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Algebraic Homotopy Theory

(John C. Moore)

1. SEMI-SIMPLICIAL COMPLEXES

In classical algebraic topology one studies simplicial complexes. However, modern developments have shown that these are inadequate, particularly for problems in homotopy theory. In recent years there has been a tendency to study the total singular complex of a space (cf. example 2 below) instead of simplicial complexes; but this method is also inconvenient from the point of view of homotopy. A more useful procedure seems to be the study of abstract semi-simplicial complexes, introduced by Eilenberg and Zilber [1], and of the sub-class consisting of semi-simplicial complexes satisfying the extension condition of Kan (cf. definition 1.2 below).

Let Z^+ denote the set of non-negative integers.

Definition 1.1: A semi-simplicial complex consists of the following:

(i) A set $X = \bigcup_{q \in Z^+} X_q$,
where the X_q are disjoint sets (an element of X_q is called a q-simplex of X);

(ii) functions $\partial_i : X_{q+1} \longrightarrow X_q$, $i = 0,\dots,q+1$, called face operators;
functions $s_i : X_q \longrightarrow X_{q+1}$, $i = 0,\dots,q$, called degeneracy operators, satisfying the relations

$$\partial_i \partial_j = \partial_{j-1} \partial_i \qquad i < j,$$
$$s_i s_j = s_{j+1} s_i \qquad i \leq j,$$

$$\partial_i s_j = s_{j-1}\partial_i \qquad i < j\ ,$$
$$\partial_j s_j = \partial_{j+1} s_j = \text{identity}$$
$$\partial_i s_j = s_j \partial_{i-1} \qquad i > j+1$$

We shall usually denote a semi-simplicial complex by its set X of simplexes.

A simplex $x \in X_{n+1}$ is called <u>degenerate</u> if there exists $y \in X_n$ and a degeneracy operator s_j such that $x = s_j y$; otherwise x is called <u>non-degenerate</u>.

<u>Example 1</u>: Recall that a simplicial complex K is a set whose elements are finite subsets of a given set $\bar{K}$, subject to the condition that if $x \in K$ and y is a non-empty subset of x, then $y \in K$. Sets with n+1 elements are called n-simplexes, and the set of n-simplexes of K is denoted by K_n.

We now define a semi-simplicial complex $X(K)$ which arises from K in a natural manner. An n-simplex of $X(K)$ is a sequence $(a_0,\ldots,a_n)$ of elements of $\bar{K}$ such that the set $\{a_0,\ldots,a_n\}$ is an r-simplex of K for some $r \leq n$. Define

$$\partial_i(a_0,\ldots,a_n) = (a_0,\ldots,a_{i-1},a_{i+1},\ldots,a_n),$$
$$s_i(a_0,\ldots,a_n) = (a_0,\ldots,a_i,a_i,a_{i+1},\ldots,a_n).$$

<u>Example 2</u>: Let Δ_n denote the standard n-simplex, so that a point of Δ_n is an (n+1)-tuple $(t_0,\ldots,t_n)$ of real numbers such that $0 \leq t_i \leq 1$, $i = 0,\ldots,n$, and $\sum t_i = 1$. Let A be a topological space. A singular n-simplex of A is a map* $u:\Delta_n \longrightarrow A$. Let $S_n(A)$ be the set of singular n-simplex in

*by "map" we shall always mean a continuous function, provided both the domain and image are topological spaces.

A, and set $S(A) = \bigcup_{n\in Z^+} S_n(A)$. Define

$$\partial_i : S_n(A) \longrightarrow S_{n-1}(A)$$

by $\partial_i\, u(t_0,\ldots,t_{n-1}) = u(t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1})$,

and define

$$s_i : S_n(A) \longrightarrow S_{n+1}(A)$$

by $s_i\, u(t_0,\ldots,t_{n+1}) = u(t_0,\ldots,t_{i-1},t_i+t_{i+1},t_{i+2},\ldots,t_{n+1})$.

It is easy to verify that $S(A)$ is a semi-simplicial complex, the <u>total singular complex</u> of the space A, [2].

In the examples we have seen two ways in which semi-simplicial complexes arise; henceforth we shall consider abstract semi-simplicial complexes. For problems in homotopy theory it is convenient to restrict attention to semi-simplicial complexes satisfying the following condition:

<u>Definition 1.2</u> A semi-simplicial complex X is said to satisfy the <u>extension condition</u> if given $x_0,\ldots,x_{k-1},x_{k+1},\ldots,x_{n+1} \in X_n$ such that $\partial_i x_j = \partial_{j-1} x_i$, $i < j$, $i, j \neq k$, then there exists $x \in X_{n+1}$ such that $\partial_i x = x_i$, $i \neq k$. Such a complex will be called a <u>Kan complex</u>.

<u>Proposition 1.3</u>: If A is a topological space, then the total singular complex $S(A)$ satisfies the extension condition.

The proposition follows from the fact that the union of n+1 faces of Δ_{n+1} is a retract of Δ_{n+1}; thus a given map defined on the union of the n+1 faces can always be extended to Δ_{n+1}.

Although it has long been realized that the total singular complex satisfies the extension condition, it was only recently that D.M.Kan pointed out that the extension condition is sufficient for the definition of homotopy groups.

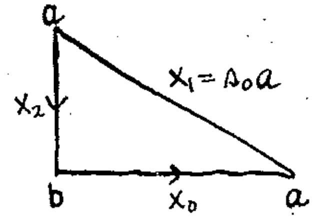
Definition 1.4: Let X be a semi-simplicial complex. A point of X is a 0-simplex, i.e. an element of X_0; and a path in X is a 1-simplex, i.e. an element of X_1. If x is a path in X, then $\partial_1 x$ is the initial point or origin of x, and $\partial_0 x$ is the final or terminal point of x.

Note that if A is a topological space, then a path in A is a map $u : \Delta_1 \longrightarrow A$, and therefore a path in $S(A)$. Further, the initial and final points of the path considered as an element of $S(A)$ are the same as when considered as the map $u : \Delta_1 \longrightarrow A$.

Let X be a Kan complex. The point $a \in X$ is said to be in the same path component as the point $b \in X$ if there exists a path with initial point a and final point b.

Proposition 1.5: The relation "to be in the same path component" is an equivalence relation.

Proof: (i) To show that the relation is symmetric, let x_2 be a path from a to b, and let $x_1 = s_0 a$. Now $\partial_1 x_2 = a = \partial_1 s_0 a = \partial_1 x_1$. Consequently there exists $x \in X_2$ such that $\partial_i x = x_i$, $i = 1,2$. Let $x_0 = \partial_0 x$. Then $\partial_0 x_0 = \partial_0 \partial_0 x = \partial_0 \partial_1 x = \partial_0 x_1 = \partial_0 s_0 a = a$, and $\partial_1 x_0 = \partial_1 \partial_0 x = \partial_0 \partial_2 x = \partial_0 x_2 = b$. Therefore x_0 is a path from b to a.



(ii) To show that the relation is transitive, let x_1, x_0 be paths from a to b and b to c respectively. Then $\partial_0 x_2 = b = \partial_1 x_0$. Let x be a 2-simplex such that $\partial_0 x = x_0, \partial_2 x = x_2$, and let $x_1 = \partial_1 x$. Then x_1 is a path in X from a to c.

(iii) That the relation is reflexive is clear.

Let $\pi_0(X)$ denote the set of path components of X. X is called <u>connected</u> if $\pi_0(X)$ has only one element.

<u>Definition 1.6</u>: If X is a semi-simplicial complex, and $x^* \in X_0$, define $\Omega(X; x^*)$ as follows:

i) $\Omega_n(X,x^*) = \{ x \mid x \in X_{n+1}, \partial_0 x = s_0^n x^*, \partial_{i_0} \cdots \partial_{i_n} x = x^* \}$

where $0 \leq i_k \leq n+1$, $k = 0,\ldots,n$.

ii) $\partial_i : \Omega_{n+1}(X,x^*) \longrightarrow \Omega_n(X,x^*)$ is the function determined by $\partial_{i+1} : X_{n+2} \longrightarrow X_{n+1}$, $i = 0,\ldots,n+1$

iii) $s_i : \Omega_n(X,x^*) \longrightarrow \Omega_{n+1}(X,x^*)$ is the function determined by $s_{i+1} : X_{n+1} \longrightarrow X_{n+2}$, $i = 0,\ldots,n$

iv) $\Omega(X,x^*) = \bigcup_{n \in Z^+} \Omega_n(X, x)$

<u>Theorem 1.7</u>: If X is a semi-simplicial complex, and $x^* \in X_0$, then

i) $\Omega(X,x^*)$ is a semi0simplcial complex

ii) If X satisfies the extension condition, then so does $\Omega(X,x^*)$.

The proof of this theorem is straightforward, and will be left to the reader.

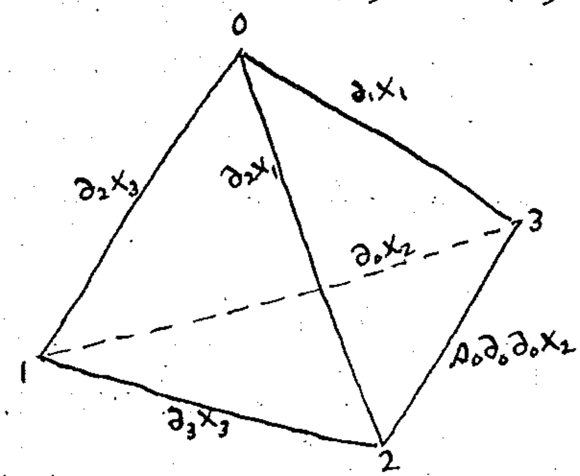
<u>Proposition 1.8:</u> If X is a Kan complex, and $x_2, x_3 \in X_2$ are such that $\partial_0 x_3 = \partial_0 x_2$, $\partial_2 x_3 = \partial_2 x_2$, then there exists $x_1 \in X_2$ such that $\partial_0 x_1 = s_0 \partial_0 \partial_0 x_2$, $\partial_1 x_1 = \partial_1 x_2$, $\partial_2 x_1 = \partial_1 x_3$

<u>Proof</u>: Let $x_0 = s_1 \partial_0 x_2$. Then $\partial_0 x_3 = \partial_0 x_2 = \partial_2 x_0$, $\partial_0 x_2 = \partial_1 x_0$, and there exists $x \in X_3$ such that $\partial_i x = x_i$ $i \neq 1$. Let $x_1 = \partial_1 x$. Then

$\partial_0 x_1 = \partial_0 \partial_1 x = \partial_0 \partial_0 x = \partial_0 s_1 \partial_0 x_2 = s_0 \partial_0 \partial_0 x_2$,

$\partial_1 x_1 = \partial_1 \partial_1 x = \partial_1 \partial_2 x = \partial_1 x_2$, and

$\partial_2 x_1 = \partial_2 \partial_1 x = \partial_1 \partial_3 x = \partial_1 x_3$.



<u>Notation and Convention</u>: If X is a semi-semplicial complex, and x^* is a point of X, let $\Omega^0(X,x^*) = X$, and let $\Omega^{n+1}(X,x^*) = \Omega(\Omega^n(X,x^*), s_0^n x^*)$. The point $s_0^n x^* \in \Omega^n{}_0(X,x^*)$ (here s_0 denotes the degeneracy operator in X) is the natural base point for $\Omega^n(X,x^*)$.

