 = 0$.

To show that this construction satisfies the condition B, suppose that $x \in B(A)_0$ and $\varepsilon(x) = 0$; then $x = d[x]$. Suppose that we also have $x = dy$, where $y \in \bar{B}(A)$; then $y = s(z)$ where $\varepsilon(z) = 0$, and $d([x] - s(z)) = ds(x-z) = 0$. However, $(x-z) = ds(x-z) + sd(x-z) = 0$, and $x = z$, so that $y = [x]$. Now suppose that $x \in B(A)_q$, $q > 0$, and that $dx = 0$. We have $x = dsx$, where $sx \in \bar{B}(A)$, and if $x = dy$

where $y \in \bar{B}(A)$ then $y = sz$, and $ds(x-z) = 0$. This means that $x-z = sd(x-z)$, $s(x-z) = ssd(x-z) = 0$, and consequently $y = sz = sx$. The proof of the theorem is now complete.

In neither of the preceding proofs have we shown how to obtain the differential operator in N in the construction (A,N,M). The construction, however, is free, so that $N = \Lambda \otimes_A M$, and the differential operator in N is the natural induced one.

<u>Proposition</u>: Let Γ be a monoid complex, and let (A,N,M) be the construction arising from the twisted Cartesian product $(\Gamma, \bar{W}(\Gamma), W(\Gamma))$. Then (A,N,M) satisfies the condition B'.

<u>Proof</u>: $\bar{W}_0$ has one element (cf. definition 2.17), and consequently $\nabla : A_0 \otimes N_0 \longrightarrow M_0$ is an isomorphism. Further if S is the contracting homotopy for $W(\Gamma)$ used in the proof of 2.15, then S satisfies the identity $S^2 = s_0 S$, and $S : W_q \longrightarrow 1_{q+1} \times \bar{W}_{q+1}$ is onto. Consequently, denoting by S the induced contracting homotopy on M, we have $S : M \longrightarrow \nabla(1 \otimes N)$, $S^2 = 0$, and $S : \hat{M} \longrightarrow \nabla(1 \otimes \hat{N})$ is an epimorphism (recall that if C is a DGA module, then $\hat{C} = \text{kernel } \varepsilon : C \longrightarrow \Lambda$). Suppose, therefore that if $x \in M_q$ is such that $\varepsilon(x) = 0$ for $q = 0$, or $dx = 0$ for $q < 0$, then $x = dSx$. If $x = dy$, where $y \in \nabla(1 \otimes N)$, then $y = Sz$ for some $d \in \hat{M}$, and $d\,S(x-z) = 0$. Consequently $x - z = Sd(x-z)$, $S(x-z) = 0$, and $y = Sx$. This proves the desired result.

<u>Definition</u>: Let (A,N,M) and (A',N',M') be constructions. Consider $(A \otimes A', N \otimes N', M \otimes M')$. Define $\nabla : A \otimes A' \otimes N \otimes N' \longrightarrow M \otimes M'$ by $\nabla(a \otimes a' \otimes n \otimes n') = (-1)^{\alpha\beta} \nabla(a \times n) \otimes \nabla(a' \times n')$ where α = dimension a' and β = dimension n. Suppose that $M \otimes M'$ is provided with the usual filtration, i.e. $F_p(M \otimes M') = \sum_{r+s=p} F_r M \otimes F_s M'$, and the usual differential operator. Consider $M \otimes M'$ as a left $A \otimes A'$ module by defining $(a \otimes a').(m \otimes m') = (-1)^{\alpha\gamma} a.m \otimes a'm'$ where α = dimension a, and γ = dimension m.

<u>Proposition</u>: If (A,N,M) and (A',N',M') are constructions whose underlying modules are free over Λ, then $(A \otimes A', N \otimes N', M \otimes M')$ is a construction whose underlying modules are free over Λ. If in addition

1) (A,N,M) and (A',N',M') are free, then $(A \otimes A', N \otimes N', M \otimes M')$ is free, and

2) if (A,N,M) and (A',N',M) are acyclic, then $(A \otimes A', N \otimes N', M \otimes M')$ is acyclic.

The proof of this proposition follows immediately from the definitons.

<u>Corollary</u>: If A, A' are DGA algebras such that $\hat{A}, \hat{A}'$ are free as Λ-modules, and $(A \otimes A', N, M)$ is an acyclic construction such that the underlying modules are free over Λ, then $H(\bar{B}(A) \otimes \bar{B}(A')) \simeq H(N)$. If $H(\bar{B}(A'))$ is a free Λ-module, then $H(\bar{B}(A)) \otimes H(\bar{B}(A')) \simeq H(N)$.