<u>Definition 1.9</u>: If X is a Kan complex, and x^* is a point of X, define $\pi_n(X,x^*)$ to be $\pi_0(\Omega^n(X,x^*))$.

Now $\pi_n(X,x^*)$ is the set we wish to make into the n-dimensional homotopy group of X. Therefore it remains to define a multiplication in $\pi_n(X,x^*)$ for $n > 0$. However, to do this it is sufficient to define a multiplication in $\pi_1(X,x^*)$ since $\pi_{n+1}(X,x^*) = \pi_1(\Omega^n(X,x_0),\ s_0^n x^*)$.

Let X be a Kan complex and $x^* \in X_0$. According to the preceding proposition there is a map of $\Omega_0(X,x^*) \times \Omega_0(X,x^*) \longrightarrow \pi_0(\Omega(X,x^*))$ defined as follows:

if $x,y \in \Omega_0(X,x^*) \subset X$, then there exists $w \in X_2$ and $z \in \Omega_0(X,x^*)$ such that $\partial_2 w = x, \partial_0 w = y, \partial_1 w = z$. Let $[z]$ denote the image of z in $\pi_0(\Omega(X,x^*))$. Although z is not unique, $[z]$ is so, according to the preceding proposition. We therefore denote $[z]$ by $x \cdot y$, and the desired map is given by $(x,y) \longrightarrow x \cdot y$.

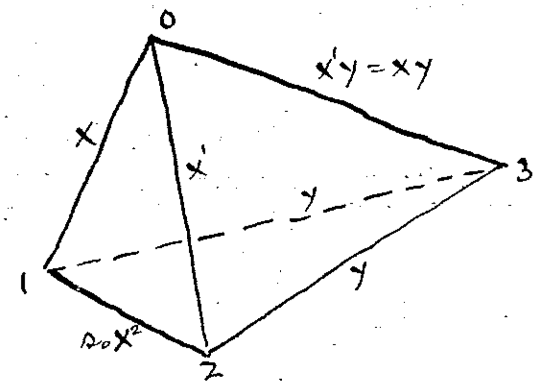
<u>Proposition 1.10</u>: If $x,x',y \in \Omega_0(X,x^*)$, and x,x' represent the same element of $\pi_0(\Omega(X,x^*))$, then $x \cdot y = x' \cdot y$.

<u>Proof</u>: Since $[x] = [x']$, there exists $z \in X_2$ such that $\partial_1 z = x', \partial_2 z = x, \partial_0 z = s_0 x^*$. By the extension condition, there exists $a \in X_2$ such that $\partial_0 a = y, \partial_2 a = x', \partial_1 a = x'y$, and there exists $b \in X_3$ such that $\partial_0 b = s_0 y, \partial_1 b = a, \partial_3 b = z$. Setting $c = \partial_2 b$, we have

$$\partial_0 c = \partial_0 \partial_2 b = \partial_1 \partial_0 b = \partial_1 s_0 y = y,$$

$$\partial_2 c = \partial_2 \partial_2 b = \partial_2 \partial_3 b = \partial_2 z = x.$$

therefore $\partial_1 c = x \cdot y$; but $\partial_1 c = \partial_1 \partial_2 b = \partial_1 \partial_1 b = \partial_1 a = x'y$, and the proposition follows.



Proposition 1.11: If $x, y, y' \in \Omega_0(X, x^*)$, and $[y] = [y']$, then $x \cdot y = x \cdot y'$.

Proof: By hypothesis there exist $a, b, z \in X_2$ such that

$$\partial_0 a = y, \partial_2 a = x, \partial_1 a = xy$$

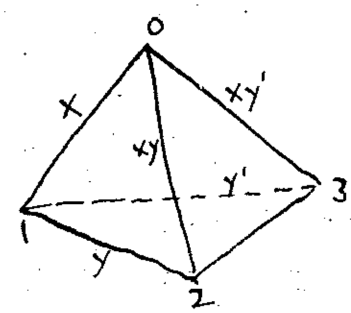
$$\partial_0 b = y', \partial_2 b = x, \partial_1 b = xy'$$

$$\partial_0 z = s_0 x_0, \partial_1 z = y', \partial_2 z = y$$

Then by the extension condition there exists $c \in X_3$ such that

$$\partial_0 c = z, \partial_2 c = b, \partial_3 c = a.$$

Let $d = \partial_1 c$; then



$$\partial_2 d = \partial_2 \partial_1 c = \partial_1 \partial_3 c = \partial_1 a = xy$$

$$\partial_1 d = \partial_1 \partial_1 c = \partial_1 \partial_2 c = \partial_1 b = xy'$$

$$\partial_0 d = \partial_0 \partial_1 c = \partial_0 \partial_0 c = \partial_0 z = \partial_0 s_0 x_0 = x_0.$$

Therefore $x \cdot y = x \cdot y'$.

According to propositions 1.10, 1.11 there is a map

$$\pi_0(\Omega_0(X, x^*) \times \pi_0(\Omega_0(X, x^*)) \longrightarrow \pi_0(\Omega(X, x^*))$$

given by $[x] \cdot [y] = x \cdot y$.

If X is a Kan complex, we shall denote by $\pi_n(X,x^*)$ the set previously defined together with this multiplication. We shall use Greek letters to denote the elements of $\pi_n(X,x^*)$.

Theorem 1.12: If X is a Kan complex, $x^* \in X_0$, then $\pi_n(X,x^*)$ is a group for $n \geq 1$.

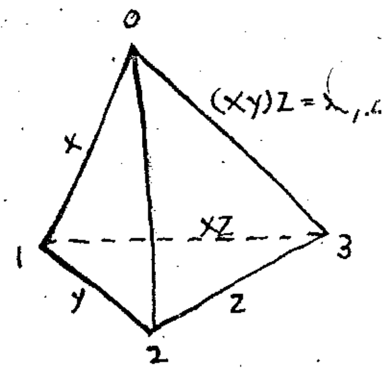
Proof: Let $\alpha, \beta, \gamma \in \pi_n(X,x^*) = \pi_0(\Omega^n(X,x^*)) = \pi_0(\Omega(\Omega^{n-1}(X,x^*)))$.

have representatives $x,y,z \in \Omega_0(\Omega^{n-1}(X,x^*)) \subset \Omega_1^{n-1}(X,x^*)$

i) Associativity: There exist $a_0, a_1, a_3 \in \Omega_2^{n-1}(X,x^*)$ such that $\partial_0 a_0 = z$, $\partial_2 a_0 = y$, $\partial_1 a_0 = yz$

$\partial_0 a_1 = z, \partial_2 a_1 = xy, \partial_1 a_1 = (yx)z$

$\partial_0 a_3 = y, \partial_2 a_3 = x, \partial_1 a_3 = xy$



By the extension condition there exists $b \in \Omega_3^{n-1}(X,x^*)$ such that $\partial_i b = a_i, i = 0,1,3$. Set $a_2 = \partial_2 b$. Then

$\partial_0 a_2 = \partial_0 \partial_2 b = \partial_1 \partial_0 b = \partial_1 a_0 = yz$

$\partial_2 a_2 = \partial_2 \partial_2 b = \partial_2 \partial_3 b = \partial_2 a_3 = x$

and therefore $\partial_1 a_2 = x(yz)$. But

$x(y_1z) = \partial_1 a_2 = \partial_1 \partial_2 b = \partial_1 \partial_1 b = \partial_1 a_1 = (xy)z$.

ii) A left identity is furnished by $s_0 x_0$; for $s_0 x \in \Omega_2^{n-1}(X,x^*)$ has as faces $\partial_0 s_0 x = x, \partial_2 s_0 x = s_0 \partial_1 x = s_0 x_0, \partial_1 s_0 x = x$;

and hence $(s_0 x) x = x$.

iii) Left inverse:

By the extension condition there exists $a \in \Omega_2^{n-1}(X,x^*)$ such that $\partial_0 a = x, \partial_1 a = s_0 x_0$;

then by definiton

$$(\partial_2 a)x = s_0 x_0,$$

so that $\partial_2 a$ is a left inverse for x.

In order to see the connection between the homotopy groups of a Kan complex X and homotopy groups as classically defined it is convenient to define $\Omega^n(X,x^*)$ directly, instead of inductively. We therefore write down the explicit definition of $\Omega^n(X,x^*)$ using elements of X, and face and degeneracy operators of X.

$\Omega^n{}_q(X,x^*) = \{x | x \in X_{n+q}, \partial_i x = s_0^{n+q-1}x^*$ for $i < n$, and $\partial_{i_0} \cdots \partial_{i_q} x = s_0^{n-1}x^*\}$. This definition is easily seen to coincide with that originally given. Now $\Omega^n{}_0(X,x^*) = \{x | x \in Xn, \partial_i x = s_0^{n-1}x^*$ for $i < n$, and $\partial_{i_0} x = s_0^{n-1}x^*\}$. Therefore an element of $\Omega^n{}_0(X,x^*)$ is an n-simplex of X all of whose faces are at the base point, and an element of $\pi_n(X,x^*)$ is an equivalence class of such simplexes. Two such simplexes x,x' are equivalent if there exists $z \in X_{n+1}$ such that $\partial_{n+1} z = x, \partial_n z = x'$, and $\partial_i z = s_0^n x^*$ for $i < n$. Futher, if x,x' are two n-simplexes all of whose faces are at the base point, then $[x \cdot x']$ is represented as follows: By the extension condition there exists $z \in X_{n+1}$ such that $\partial_{n+1} z = x, \partial_{n-1} z = x'$, and $\partial_i z = s_0^n x^*$ for $i < n$. $[x \cdot x']$ is represented by $\partial_n z$.

<u>Definition 1.13:</u> If X,Y are semi-simplicial complexes, then $f : X \longrightarrow Y$ is a <u>semi-simplicial map</u> if

1) $f(X_q) \subset Y_q$,

2) $f\partial_i = \partial_i f$, all i, and

3) $s_i f = f s_i$, all i .

We shall often denote $f|X_q$ by f_q .

<u>Definition 1.14:</u> If X, Y are Kan complexes and $f:X \longrightarrow Y$ is a semi-simplicial map, then for every $q \geq 0$ f induces a function

$$f_q^{\#} : \pi_q(X,x^*) \longrightarrow \pi_q(Y,f(x^*))$$

by $f_q^{\#}[x] = [f_q x]$, for $x \in \Omega^q_0(X,x^*)$.

<u>Proposition 1.15</u>: The function $f_q^{\#}$ is a homomorphism for $q > 0$.

The proof is evident from the definition.

<u>Proposition 1.16</u>: Let A, B, C be Kan complexes

i) If $f: A \longrightarrow B$, $g : B \longrightarrow C$, are semi-simplicial maps, and $a^* \in A_0$, then

$$(gf)_q^{\#} = g_q^{\#} f_q^{\#} : \pi_q(X,a^*) \longrightarrow \pi_q(C,gf(a^*)).$$

ii) If 1 is the identity map of A, then $1_q^{\#}$ is the identity automorphism of $\pi_q(A,a^*)$.

It is convenient to derive some of the relations between the faces of a 3-simplex. For the following five propositions let X be a Kan complex, $x \in X_0$.

<u>Proposition 1.17</u>: Let x_3 be 3-simplex such that $\partial_i \partial_j x_3 = s_0 x$, all i, j. Let the faces of x_3 be a, b, c, $s_0^2 x$, in order. Then [a] [c] = [b].

Proof: It is straightforward to check that the following four 3-simplexes satisfy the extension condition:

$x_0 = s_2 a$

x_1 has faces $\partial_0 x_1 = s_0^2 x, \partial_2 x_1 = b, \partial_3 x_1 = a$, and is then obtained by the extension condition. Set $w = \partial_1 x_1$.

x_3 as given

$x_4 = s_0 a$.

Therefore there exists a 4-simplex z such that $\partial_i z = x_i$, $i \neq 2$. Let $x_2 = \partial_2 z$.

Then $\partial_0 x_2 = s_0^2 x, \partial_1 x_2 = w, \partial_2 x_2 = c, \partial_3 x_2 = s_0^2 x$.

Therefore, by the rule for addition, $[w] = [c]$. But from x_1 we have $[a][w] = [b]$; therefore $[a][c] = [b]$.

Proposition 1.18: Let x_2 be a 3-simplex such that $\partial_i \partial_j x_2 = s_0 x$, all i, j. Let the faces of x_2 be $a, s_0^2 x, c, d$, in order. Then $[c][a] = [d]$.

Proof: The following four 3-simplexes satisfy the extension condition:

$x_0 = s_0 a$

x_1 has faces $\partial_0 x_1 = a, \partial_1 x_1 = \partial_3 x_1 = s_0^2 x$, and is obtained by extension. Let $y = \partial_2 x_1$.

x_2 as given

$x_4 = s_2 d$

Therefore there exists a 4-simplex z such that $\partial_i z = x_i$, $i \neq 3$. Set $x_3 = \partial_3 z$

Then $\partial_0 x_3 = s_0^2 x, \partial_1 x_3 = y, \partial_2 x_3 = c, \partial_3 x_3 = d$. Therefore $[d][y] = [c]$. From x_1 and 1.17 we have $[y] = [a]^{-1}$. Therefore $[d] = [c][a]$.

<u>Proposition 1.19:</u> Let x_4 be a 3-simplex such that $\partial_i \partial_j x_4 = s_0 x$, all i, j. Let the faces of x_4 be a,b,c,d in order. Then $[d][b][a]^{-1} = [c]$.

<u>Proof</u>: The following four 3-simplexes satisfy the extension condition:

x_0 has faces $\partial_1 x_0 = \partial_2 x_0 = s_0 x, \partial_3 x_0 = a$, and is obtained by extension.

set $v = \partial_0 x_0$.

x_1 has faces $\partial_0 x_1 = v, \partial_1 x_1 = s_0 x, \partial_3 x_0 = b$, and is obtained by extension.

set $w = \partial_2 x_1$.

$x_2 = s_2 c$

x_4 as given.

Therefore there exists a 4-simplex z such that $\partial_i z = x_i, i \neq 3$. Set $x_3 = \partial_3 z$.

By 1.18, $[v] = [a]$, and $[w] = [b][v]^{-1} = [b][a]^{-1}$.

x_3 has faces $\partial_0 x_3 = s_0^2 x, \partial_1 x_3 = w, \partial_2 x_3 = c, \partial_3 x_3 = d$.

Therefore $[c] = [d][w] = [d][b][a]^{-1}$.

Setting $d = s_0^2 x$ in 1.19, x_4 then has faces $a, b, c, s_0^2 x$, in order, and the relation $[c] = [b][a]^{-1}$ holds. But 1.17 applies to the simplex x_4 to give the relation $[c] = [a]^{-1}[b]$. Therefore, for arbitrary [a] and [b], $[b][a]^{-1} = [a]^{-1}[b]$, or

$[a][b] = [b][a]$, and π_2 is therefore abelian. Since the higher homotopy groups were defined by iteration, we have

<u>Corollary 1.20:</u> $\pi_n(X,x)$ is abelian for $n \geq 2$.

We shall henceforth write π_n additively for $n \geq 2$.

<u>Proposition 1.21</u>: Let $z \in X_{q+1}, q \geq 2$, be such that

(1) $\partial_r z = a, \partial_{r+1} z = b, \quad 0 \leq r \leq q$

(2) $\partial_i z = s_0^q x, \quad i \neq r, r+1$

(3) $\partial_j \partial_k z = s_0^{q-1} x$, all j,k.

Then $[a] = [b]$.

<u>Proof</u>: If $r = q$, the proposition follows from the definition of homotopy classes. Suppose $r < q$; then the following set of q+1 (q+1)-simplexes satisfies the extension condition:

$$y_i = s_0^{q+1} x \quad \text{for } i < r \quad \text{and } i > r+3$$

$$y_{r+1} = s_{r+1} b$$

$$y_{r+z} = z$$

$$y_{r+3} = s_r b$$

Then there exists $y \in X_{q+2}$ such that $\partial_i y = y_i$, $i \neq r$. $y_r = \partial_r y$ has faces $\partial_i y_r = s_0^q x$, $i \neq r+1, r+2$; $\partial_{r+1} y = a$, $\partial_{r+2} y = b$.

If we iterate this process q-r times we obtain a (q+1)-simplex y' such that

$$\partial_i y' = s_0^q x, \quad , i < q, \ \partial_q y' = a, \ \partial_{q+1} y' = b.$$

Hence $[a] = [b]$.

<u>Proposition 1.22:</u> Let X be a Kan complex $x \in X_0$. Let $z \in X_{q+1}$, $q \geq 2$, be such that (1) $\partial_{r-1} z = a$, $\partial_r z = b$, $\partial_{r+1} z = c$, where $1 \leq r \leq q$, (2) $\partial_i z = s_0^q x$, $i \neq r-1, r, r+1$. (3) $\partial_j \partial_k z = s_0^{q-1} x$, all j,k.

Then $[b] = [c][a] = [a][c]$.

<u>Proof</u>: Hypothesis (3) implies that a,b,c represent elements of $\pi_q(X,x)$; and since this group is abelian, $[c][a] = [a][c]$. If $r = q$, $[b] = [c][a]$ is just the definition of the group operation. If $r < q$, then the following set of q+1 (q+1)-simplexes satisfies the extension condition:

$y_i = s_0^{q+1} x$ for $i < r$ and $i > r+4$

$y_i = s_{r+2} a$

y_{r+2} has faces $\partial_i y_{r+2} = s_0^q x$, $i \neq r+1, r+2$, $\partial_{r+2} y_{r+2} = c$, and is obtained by extension. Let $w = \partial_{r+1} y_{r+2}$.

$y_{r+3} = z$

$y_{r+4} = s_r a$.

Then there exists $y \in X_{q+2}$ such that $\partial_i y = y_i$, $i \neq r+1$.

$y_{r+1} = \partial_{r+1} y$ has faces $\partial_i y_{r+1} = s_0^q x$, $i < r+1$ or $i > r+3$, $\partial_{r+1} y_{r+1} = w$, $\partial_{r+2} y_{r+1} = b$, $\partial_{r+3} y_{r+1} = a$.

By the previous proposition, $[w] = [c]$.

It is easy to see that by iterating this process q-r-1 times we obtain a (q+1)-simplex y' such that

$\partial_i y' = s_0^q x$, $i < q-1$, $_{q-1} y' = w'$,

$\partial_q y' = b$, $\partial_{q+1} y' = w''$, and such that either $[w'] = [a], [w''] = [c]$, or $[w'] = [c], [w''] = [a]$.

In either case $[b] = [c][a] = [a][c]$.

Definition 1.23: A semi-simplicial fiber space is a triple (E,p,B) where E,B are semi-simplicial complexes, and $p: E \longrightarrow B$ is a semi-simplicial map, satisfying the following condition: if $x \in B_{q+1}$, $y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_{q+1} \in E_q$ are such that $p(y_i) = \partial_i x$ for $i \neq k$, and $\partial_i y_j = \partial_{j-1} y_i$, for $i < j$, $i,j \neq k$, then there exists $y \in E_{q+1}$ such that $p(y) = x$, and $\partial_i y = y_i$ for $i \neq k$.

Let b be a point of B, and let $F_q = \{x | x\ E_q, p(x) = s_0^q\, b\}$. Let $F = \bigcup F_g$, and defines $\partial_i : F_{g+1} \longrightarrow F_q$ to be the function induced by $\partial_i : E_{q+1} \longrightarrow E_q$, and $s_i : F_q \longrightarrow F_{q+1}$ to be the function induced by $s_i : E_q \longrightarrow E_{q+1}$. Now F is a semi simplicial complex called the fibre over b.

Proposition 1.24: F is a Kan complex.

Proof: Suppose $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{q+1} \in F_q$ are such that $\partial_i x_j = \partial_{j-1} x_i$ for $i < j, i, j \neq k$. Then $p(x_i) = s_0^q b$; and since (E,p,B) is a fibre space, there exists $x \in E_{q+1}$ such that $p(x) = s_0^{q+1} b$, and $\partial_i x = x_i$ for $i \neq 1$. Since $p(x) = s_0^{q+1} b$, $x \in F_{q+1}$, which proves the proposition.

Now let (E,p,B) be a semi-simplicial fibre space in which E and B are Kan complexes, and let F be the fibre over a point b of B. Let a be a point of F, which we assume to be non-empty.

For $q \geq 2$ we define a homomorphism

$$\partial^{\#} : \pi_q(B,b) \longrightarrow \pi_{q-1}(F,a)$$

as follows. Recall that an element $\alpha \in {}_q(B,b)$ is represented by $x \in B_q$ such that $\partial_i = s_0^{q-1}b$ for all i. Since p is a fibre map, there exists $y \in E_q$ such that $p(y) = x$ and $\partial_i y = s_0^{q-1}a$ for $i > 0$. Then $\partial_0 y$ is contained in F_{q-1}, and represents an element of $\pi_{q-1}(F,a)$. Suppose $x' \in B_q$ also represents α. Then there exists $z \in B_{q+1}$ such that $\partial_i z = s_0^q b$, $i < q$, $\partial_q z = x$, $\partial_{q+1} z = x'$. Let $y' \in E_q$ be such that $p(y) = x'$ and $\partial_i y = s_0^{q-1}a$ for $i > 0$. Since p is a fibre map, there exists $w \in E_{q+1}$ such that $p(w) = z$, $\partial_i w = s_0^q a$, $0 < i < q$, $\partial_q w = y$, $\partial_{q+1} w = y'$. Now $p(\partial_0 w) = s_0^q b$: and $\partial_i \partial_0 w = \partial_0 \partial_{i+1} w = s_0^q a$, $i < q-1$, $\partial_{q-1}\partial_0 w = \partial_0 y$, $\partial_q \partial_0 w = \partial_0 y'$. Therefore $[\partial_0 y] = [\partial_0 y']$ in $\pi_{q-1}(F,a)$. Since in particular we may take $x' = x$, the element $[\partial_0 y]$ is independent of both the choice of x representing $[x]$ and the choice of y. We set $\partial^{\#}\alpha = [\partial_0 y]$.