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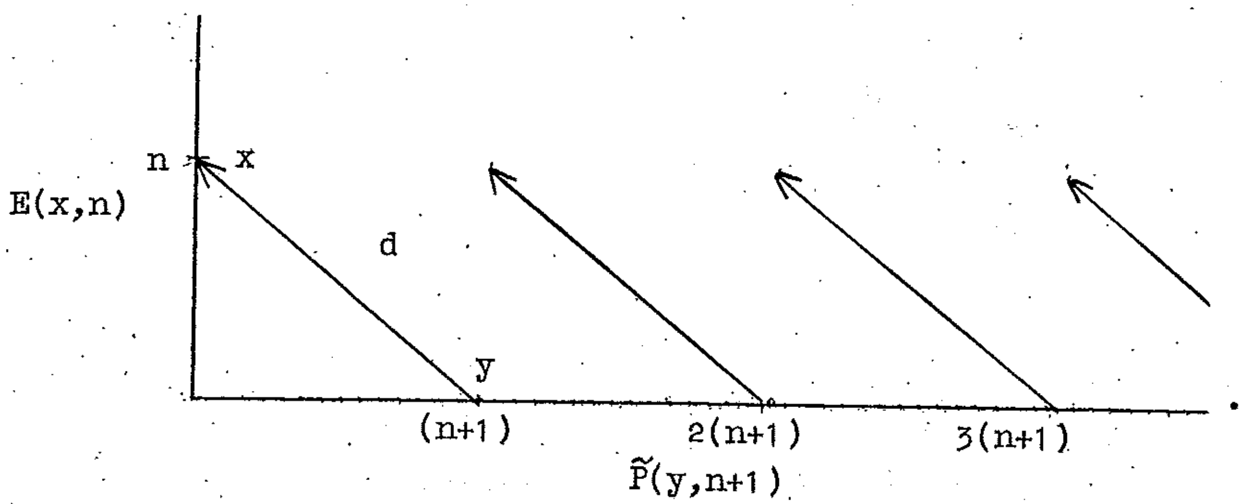
<u>Notation</u>: Let $E(x,n)$ denote the <u>exterior algebra</u> over Λ <u>with one generator x of dimension n</u>. In other words $E(x,n)_q = 0$ for $q \neq 0,n$; $E(x,n)_0 \simeq \Lambda$ with basis element 1, the unit of $E(x,n)$; and $E(x,n)_n \simeq \Lambda$, with basis element x. In the algebra $x^2 = 0$.

Let $\widetilde{P}(y,n)$ denote the <u>divided polynomial ring</u> with basic element y in dimension n. In other words $\widetilde{P}(y,n)_q = 0$ unless q is of the form kn for some non-negative integer k, $\widetilde{P}(y,n)_{kn} \simeq \Lambda$ with basis element y_k, $y_0 = 1$ is the unit of the algebra $\widetilde{P}(y,n)$, $y_1 = y$, and the product in the algebra is defined by $y_i y_j = \binom{i+j}{i} y_{i+j}$.

Notice that for n odd, both $E(x,n)$ and $\widetilde{P}(y,n+1)$ are anti-commutative. For each n we define a free acyclic construction $(E(x,n),\widetilde{P}(y,n+1),M)$ as follows: since the construction is free $\nabla : E(x,n) \otimes \widetilde{P}(y,n+1) \xrightarrow{\approx} M$, and we will assume that ∇ is the identity map as far as modules are concerned. Define $d(1 \otimes y_{k+1}) = x \otimes y_k$, $d(x \otimes y_k) = 0$. Now M is an algebra with an additive base $\{x \otimes y_k, 1 \otimes y_k\}$. Further $d((1 \otimes y_i)(1 \otimes y_j)) = d(1 \otimes y_i y_j) = d(\binom{i+j}{i}(1 \otimes y_{i+j}) = \binom{i+j}{i} x \otimes y_{i+j-1}$, and $d(1 \otimes y_i)(1 \otimes y_j) + (1 \otimes y_i) d(1 \otimes y_j) = (x \otimes y_{i-1})(1 \otimes y_j) + (1 \otimes y_i)(x \otimes y_{j-1}) =$
$(\binom{i+j-1}{i-1} + \binom{i+j-1}{i}) x \otimes y_{i+j-1} = \binom{i+j}{i} x \otimes y_{i+j-1}$.
These calculations show that d is an anti-derivation on the algebra M. Moreover, it is clear that the algebra M is acyclic. Its structure is described by the diagram