We now show that $\partial^{\#}$ is a homomorphism. Let $\alpha, \beta \in \pi q(B,b)$ have representatives x, x' respectively. Let $z \in B_{q+1}$ have faces

$$\partial_i z = s_0^q b, \quad i < q-1, \quad \partial_{q-1} z = x', \ \partial_{q+1} z = x.$$

Then $\partial_q z$ represents $\alpha + \beta$. Let $v \in E_{q+1}$ be such that $p(v) = z$, $\partial_i v = s_0^q a$, $0 < i < q-1$, and

$\partial_i \partial_j v = s_0^{q-1} a$ for $j = q-1, q, q+1$, and $i > 0$.

Then $\partial_0 v \in F_q$, and

$\partial_i \partial_0 v = s_0^{q-1} a$ for $i < q-2$, $\partial_{q-2}\partial_0 v = \partial_0 \partial_{q-1} v$,

$\partial_{q-1}\partial_0 v = \partial_0 \partial_q v, \partial_q \partial_0 v = \partial_0 \partial_{q+1} v$.

Since $\partial_0\partial_{q-1}v, \partial_0\partial_q v, \partial_0\partial_{q+1}v$ represent $\partial^{\#}\beta$, $\partial^{\#}(\alpha+\beta)$, $\partial^{\#}\alpha$ respectively, from their relationship as faces of $\partial_0 v$ it follows that

$\partial^{\#}(\alpha+\beta) = \partial^{\#}\alpha + \partial^{\#}\beta$.

<u>Theorem 1.25</u>: Let (E,p,B) be a semi-simplicial fibre space in which E and B are Kan complexes. Let $b \in B_0$, F the fibre over b, $a \in F_0$ (we assume F non-empty). Let $i : F \longrightarrow E$ be the inclusion map. Then the following sequence is exact:

$$\cdots \longrightarrow \pi_q(F,a) \xrightarrow{i^{\#}} \pi_q(E,a) \xrightarrow{p^{\#}} \pi_q(B,b) \xrightarrow{\partial^{\#}} \pi_{q-1}(F,a) \longrightarrow \cdots$$

<u>Proof</u>: Let x represent $\alpha \in \pi_q(F,a)$. Then $pix = s_0^q b$, and consequently $p^{\#} i^{\#} = 0$. If x represents $\alpha \in \pi_q(E,a)$, then $\partial_0 x = s_0^{q-1}a$ represents $\partial^{\#}p^{\#}\alpha$ and $\partial^{\#}p^{\#} = 0$. Again, let x represent $\alpha \in \pi_q(B,b)$. Let $y \in E_q$ be such that $\partial_i y = s_0^{q-1}a$, $0 < i$, and $p(y) = x$. Then $\partial_0 y$ represents $i^{\#}\partial^{\#}\alpha$; but as an element of $\pi_{q-1}(E,a)$, by proposition 1.21, $[\partial_0 y] = [\partial_1 y] = [s_0^{q-1}a] = 0$, and $i^{\#}\partial^{\#} = 0$.

If x represents $\alpha \in \pi_q(F,a)$ and $i^{\#}(\alpha) = 0$, then there exists $y \in E_{q+1}$ such that $\partial_i y = s_0^q a$, $i < q+1$, and $\partial_{q+1} y = x$. Therefore $\partial_i p(y) = s_0^q b$, $i \leq q+1$, and $\partial^{\#}[p(y)] = \alpha$.

Suppose that x represents $\alpha \in \pi_q(E,a)$ such that $p^{\#}\alpha = 0$. Then we may assume that $x \in F_q$, and thus $i^{\#}[x] = \alpha$.

Finally suppose that x represents $\alpha \in \pi_q(B,b)$ such that $\partial^{\#}\alpha = 0$. Then there exists $y \in E_q$ such that $p(y) = x, \partial_i y = s_0^{q-1}a$, $0 < i$, and $[\partial_0 y] = [s_0^{q-1}a]$ in $\pi_{q-1}(F,a)$. Therefore, since p is a fibre map, there exists $y' \in E_q$ such that $p(y') = x$ and $\partial_i y' = s_0^{q-1}a$, all i. Then $p^{\#}[y'] = \alpha$.

This completes the proof of the theorem.

Proposition 1.26: Let (E,p,B) be a fibre space, $p: E \longrightarrow B$ be onto, $x \in B_q$, and let $y_{i_0} \ldots, y_{i_r} \in E_{q-1}, 0 \leq i_0 < \ldots < i_r \leq q$, be such that $\partial_{i_s} y_{i_t} = \partial_{i_t -1} y_{i_s}$ for $s < t$, $\{i_0, \ldots, i_r\} \neq \{0, \ldots, q\}$, and $p(y_{i_s}) = \partial_{i_s} x$; then there exists $y \in E_q$ such that $p(y) = x$, and $\partial_{i_s} y = y_{i_s}$, $s = 0, \ldots, r$.

Proof: If $q = 1$, then the proposition follows immediately from the definition of fibre space. Consequently suppose that the proposition is true for $q \leq n$, and that $q = n+1$. If the set $\{i_0, \ldots, i_r\}$ has q elements, the result follows immediately from the definition of fibre space. In this case $r = q-1$. Suppose then that the proposition is true for $r \geq m$, $m \leq q-1$, and that $r = m-1 \geq 0$. Let $t \in \{0, \ldots, q\}$ be the least integer such that $t \notin \{i_0, \ldots, i_r\}$. Define

$j_s = i_s$ for $i_s < t$. Let s' be the largest integer s such that $i_s < t$. Define $j_{s'+1} = t$, and $j_s = i_{s-1}$ for $s' + 1 < s \leq r+1 = m$. We now wish to define y_t such that

$$\partial_{j_s} y_t = \partial_{t-1} y_{j_s} \quad s \leq s', \partial_{j_s 1} y_t = \partial_t y_{j_s} \quad s \geq s'+2.$$

The set $\{j_0, \ldots, j_{s'}, j_{s'+2}, \ldots, j_{r+1}\}$ has at most $(q-1)$ elements. Therefore, using the inductive hypothesis, we may choose $y_t = y_{j_{s'+1}}$ such that $\partial_{j_s} y_t = \partial_{t-1} y_{j_s} \quad s \leq s', \partial_{j_{s-1}} y_t = \partial_t y_{j_s} \quad s \geq s'+2$, and $p(y) = \partial_t x$. Now the set $\{j_0, \ \ldots, j_{r+1}\}$ has m elements; therefore by inductive hypothesis there exists $y \in E_q$ such that $p(y) = x$, and $\partial_{j_s} y = y_{j_s} \quad s = 0, \ldots, r+1$. Then $p(y) = x$, and $\partial_{i_s} y = y_{i_s} \quad i = 0, \ldots, r$.

<u>Proposition 1.27</u>: If (E,p,B) is a fibre space, and p is onto, then B is a Kan complex if and only if E is a Kan complex.

<u>Proof</u>: Let E be a Kan complex, and let $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_q$ be elements of B_{q-1} such that $\partial_i x_j = \partial_{j-1} x_i \; i < j, \; i j \neq k$. Choose $y_0 \in E_{q-1}$ such that $p(y_0) = x_0$, choose $y_1 \in E_{q-1}$ such that $p(y_1) = x_1$ and $\partial_0 y_1 = \partial_0 y_0$, and continue in this manner until $y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_q$ have been chosen such that $\partial_i y_j = \partial_{j-1} y_i, i < j, \; i, j \neq k$, and $p(y_i) = x_i \quad i \neq k$. This procedure is possible by the preceding proposition. Now choose $y \in E_q$

such that $\partial_i y = y_i$ for $i \neq k$, and let $x = p(y)$. Then $\partial_i x = x_i$ for $i \neq k$, and B is a Kan complex

Now let B be a Kan complex, and let $y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_q$ be elements of E_{q-1} such that $\partial_i y_j = \partial_{j-1} y_i$, $i < j$, $i, j \neq k$. Let $x_i = p(y_i)$, and let $x \in B_q$ be an element such that $\partial_i x = x_i$ for $i \neq k$. Since p is a fibre map, there exists $y \in E_q$ such that $p(y) = x$ and $\partial_i y = y_i$ for $i \neq k$. Therefore E is a Kan complex.

Definition 1.28: Let X be a semi-simplicial complex. If $x, x' \in X_q$; and n is a non-negative integer then $x \overset{n}{\sim} x'$ if and only if $\partial_{i_1} \ldots \partial_{i_r} x = \partial_{i_1} \ldots \partial_{i_r} x'$ for every iterated face operator $\partial_{i_1} \ldots \partial_{i_r}$ such that $n + r \geq q$.

Lemma 1.29: If $x, x', x'' \in X_q$, then

1) $\overset{n}{\sim}$ is an equivalence relation
2) If $x \overset{n}{\sim} x'$, then $\partial_i x \overset{n}{\sim} \partial_i x'$, and
3) If $x \overset{n}{\sim} x'$, then $s_i x \overset{n}{\sim} s_i x'$.

Definition 1.30: Let X be a semi-simplicial complex. Define a semi-simplicial complex $X^{(n)}$ as follows:

1) An element of $X_q^{(n)}$ is an equivalence class of q-simplexes of X, $x, x' \in X_q$ being equivalent if $x \overset{n}{\sim} x'$,
2) $\partial_i : X_{q+1}^{(n)} \longrightarrow X_q^{(n)}$ is induced by $\partial_i : X_{q+1} \longrightarrow X_q$ and
3) $s_i : X_q^{(n)} \longrightarrow X_{q+1}^{(n)}$ is induced by $s_i : X_q \longrightarrow X_{q-1}$

Let $X^{(\infty)} = X$, and let $p_k^n : X^{(n)} \longrightarrow X^{(k)}$ be the natural map for $n \geq k$, where $\infty \geq k$, for every k. When there is no danger of confusion, p_k^n will be abbreviated by p.

Theorem 1.31: If X is a Kan complex, then $(X^{(n)}, p_k^n, X^{(k)})$ is a fibre space for $n \geq k$, and $X^{(n)}$ is a Kan complex.

Proof: We will first prove that $(X^{(\infty)}, p_k^{\infty}, X^{(k)})$ is a fibre space. Suppose that $x \in X_q^{(k)}$, and that $y_0, \ldots, y_{k-1}, y_{k+1}, \ldots, y_q \in X_{q-1}$ are such that $\partial_i y_j = \partial_{j-1} y_i$, $i, j \neq k$, $i < j$, and $p(y_i) = \partial_i x$. Now if $q \leq k$, then $X_q^{(k)} = X_q$, and $y_i = \partial_i x$. Therefore if we choose $y = x$, then $y \in X_q$, $\partial_i y = y_i$, and $p(y) = x$. Assume therefore that $q > k$. Since X is a Kan complex there exists $y \in X_q$ such that $\partial_i y = y_i$ for $i \neq k$. Further any face of dimension $\leq n$ of y is also a face of some y_i. Therefore $p(y) = x$, and p is a fibre map. Now $X^{(\infty)} = X$ is a Kan complex, and p_k^{∞} is a fibre map. Therefore, $X^{(k)}$ is a Kan complex. The fact that $(X^{(n)}, p_k^n, X^{(k)})$ is a fibre space follows similarly, and the details will be left to the reader.

The fibre spaces $(X, p, X^{(n)})$ are closely related to the construction (II) of Cartan and Serre [2].

Notation: If X is a Kan complex, $x \in X_0$, let $E_n(X,x)$ denotes the fibre of $p : X \longrightarrow X^{(n-1)}$.

The complex $E_n(X,x)$ is the n-th Eilenberg subcomplex of X based at x.[3].

Theorem 1.32. Let X be a Kan complex, $x \in X_0$, and $i : E_{n+1}(X,x) \longrightarrow X$ the natural inclusion map. Then

1) $p^{\#} : \pi_q(X,x) \xrightarrow{\approx} \pi_q(X^{(n)},x)$ for $q \leq n$,

2) $\pi_q(X^{(n)},x) = 0$ for $q > n$,

3) $i^{\#} : \pi_q(E_{n+1}(X,x),x) \xrightarrow{\approx} \pi_q(X,x)$ for $q > n$.

4) $\pi_q(E_{n+1}(X,x),x) = 0$ for $q \leq n$.

Proof: Notice that $E_{n+1}(X,x)_q$ has a single element for $q \leq n$. This implies (4), and (4) implies (1) since $(X,p,X^{(n)})$ is a fibre space with fibre $E_{n+1}(X,x)$.

Let y represent $\alpha \in \pi_q(X^{(n)},x)$; then $\partial_i y = s_0^{q-1}x$ for all i. Now y is an equivalence class of simplexes $z \in X_q$, and the above condition on the faces of y implies that all faces of dimension $r \leq n$ of z are $s_0^r x$. Therefore $s_0^q x$ is in the class y, and $\alpha = 0$. This proves (2), which implies (3), using the exact sequence of the fibre space.

Definition 1.33: If X is a Kan complex, let $\mathfrak{X}^n = (X^{(n+1)}, p, X^{(n)})$. The sequence $\mathfrak{X} = (\mathfrak{X}^0, \mathfrak{X}^1, \ldots, \mathfrak{X}^n, \ldots)$ is defined to be the natural Postnikov system of X. [4].

Theorem 1.34. If X is a Kan complex, $\mathfrak{X}$ is the natural Postnikov system of X, x is a part of X, and if $F^{(n+1)}$ is the fibre over x in the fibre space $\mathfrak{X}^n$, then

$$\pi_q (F^{n+1}, x) = 0 \quad \text{for} \quad q \neq n+1$$
$$\pi_{n+1}(F^{(n+1)}, x) = \pi_{n+1} (X, x).$$

The proof, which follows easily from the previous theorems, will be omitted.

Definition 1.35: If X is a connected Kan complex, n is a positive integer, $\pi_q(X,x) = 0$ for $q \neq n$, and $\pi_n(X,x) = \pi$; then X will be called an Eilenberg-MacLane complex of type (π, n).

Thus what we have shown is that, in some sense, any Kan complex X can be constructed from Eilenberg-MacLane complexes, and that this is done by means of the natural Postnikov system of X.

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References

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Chapter 1. Appendix A.

In Chapter 1, no general definition was given of homotopy between maps of one semi-simplicial complex into another. The purpose of this appendix is to rectify that situation, and further to prove after the manner of Eilenberg-Zilber ([1]), that every Kan complex is equivalent to a minimal subcomplex.

Definition: If X and Y are semi-simplicial complexes, the Cartesian product of X and Y is the semi-simplicial complex $X \times Y$ given by

1) $(X \times Y)_q = \{ (a,b) \mid a \in X_q, b \in Y_q \}$,
2) if $(a,b) \in (X \times Y)_{q+1}$, then $\partial_i(a,b) = (\partial_i a, \partial_i b)$ for $i = 0,\ldots,q+1$, and
3) if $(a,b) \in (X \times Y)_q$, then $s_i(a,b) = (s_i a, s_i b)$ for $i = 0,\ldots,q$.

Notation and Convention: Let Δ_q denote the semi-simplicial complex defined by the following:

1) an n-simplex is an (n+1)-tuple $(a_0,\ldots,a_n)$ of integers a_i such that $0 \leq a_0 \leq \ldots \leq a_i \leq a_{i+1} \leq \ldots \leq a_n \leq q$,
2) $\partial_i (a_0,\ldots,a_n) = (a_0,\ldots,a_{i-1},a_{i+1},\ldots,a_n)$, and
3) $s_i(a_0,\ldots,a_n) = (a_0,\ldots,a_{i-1},a_i,a_i,a_{i+1},\ldots,a_n)$.

The semi-simplicial complex Δ_q is the standard q-simplex, and itself has a canonical element of dimension q, namely

$(0,\ldots,q)$. If X is any semi-simplicial complex, and $x \in X_q$, there is a unique semi-simplicial map $f : \Delta_q \longrightarrow X$ such that $f(\,(0,\ldots,q)\,) = x$. The semi-simplicial complex Δ_1 will also be denoted by I.

<u>Definition:</u> If X, Y are semi-simplicial complexes, $f_0, f_1 : X \longrightarrow Y$ are homotopic if there exists $F : X \times I \longrightarrow Y$ such that for any simplex σ of X,

1) $F(\sigma \times (0,\ldots,0)\,) = f_0(\sigma)$, and
2) $F(\sigma \times (1,\ldots,1)\,) = f_1(\sigma)$

The map F is a <u>homotopy from</u> f_0 to f_1. If A is a subcomplex of X, and $f_0|A = f_1|A$, then f_0 is said to be <u>homotopic to</u> f_1 <u>relative to</u> A if there exists a homotopy F from f_0 to f_1 such that $F(\sigma \times \tau) = f_0(\sigma)$ for $\sigma \in A$. The subcomplex A is a <u>deformation retract</u> of X, if the identity map of $X \longrightarrow X$ is homotopic relative to A to a map of X into A.

<u>Proposition 1:</u> If X and Y are semi-simplicial complexes, then $f_0, f_1 : X \longrightarrow Y$ are homotpic if and only if there exist functions $k_i : X_q \longrightarrow Y_{q+1}$ defined for $i = 0,\ldots,q$, and all q such that

1) $\partial_0 k_0 = f_1$,
2) $\partial_{q+1} k_q = f_0$,
3) $\partial_i k_j = k_{j-1} \partial_i \quad i < j.$
4) $\partial_{j+1} k_{j+1} = \partial_{j+1} k_j$,
5) $\partial_i k_j = k_j \partial_{i-1} \quad$ for $i > j+1$,
6) $s_i k_j = k_{j+1} s_i \quad$ for $i \leq j$, and
7) $s_i k_j = k_j s_{i-1} \quad$ for $i > j$.

If A is a subcomplex of X, and $f_0|A = f_1|A$, then f_0 is homotopic to f_1 relative to A if and only if $k_i(\sigma) = f_0(s_i(\sigma))$ for $\sigma \in A$.

<u>Proof</u>: Suppose that F is a homotopy connecting f_0 and f_1. Define $k_i(\sigma) = F(s_i\sigma \times s_q \ldots s_{i+1} s_{i-1} \ldots s_0(0,1))$ for $\sigma \in X_q$, $i = 0, \ldots, q$. The verification that the k_i's satisfy relation 1) - 7) is now a routine matter.

Suppose that there exist functions k_i satisfying 1) -7). Define $F(\sigma \times s_{q-1} \ldots s_{i+1} s_{i-1} \ldots s_0(0,1)) = \partial_{i+1} k_i(\sigma)$ for $\sigma \in X_q$, $i = 0, \ldots, q-1$, $F(\sigma \times s_0^q(0)) = f_0(\sigma)$ and $F(\sigma \times s_0^q(1)) = f_1(\sigma)$. Using relations 1) - 7), one sees readily that F is a semi-simplicial map, and hence a homotpy from f_0 to f_1.

<u>Notation and Convention:</u> For $i = 0, \ldots, q+1$, let $\lambda^i : \{0, \ldots, q\} \longrightarrow \{0, \ldots, q+1\}$ be the function defined by

$$\lambda^i(j) = j \qquad j < i, \quad \text{and}$$

$$\lambda^i(j) = j+1 \qquad j \geq i\,.$$

Similarly let $\eta^i : \{0, \ldots, q+1\} \longrightarrow \{0, \ldots, q\}$ be defined by

$$\eta^i(j) = j \qquad j \leq i, \quad \text{and}$$

$$\eta^i(j) = j-1 \qquad j > i \text{ for } i = 0, \ldots, q.$$

Further denote by $\lambda^i : \Delta_q \longrightarrow \Delta_{q+1}$ the semi-simplicial map defined by the function λ^i, and by $\eta^i : \Delta_{q+1} \longrightarrow \Delta_q$ the map defined by η^i.

We now wish to translate these definitions into a slightly different framework. In ordinary topology, if A and B are spaces, a map of A into B is a point in the function-space of maps of A into B, and this function-space is usually denoted by B^A. Following an idea of A. Heller, we shall now define the semi-simplicial analogue of a function-space.

<u>Definition:</u> If X and Y are semi-simplicial complexes, then Y^X is the semi-simplicial complex defined as follows:

1) $(Y^X)_q$ is the set of semi-simplicial maps $f : X \times \Delta_q \longrightarrow Y$, and

2) if $f : X \times \Delta_q \longrightarrow Y$, then $\partial_i f : X \times \Delta_{q-1} \longrightarrow Y$ is defined by

$\partial_i f = f(i \times \lambda^i)$, where $i : X \longrightarrow X$ is the identity map, and $s_i f : X \times \Delta_{q+1} \longrightarrow Y$ is defined by $s_i f = f(i \times \eta^i)$.

Now, as in the geometric case, a homotopy between between $f_0, f_1 : X \longrightarrow Y$ is just a path in Y^X which starts at the point f_0 and ends at the point f_1. Consequently, for homotopy to be an equivalence relation it would suffice for Y^X to be a Kan complex. (cf. definition of π_0 in Chapter 1). This is indeed the case if Y is a Kan complex. The next few pages will therefore be devoted to the proof of this theorem.

Definition: A (p,q) "shuffle" is a partition (μ,ν) of the set $\{0,\ldots,p+q-1\}$ of integers into two disjoint sets such that $\mu_1 < \ldots < \mu_p$ and $\nu_1 < \ldots < \nu_q$. The (p,q) shuffle is determined by μ or ν.

The reason for introducing (p,q)-shuffles is the following: If τ is a non-degenerate p-simplex of K, let $\hat{\tau}$ denote the smallest subcomplex of K containing τ. Then the non-degenerate (p+q)-simplexes of $\bar{\tau} \times \Delta_q$ are of the form

$$s_{\nu_q} \cdots s_{\nu_1} \times s_{\mu_p} \cdots s_{\mu_1}(0,\ldots,q)$$

where (μ,ν) is a (p,q)-shuffle; and the set of such simplex is thus in a natural 1-1 correspondence with the set of (p,q)-shuffles.

Let $i \in \{0,\ldots,p+q\}$. The (p,q) shuffle (μ,ν) is of type I relative to i if either

1) $i < \mu_1$, or
2) $i, i-1 \in \{\nu_1,\ldots,\nu_q\}$, or
3) $i = p+q$, $i-1 = \nu_q$

It is of type II relative to i if either

1) $i < \nu_1$, or
2) $i, i-1 \in \{\mu_1,\ldots,\mu_p\}$, or
3) $i = p+q$, $i-1 = \mu_p$.

If the (p,q) shuffle (μ,ν) is not of type I or II relative to i, then it is said to be of type III relative to i.