Combining the results of the calculation just made, the comparison theorem for constructions, and the previous proposition concerning constructions over tensor products, we obtain the following result. Suppose that $(E(x_1,n_1) \otimes \ldots \otimes E(x_k,n_k), N, M)$ is an acyclic construction with N and M free Λ-modules. Suppose further that n_i is odd for $i = 1,\ldots,k$. In this case $H(N) \simeq \widetilde{P}(y_1,n_1+1) \otimes \ldots \otimes \widetilde{P}(y_k,n_k+1)$. This result is quite weak, but we have a much stronger result due to A. Borel [2].

<u>Theorem</u>: Suppose that (A,N,M) is an acyclic construction such that the underlying Λ-modules are free, and that $H(A) \simeq E(x_1,n_1) \otimes \ldots \otimes E(x_k,n_k)$, where n_i is odd, for $i = 1,\ldots,k$. In this case $H(N) \simeq \widetilde{P}(y_1,n_1+1) \otimes \ldots \otimes \widetilde{P}(y_k,n_k+1)$.

<u>Proof</u>: It is sufficient to prove this theorem for the construction $(A,\bar{B}(A),B(A))$. In other words it is sufficient to prove that $H(\bar{B}(A)) \simeq \widetilde{P}(y_1,n_1+1) \otimes \ldots \otimes \widetilde{P}(y_k,n_k+1)$. To do this we shall look at a spectral sequence for $\bar{B}(A)$.

As usual let $\hat{A}$ = kernel $\varepsilon : A \longrightarrow \Lambda$, and recall that if we define $\bar{B}^k(A) = \hat{A} \otimes \ldots \otimes \hat{A}$, the tensor product being taken k-times, then $\bar{B}(A) = \Lambda + \bar{B}^1(A) + \ldots + \bar{B}^k(A) + \ldots$. In $\bar{B}^k(A)$ the dimension of a typical element $[a_1, \ldots, a_k]$ is $\Sigma \alpha_i + k$, where α_i is the dimension of a_i .

Define $F_p(\bar{B}(A)) = \Sigma_{k \leq p} \bar{B}^k(A)$. Then $E'(\bar{B}(A)) = \Lambda + H(\hat{A}) + \ldots H(\hat{A} \otimes \ldots \otimes \hat{A}) + \ldots$ with the appropriate conventions concerning dimensions. Now if $\hat{H}(A)$ is a free Λ-module, then $H(\hat{A} \otimes \ldots \otimes \hat{A}) \simeq \hat{H}(A) \otimes \ldots \otimes \hat{H}(A)$, and $E'(\bar{B}(A)) = \bar{B}(H(A))$. Further it is not difficult to verify that in this case $E^2(\bar{B}(A)) = H(\bar{B}(H(A)))$. However, we have more data available. We have assumed that $H(A) \simeq E(x_1, n_1) \otimes \ldots \otimes E(x_k, n_k)$. Consequently by our earlier remark $H(\bar{B}(H(A))) \simeq \tilde{P}(y_1, n_1 + 1) \otimes \ldots \otimes \tilde{P}(y_k, n_k + 1)$. This means that the total degree or dimension of every element of $E^2(\bar{B}(A))$ is even, and therefore that $E^2(\bar{B}(A)) = E^\infty(\bar{B}(A))$. We then have $\tilde{P}(y_1, n_1 + 1) \otimes \ldots \otimes P(y_k, n_k + 1) = E^\infty(\bar{B}(A)) = E^0(H\bar{B}(A))$. Since $E^0(H(\bar{B}(A)))$ is a free Λ-module we now see that $H(\bar{B}(A)) \simeq \tilde{P}(y_1, n_1 + 1) \otimes \ldots \otimes \tilde{P}(y_k, n_k + 1)$, which is the desired result. Note that this last isomorphism is not natural.

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Errata to Chapter IV

p.4-2 line 9: read "functor $H:A \longrightarrow \mathcal{G}_\wedge$" instead of "functor $H:A \longrightarrow \wedge$."

p.4-10 line 5: read "theorem of Serre [5]" instead of "theorem of Serre [ ]."

References for Chapter IV

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