In this case $\max\{\mu_1, \nu_1\} \leq i < p+q$ and either

1) $i \in \{\mu_1, \ldots, \mu_p\}$ and $i-1 \in \{\nu_1, \ldots, \nu_q\}$, on

2) $i \in \{\nu_1, \ldots, \nu_q\}$ and $i-1 \in \{\mu_1, \ldots, \mu_p\}$.

Now we wish to define a new shuffle $(\bar{\mu}, \bar{\nu})$ associated with (μ, ν) and i.

If (μ, ν) is of type I relative to i, then $(\bar{\mu}, \bar{\nu})$ is a $(p,q-1)$ shuffle. Let k be the integer such that $\nu_k = i$ in case 1 or case 2, and let $k = q$ in case 3. Let $\bar{\nu}_j = \nu_j$ for $j < k$, $\bar{\nu}_j = \nu_{j+1} - 1$ for $k \leq j \leq q-1$; $(\bar{\mu}, \bar{\nu})$ is the corresponding $(p,q-1)$ shuffle. There is an integer r, called the index of i in (μ, ν), such that $\bar{\mu}_j = \mu_j$ for $j \leq r$, and $\bar{\mu}_j = \mu_j - 1$ for $r < j \leq p$.

If (μ, ν) is of type II relative to i, then $(\bar{\mu}, \bar{\nu})$ is a $(p-1,q)$ shuffle. Let k be the integer such that $\mu_k = i$ in case 1 or case 2, and let $k = p$ in case 3. Let $\bar{\mu}_j = \mu_j$ for $j < k$, $\bar{\mu}_j = \mu_{j+1} - 1$ for $k \leq j \leq p-1$; $(\bar{\mu}, \bar{\nu})$ is the corresponding $(p-1,q)$ shuffle. There is an integer r, called the index of i in (μ, ν) such that $\bar{\nu}_j = \nu_j$ for $j \leq r$, and $\bar{\nu}_j = \nu_j - 1$ for $r < j \leq q$.

If (μ, ν) is of type III relative to i, then in case 1, $i = \mu_r$, $i-1 = \nu_s$. Let $\bar{\mu}_j = \mu_j$ for $j \neq r$, $\bar{\mu}_r = i-1$, and let $(\bar{\mu}, \bar{\nu})$ be the corresponding (p,q) shuffle. In case 1, $i-1 = \mu_i$, $i = \nu_s$. Let $\bar{\mu}_j = \mu_j$ for $j \neq r$, $\bar{\mu}_i = i$, and let $(\bar{\mu}, \bar{\nu})$ be the corresponding (p,q) shuffle.

Now the associated shuffle $(\bar{\mu}, \bar{\nu})$ of (μ, ν) relative to i is defined for all (μ,ν) and i. However, we want a <u>second associated shuffle</u> $(\bar{\mu},\bar{\nu})$ relative to i; it is to be a $(p+1,q)$ shuffle, defined as follows. If r is the largest integer such that $\mu_j < i$ for $j < r$, then $\bar{\mu}_j = \mu_j$ for $j < r$, $\bar{\mu}_r = i$, and $\bar{\mu}_j = \mu_{j-1}+1$ for $j > r$. The <u>second index</u> of i in (μ,ν) is the number of ν_j such that $i > \mu_j$.

<u>Definition:</u> Let X and Y be semi-simplicial complexes, and $F : X \times \Delta_q \longrightarrow Y$ a semi simplicial map. If (μ,ν) is a (p,q) shuffle, define

$$F_{(\mu,\nu)} : X_p \longrightarrow Y_{p+q} \quad \text{by}$$

$$F_{(\mu,\nu)}a = s_{\nu_q}\cdots s_{\nu_1} a \times s_{\mu_p}\cdots s_{\mu_1}(0,\ldots,q).$$

Further define

$$F^i_{(\mu',\nu')} : X_p \longrightarrow Y_{p+q-1}$$

by $F^i_{(\mu',\nu')}a = s_{\nu'_{q-1}}\cdots s_{\nu'_1} a \times s_{\mu'_p}\cdots s_{\mu'_1}(0,\ldots,i-1,i+1,\ldots,q)$ where (μ',ν') is a $(p,q-1)$ shuffle, and $i = 0,\ldots,q$.

<u>Proposition 2:</u> If $F : X \times \Delta_q \longrightarrow Y$ is a semi simplicial map, then

1) $\partial_i F_{(\mu,\nu)} = F_{(\bar{\mu},\bar{\nu})}\partial_{i-r}$ is (μ,ν)

is a (p,q) shuffle of type II relative to i, r is the index of i in (μ,ν), and $(\bar{\mu},\bar{\nu})$ is the associated shuffle of (μ,ν) relative to i,

2) $\partial_i F_{(\mu,\nu)} = \partial_i F_{(\bar{\mu},\bar{\nu})}$ if (μ,ν)

is a (p,q) shuffle of type III relative to i, and

$(\bar{\mu}, \bar{\nu})$ is the associated shuffle of (μ, ν) relative to i,

3) $s_i F_{(\mu,\nu)} = F_{(\bar{\mu},\bar{\nu})}$ is the second associated shuffle of (μ, ν) relative to i, and r is the second index of i in (μ, ν), and

4) $\partial_i F_{(\mu,\nu)} = F^{i-r}_{(\bar{\mu},\bar{\nu})}$ if (μ, ν) is a (p,q) shuffle of type I relative to i, $(\bar{\mu}, \bar{\nu})$ is the associated (p,q-1) shuffle, and r is the index of i in (μ, ν).

Further, a set $\{F_{(\mu,\nu)}\}$ of functions $F_{(\mu,\nu)} : X_p \longrightarrow Y_{p+q}$ indexed on the (p,q) shuffles for fixed q, and satisfying conditions 1)-3) above, determine a map $F : X \times \Delta_q \longrightarrow Y$.

The proof is entirely similar to the proof of the first proposition of this appendix, but more tedious. It will be omitted.

<u>Theorem 3:</u> If X is a semi-simplicial complex, and Y is a Kan complex, then Y^X is a Kan complex.

<u>Proof</u>: Let $F_0, \ldots, F_{k-1}, F_{k+1}, \ldots, F_q \in (Y^X)_{q-1}$ be such that $\partial_i F_j = \partial_{j-1} F_i$, $i < j$, $i, j \neq k$. Let $\{F^i_{(\mu,\nu)}\}$ be the functions indexed on the (p,q-1) shuffles determined by F_i for $i \neq k$. We wish to produce a set of functions $F_{(\mu,\nu)}$, indexed on the (p,q) shuffles, and satisfying relation 1)-4). Order the shuffls as follows: an (r,q) shuffle precedes a (p,q) shuffle if $r < p$. A (p,q) shuffle

(μ,ν) precedes a (p,q) shuffle (μ^*, ν^*) if $\mu_i = \mu_i^*$ for $i < j$, and $\mu_j < \mu_j^*$. The first shuffle is a $(0,q)$ shuffle, and this unique. Therefore, if $a \in X_0$ we must find an element $b \in Y_p$ such that $\partial_i b = F^i_{(0,\ldots,q-1)}a$ for $i \neq k$; and we can do so since Y is a Kan complex. Define $F_{(0,\ldots,q)}a = b$.

Suppose now that $F_{(\mu,\nu)}$ is defined for $(\mu,\nu) < (\mu^*, \nu^*)$.

<u>Case 1:</u> (μ^*, ν^*) is the first (p,q) shuffle; i.e. $\mu^*_i = i-1$ for $i = 1,\ldots,p$, $\nu^*_2 = i+p-1$. This shuffle is of type III with respect to p, and $(\bar\mu, \bar\nu)$ the associated (p,q) shuffle relative to p, is given by $\bar\mu_i = i-1$ for $i < p$, $\bar\mu_p = p$, $\bar\nu_1 = p-1$, $\bar\nu_i = i+p-1$ for $i > 1$. Therefore (μ,ν) precedes $(\bar\mu,\bar\nu)$. Consequently if $a \in X_p$, $\partial_p F_{(\mu^*, \nu^*)}a$ is not specified. Therefore if a is non-degenerate we may use the extension condition to define $F_{(\mu^*,\nu^*)}a$; while if a is degenerate we may condition 3) of the proposition to make the definition.

<u>Case 2:</u> For some integer i, and for some $r \in \{1,\ldots,p\}$ $s \in \{1,\ldots,q\}$, we have $\mu^*_r = i-1$, and $\nu^*_s = i$. Now (μ^*, ν^*) precedes $(\bar\mu, \bar\nu)$, the associated (p,q) shuffle relative to i, and $\partial_i F_{(\mu^*,\nu^*)}$ is not specified. The proof for this case is then completed as in case 1.

Case 3: $\mu_i^* = i+q-1$, $\nu_i^* = i-1$, $k < q$. In this case we must have $\partial_k F_{(\mu^*, \nu^*)} = F^k_{(\bar{\mu}, \bar{\nu})}$ where $(\bar{\mu}, \bar{\nu})$ is associated with (μ^*, ν^*) relative to k. But $F^k_{(\bar{\mu}, \bar{\nu})}$ is undefined, so that ${}_kF_{(\mu^*, \nu^*)}$ is free, and we may proceed as before.

If $k < q$, cases 1,2,3 are exhaustive. Therefore it remains to prove the extension condition in case $k = q$. To do this we reorder the $F_{(\mu,\nu)}$'s by simply reversing the ordering of the (p,q)-shuffles for each fixed p.

Now in the inductive step, $\mu_i^* = i+q-1$, $\nu_i^* = i-1$ is the first case to be considered, and this may be carried through. The reverse of the previous case 2) is now case 2), i.e. for some i,r,s, $r \in \{1,\ldots,p\}$, $s \in \{1,\ldots,q\}$ $\mu_r^* = i$, $\nu_s^* = i-1$, and we proceed as in case 2. The last case is now $\mu_i^* = i-1$, $\nu_i^* = i+p-1$, and by the relations we see that $\partial_{p+q} F_{(\mu^*, \nu^*)}$ is unspecified, and the proof may be completed.

Theorem 4: If X is a semi simplicial complex, A is a subcomplex of X, and Y is a Kan complex, then the map $p : Y^X \longrightarrow Y^A$ given by $p(f) = f|A \times \Delta_q$, where $f : X \times \Delta_q \longrightarrow Y$, is a fibre map.

Proof: The proof of this theorem is essentially the same as the proof of the preceding theorem.

Corollary 5: (Homotopy Extension Theorem) Let (X,A) be a semi-simplicial pair, Y a Kan complex. Let $f : X \longrightarrow Y$, and let $F : A \times I \longrightarrow Y$ be a homotopy such that $F(\tau \times (0_0, \ldots, 0_r)) = f(\tau)$ for $\tau \in A_r$, all r. Then there exists a homotopy $\bar{F} : X \times I \longrightarrow Y$ which agrees with F on $A \times I$ and such that $F(\sigma \times (0_0, \ldots, 0_r) = f(\sigma)$ for $\sigma \in X_r$, all r.

Now following Eilenberg and Zilber ([1]) we shall show the existence of a minimal subcomplex of any Kan complex which is equivalent to that Kan complex up to homotopy. We first give some preliminary definitions and lemmas.

Definition: If X is a semi-simplicial complex, then $x, y \in X_q$ are compatible if $\partial_i x = \partial_i y$ for $i = 0, \ldots, q$. Now x defines a unique map $\bar{x} : \Delta_q \longrightarrow X$, determined by $\bar{x}(0, \ldots, q) = x$, and similarly for y. The simplexes x and y are said to be homotopic if $\bar{x}$ and $\bar{y}$ are homotopic rel $\dot{\Delta}_q$.

Lemma 6: If X is a Kan complex, then X is minimal if and only if for each compatible pair $x, y \in X_q$ such that x is homotopic to y, we have $x = y$.

<u>Proof:</u> Suppose first that X is minimal, and $x, y \in X_q$ with x homotopic to y. Let $k_i : (\Delta_q)_r \longrightarrow X_{r+1}$ be functions satisfying the conditions of Proposition 1 generating a homotopy from $\bar{x}$ to $\bar{y}$ rel $\dot{\Delta}_q$.

We then have $\partial_0 k_0(0,\ldots,q) = x$,

$$\partial_i k_0(0,\ldots,q) = k_0(0,\ldots,i-2,i,\ldots,q) = \bar{x}(s_0(0,\ldots,i-2,i,\ldots,q))$$
$$= s_0 \partial_{i-1} x = \partial_i s_0 x \quad \text{for} \quad i > 1.$$

Therefore $k_0(0,\ldots,q)$ has the same faces, other than the first, as does $s_0 x$. Since X is minimal, we have therefore

$$\partial_1 k_1(0,\ldots,q) = \partial_1 k_0(0,\ldots,q) = \partial_1 s_0 x = x.$$

By an inductive argument of this nature it is easy to show that $\partial_{i+1} k_i(0,\ldots,q) = x$ for all i. Hence $x = \partial_{q+1} k_q(0,\ldots,q) = y$.

The converse is proved in a similar manner.

<u>Lemma 7:</u> If X is a semi-simplicial complex, $x, y \in X_q$, and x and y are compatible and degenerate, then $x = y$.

<u>Proof</u>: Let $x = s_m z$, $y = s_n z'$. Then either $m = n$, in which case $\partial_m x = z$ and $\partial_m y = z'$ implies $z = z'$, or $m \neq n$. In this latter case suppose $m < n$. Now $z = \partial_m s_m z = \partial_m x = \partial_m y = \partial_m s_n z' = s_{n-1} \partial_m z'$. Therefore $x = s_m s_{n-1} \partial_m z' = s_n s_m \partial_m z'$, and $\partial_n x = s_m \partial_m z'$. Since $z' = \partial_n y = \partial_n x = s_m \partial_m z'$, $z' = s_m \partial_m z'$. Then $x = s_n s_m \partial_m z' = s_n z' = y$.

Now let X be a Kan complex, and define a new seim-simplicial complex M as follows. For each component of X choose a representative point. These are to be the elements of M_0. Suppose now that M_r is is defined with face operators for $r \leq n$, so that $M_r \subset X_r$ and the face operators agree. Consider the homotopy classes of (n+1)-simplexes of X, each simplex having all its faces in M_n. We choose one representative from each such class, always choosing a degenerate representative if such exists; these are to be the elements of M_{n+1}. ∂_i and s_i are induced by the corresponding operators in X. Thus we obtain by induction a semi-simplicial complex $M \subset X$ which is clearly minimal. We now define by induction a set of functions

$$k_i : X_n \longrightarrow X_{n+1}, \; i = 0, \ldots, n$$

for each dimension $n = 0, 1, \ldots$, satisfying the relations of proposition 1, and such that $\partial_0 k_0(x) = x$, $\partial_{n+1} k_n(x) \in M_n$ for $x \in X_n$, and $k_i(x) = s_i(x)$ if $x \in M_n$.

1) If $x \in X_0$, $k_0(x)$ is to be a path such that $\partial_0 k_0 = x$, $\partial_1 k_0(x) \in M_0$.

   Further, if $x \in M_0$, we take $k_0(x) = s_0(x)$.

2) Suppose that the functions k_i have been defined for X_n for $n \leq r$, satisfying the above conditions. Let $x \in X_{r+1}$. If x is degenerate, then $k_0(x)$ is defined by the relations, while if $x \in M_{r+1}$ we set $k_0(x) = s_0(x)$. Otherwise we must find an element $y = k_0(x)$ such that $\partial_0 y = x$ and $\partial_i y = k_0(\partial_{i-1} x)$ for $i > 1$. We may choose such a y using the extension condition.

3) Suppose further that $k_i : X_{r+1} \longrightarrow X_{r+2}$ has been defined for $i < j$. Then for $x \in X_{r+1}$ we must find $y = k_j(x)$ such that
$\partial_i y = k_{j-1}\partial_i x$ for $i < j$, $\partial_j k_j(x) = \partial_j k_{j-1}(x)$, and $\partial_i k_j(x) = k_j \partial_{i-1} x$ for $i > j+1$. If x is degenerate, define $k_j(x)$ using the relations. If $x \in M_{r+1}$, set $k_j(x) = s_j(x)$. Otherwise apply the extension condition and choose $k_j(x)$ arbitrarily, provided $j \neq r+1$. If $j = r+1$, we must have the further condition $\partial_{r+2}k_{r+1}(x) \in M_{r+1}$. First choose $y = k_{n+1}(x)$ by the extension condition to satisfy all the above conditions except that on $\partial_{r+1}y$.
Then

$$\partial_i\partial_{r+2}k_{r+1}(x) = \partial_{r+1}\partial_i k_{r+1}(x) = \partial_{r+1}k_r(\ {}_i x) \in M_r \text{ for } i < r+1;$$

$$\partial_{r+1}\partial_{r+2}k_{r+1}(x) = \partial_{r+1}\partial_{r+1}k_{r+1}(x) = \partial_{r+1}\partial_{r+1}k_r(x)$$
$$= \partial_{r+1}\partial_{r+2}k_r(x) = \partial_{r+1}k_r(\partial_{r+1}x) \in M_r.$$

Thus $\partial_{r+2}y$ has all its faces in M_r, and there is therefore a unique $z \in M_{r+1}$ which is compatible with and homotopic to $\partial_{r+2}y$. Then by an obvious modification of the homotopy extension theorem, there exists $y' \in X_{r+2}$ such that $\partial_i y = \partial_i y'$, $i < r+2$, and $\partial_{r+2}y' = z$. We finally define $k_{r+1}(x) = y'$. This completes the induction.

<u>Theorem 8:</u> If X is a Kan complex, then there exists a minimal subcomplex M of X which is a deformation retract of X. Further, if M' is another such subcomplex, then M is isomorphic to M'.

Proof: The existence of M has already been proved, so suppose that M' is another such complex

Let $r : X \longrightarrow M$, $r' : X \longrightarrow M'$ be deformation retractions. Then we have maps

$$M \xrightarrow{i} X \xrightarrow{r'} M', \quad \text{and}$$

$$M' \xrightarrow{i'} X \xrightarrow{r} M$$

where i and i' are inclusions.

The map $i \circ r$ is homotopic to the identity map of X, and hence $r' \circ i \circ r \circ i' \simeq r' \circ i' = 1$. One verifies readily that the identity is the only map of a minimal complex into itself which is homotopic to the identity, and hence $r' \circ i \circ r \circ i' = 1$. Similarly $r \circ i' \circ r' \circ i = 1$, and hence $r' \circ i$ is an isomorphism. This completes the proof.

Reference

[1] S. Eilenberg and J. A. Zilber, Semi-simplicial complexes and singular homology, Annals of Math. 51 (1950), pp. 499-513.

Appendix 1 B. Definition of Homotopy Groups by Mappings of Spheres

W. Barcus

Let Δ_q denote the semi-simplicial complex on the standard q-simplex; an r-simplex of Δ_q is a sequence $(i_0,\ldots,i_r)$ with $0 \leq i_0 \leq \ldots \leq i_r \leq q$, the i_j being the "vertices" of the simplex. We shall also denote the complex Δ_1 by I. Similarly, let $\dot{\Delta}_q$ denote the usual semi-simplicial complex on the boundary of the standard q-simplex, so that $\dot{\Delta}_q$ is a subcomplex of Δ_q. $\dot{\Delta}_{q+1}$ is the analogue of a q-sphere, for semi-simplicial theory. Let σ_i denote the simplex $(0,\ldots,i-1,i+1,\ldots,q+1)$ of $\dot{\Delta}_{q+1}$, and let $\partial_i \Delta_{q+1}$ denote the subcomplex of Δ_{q+1} consisting of simplexes which do not comtain the vertex i. We may embed Δ_q in Δ_{q+1} as $\partial_{q+1}\Delta_{q+1}$.

Let X be a Kan complex, $x^* \in X_0$. It is clear that $\pi_q(X,x^*)$, the q^{th} homotopy group of X based at x^*, as previously defined, may be considered as the set of equivalence classes of maps[1] $h:(\Delta_q,\dot{\Delta}_q) \longrightarrow (X,\bar{x}^*)$, two maps h, k being equivalent ("simplicially homotopic") if there exists a map $F:\Delta_{q+1} \longrightarrow X$ such that $F(\partial_j \sigma_i) = s_0^{q-1} x^*$, all i, j; $F(\sigma_i) = s_0^q x^*$, $i \neq q, q+1$; $F(\sigma_q) = h(0,\ldots,q)$, $F(\sigma_{q+1}) = k(0,\ldots,q)$.

[1]For any simplex $\gamma \in X$, let $\bar{\gamma}$ denote the smallest subcomplex of X containing γ.

Let $\overline{\pi}_q(X,x^*)$ denote the set whose elements are the homotopy classes[1] rel(q+1) of maps $f:(\dot{\Delta}_{q+1},(q+1)) \longrightarrow (X,x^*)$, (q+1) being the 0-simplex consisting of just the vertex q+1.

Lemma 1B.1: Any map $g:(\dot{\Delta}_{q+1},(q+1)) \longrightarrow (X,x^*)$ is homotopic rel (q+1) to a map $\overline{g}:(\dot{\Delta}_{q+1},(q+1)) \longrightarrow (X,x^*)$ such that $\overline{g}(\sigma_i) = s_0^q x^*$ for $i < q+1$.

Lemma 1B.2: Let $h,k:(\dot{\Delta}_{q+1},(q+1)) \longrightarrow (X,x^*)$ be maps such that $h(\sigma_i) = k(\sigma_i) = s_0^q x^*$, $i < q+1$, and suppose that $h \sim k$ rel (q+1). Then $h \sim k$ rel $\sigma_0 \cup \ldots \cup \sigma_q$.

The proofs of the above two lemmas are straightforward; one need only extend maps defined on subcomplexes of $\dot{\Delta}_{q+1} \times I$ and $\dot{\Delta}_{q+1} \times I \times I$. Details will be omitted.

Lemma 1B.3: Let $h,k:(\Delta_q,\dot{\Delta}_q) \longrightarrow (X,\overline{x}^*)$. Then $h \sim k$ rel $\dot{\Delta}_q$ if and only if $h \underset{s}{\sim} k$.

Proof: Suppose that $h \sim k$ rel $\dot{\Delta}_q$ under a homotopy $F:\Delta_q \times I \longrightarrow X$. The non-degenerate (q+1)-simplexes of $\Delta_q \times I$ are

$$\tau_i = (0,\ldots,i-1,i,i,i+1,\ldots,q)\times(0_0,\ldots,0_i,1_{i+1},\ldots,1_{q+1}).$$

For each i, $\partial_k \partial_j \tau_i \in \dot{\Delta}_q \times I$ for all k, j, and $\partial_j \tau_i \in \dot{\Delta}_q \times I$ for $j \neq i, i+1$. Applying lemma (1.21), from τ_0 we have

[1] In the sense of Appendix 1A. Homotopy in this sense will be denoted $\sim$; in the simplicial sense, $\underset{s}{\sim}$.

$k \underset{s}{\sim} F|\partial_1\tau_0$; from τ_1 we have $F|\partial_1\tau_0 = F|\partial_1\tau_1 \underset{s}{\sim} F|\partial_2\tau_1$; hence by the transitivity of $\underset{s}{\sim}$, $k \underset{s}{\sim} F|\partial_2\tau_1$. Proceeding inductively, $k \underset{s}{\sim} F|\partial_{q+1}\tau_q = h$.

Conversely, let $h \underset{s}{\sim} k$. Then we define $F: \Delta_q \times I \longrightarrow X$ as follows. $\beta_q = F(\tau_q)$ is to have faces $\partial_{q+1}\beta_q = h(0,\ldots,q), \partial_q\beta_q = k(0,\ldots,q), \partial_i\beta_q = s_0^q x^*, i \leq q$. Let $\beta_i = F(\tau_i) = s_i k(0,\ldots,q), i < q$. F is then determined, and is a homotopy from h to k rel $\dot{\Delta}_q$.

Define a function $\psi: \pi_q \longrightarrow \overline{\pi}_q$ as follows: $\psi[h]$ is represented by the map $h': (\dot{\Delta}_{q+1}, (q+1)) \longrightarrow (X, x^*)$ determined by $h'(\sigma_i) = s_0^q x^*, i \neq q+1$; $h'(\sigma_{q+1}) = h(0,\ldots,q)$.

<u>Theorem 1B.4:</u> ψ is 1-1.

A group structure is therefore induced in $\overline{\pi}_q$ such that ψ is an isomorphism.

<u>Proof of 1B.4:</u> To show that ψ is single-valued, suppose that $h \underset{s}{\sim} k$. Then by (1B.3), $h \sim k$ rel $\dot{\Delta}_q$. If the homotopy is $F: \Delta_q \times I \longrightarrow X$, then F can be extended to $F': \dot{\Delta}_{q+1} \times I \longrightarrow X$ by setting $F'(\omega_r) = s_0^r x^*$ for any simplex ω_r of $\dot{\Delta}_{q+1} \times I - \Delta_q \times I$. F' is then a homotopy from h' to k'.

Define $\phi: \overline{\pi}_q \longrightarrow \pi_q$ by $\phi[g] = [\overline{\overline{g}}]$, where $\overline{\overline{g}}: (\Delta_q, \dot{\Delta}_q) \longrightarrow (X, \overline{x}^*)$ is the restriction of the map $\overline{g}$ of (1B.1). ϕ is single-valued by (1B.2),(1B.3). It is clear that $\phi\psi$ = identity, and by (1B.1) ψ is onto. Therefore ψ is 1-1, which proves (1B.4).

Using the representation of the elements of π_q as homotopy classes of mappings of $\dot{\Delta}_{q+1}$, it is easy to define the isomorphism[1] induced by a path α in X from x_0 to x_1:

$$\alpha_{\#} : \pi_q(X,x_0) \xrightarrow{\approx} \pi_q(X,x_1).$$

Let $\xi \in \pi_q(X,x_0)$ have representative map $f_0:(\dot{\Delta}_{q+1},(q+1)) \longrightarrow (X,x_0)$. Define $F:\dot{\Delta}_{q+1} \times I \longrightarrow X$ by $F(\tau \times (0_0,\ldots,0_r)) = f_0(\tau)$, $\tau \in (\dot{\Delta}_{q+1})_r$, all r; $F((q+1,q+1)\times(0,1)) = \alpha$; and extend by the homotopy extension theorem. Define $f_1:(\dot{\Delta}_{q+1},(q+1)) \longrightarrow (X,x_1)$ by $f_1(\tau) = F(\tau \times (1_0,\ldots,1_r))$; then $\alpha_{\#}\xi = [f_1]$. That $\alpha_{\#}$ is an isomorphism follows by applying the homotopy extension theorem. The usual properties of the induced isomorphism may also be demonstrated.

1) It is more convenient to define this isomorphism rather than its inverse, as is usually done.

Chapter 1. Appendex C.

In the preceding parts of chapter 1, a good deal of elementary homotopy theory has been developed, but some standard and necessary properties have not yet been stated. This section will first take up a few of these, and then pass on to a proof of the Hurewicz Theorem.

Theorem: If X,Y are Kan complexes, and $f,g:X \longrightarrow Y$ are semi-simplicial maps homotopic relative to $[x]$ (the subcomplex of X generated by $x \in X_0$), then $f^{\#} = g^{\#} : \pi_q(X,x) \longrightarrow \pi_q(Y,f(x))$.

Proof: The theorem follows immediately from the fact that elements of $\pi_q(X,x)$ correspond to homotopy classes of maps $\phi:(\Delta_q, \dot{\Delta}_q) \longrightarrow (X,x)$ (see appendix B); since f,g are homotopic relative to $[x]$, $f\circ\phi$, $g\circ\phi : (\Delta_q, \dot{\Delta}_q) \longrightarrow (Y,f(x)$ are homotopic.

Definition: Two Kan complexes X and Y are said to have the same homotopy type if and only if there exist maps $f:X \longrightarrow Y$ and $g:Y \longrightarrow X$ such that fg is homotopic to the identity map of Y and gf is homotopic to the identity map of X.

Proposition: If X and Y are connected minimal Kan complexes such that $\pi_q(X,x) = \pi_q(Y,y) = 0$ for $g \neq n$, and $\phi: \pi_n(X,x) \longrightarrow \pi_n(Y,y)$ is a homomorphism, then there is a unique semi-simplicial map $f:X \longrightarrow Y$ such that

$f^{\#} = \phi: \pi_n(X,x) \longrightarrow \pi_n(Y,y).$

<u>Proof:</u> Since X and Y are minimal they both have exactly one simplex in each dimension $< n$. Further there is a natural 1:1 correspondence between $\pi_n(X,x)$ and X_n, and between $\pi_n(Y,y)$ and Y_n. Therefore f is defined and is unique in dimension $\leq n$. Suppose now that f is defined in dimension $\leq q$, where $q \geq n$, and let $\sigma \in X_{q+1}$. Then $f(\partial_i \sigma)$ is defined for $i = 0,\ldots,q+1$, and there is a unique element τ of Y_{q+1} such that $\partial_i \tau = f(\partial_i \sigma)$ for $i = 0,\ldots,q$. Set $f(\sigma) = \tau$. Thus f is defined inductively and satisfies the condition $\partial_i f = f \partial_i$. Suppose that $s_i f = f s_i$ in dimension $\leq q$ (we may suppose that $q \geq n$), and $\sigma \in X_{q+1}$. Then $\partial_j s_i f(\sigma) = s_{i-1} \partial_j f(\sigma) = s_{i-1} f(\partial_j \sigma) = f(s_{i-1} \partial_j \sigma) = f(\partial_j s_i \sigma) = \partial_j f(s_i \sigma)$ for $j < i$, $\partial_i s_i f(\sigma) = f(\sigma) = f(\partial_i s_i \sigma) = \partial_i f(s_i \sigma)$, $\partial_{i+1} s_i f(\sigma) = f(\sigma) = f(\partial_{i+1} s_i \sigma) = \partial_{i+1} f(s_i \sigma)$, $\partial_j s_i f(\sigma) = s_i \partial_{j-1} f(\sigma) = s_i f(\partial_{j-1}\sigma) = f(s_i \partial_{j-1} \sigma) = \partial_j f(s_i \sigma)$ for $j > i+1$. Consequently $s_i f(\sigma)$ and $f(s_i \sigma)$ have the same faces, and since $q+1 > n$, and $\pi_{q+1}(Y,y) = 0$, we have $f(s_i \sigma) = s_i f(\sigma)$. This last assertion completes the inductive step in the proof.

<u>Corollary</u>: If X and Y are connected minimal Kan complexes such that $\pi_q(X,x) = \pi_q(Y,y) = 0$ for $q \neq n$, and $\pi_n(X,x) \simeq \pi_n(Y,y)$, then X and Y are isomorphic.

<u>Theorem</u>: If X and Y are connected minimal Kan complexes, and $f:X \longrightarrow Y$ is a semi-simplicial map such that $f^{\#}: \pi_q(X,x) \xrightarrow{\approx} \pi_q(Y,y)$ for all q, then f is an isomorphism.

<u>Proof:</u> Let $\mathfrak{X}^n = (X^{(n+1)}, p, X^{(n)})$ be the n'th term in the natural Postnikov system of X, and $\mathfrak{Y}^n = (Y^{(n+1)}, p', Y^{(n)})$ that for Y (Chapter 1, p. 23). Now it is evident that all the terms in the Postnikov system of a minimal complex are minimal. Using the preceding corollary, we may make the inductive hypothesis that $f^{(n)}:X^{(n)} \longrightarrow Y^{(n)}$ is an isomorphism. There is a commutative diagram

$$\begin{array}{ccc} X^{(n+1)} & \xrightarrow{\quad f^{(n+1)} \quad} & Y^{(n+1)} \\ \downarrow p & & \downarrow p' \\ X^n & \xrightarrow{\quad f^{(n)} \quad} & Y^{(n)} \end{array}$$

Suppose that $\sigma, \tau \in X_q^{(n+1)}$, and that $f^{(n+1)}(\sigma) = f^{(n+1)}(\tau)$. Then $p'f^{(n+1)}(\sigma) = p'f^{(n+1)}(\tau)$, and $p(\sigma) = p(\tau)$. Therefore $\sigma = \tau$ if $q \leq n$. Suppose we have proved that $f^{(n+1)}(\sigma') = f^{(n+1)}(\tau')$ implies $\sigma' = \tau'$ when $\dim \sigma' = \dim \tau' < q$. We then have $\partial_i \sigma = \partial_i \tau$, $i = 0, \ldots, q$, and $\sigma = \tau$ unless $q = n+1$. If $q = n+1$ we recall that the simplexes of dimension $(n+1)$ with a given boundary in a minimal complex are in a natural 1:1 correspondence with π_{n+1}. Let $[\sigma]$, $[\tau]$ be the element of π_{n+1} corresponding to

σ and τ respectively. Since $f(\sigma) = f(\tau)$, by naturality $f^{\#}[\sigma] = f^{\#}[\tau]$; since $f^{\#}$ is an isomorphism, $[\sigma] = [\tau]$, and hence σ is homotopic to τ. Since σ and τ are compatible (1A-11) and homotopic, $\sigma = \tau$.

The fact that $f^{(n+1)}$ is onto may be proved similarly. It then follows that f is an isomorphism, since $X_q = X_q^{(n)}$ for $q \leq n$.

Theorem: Let X and Y be connected Kan complexes. Then the following conditions are equivalent

1) X and Y have the same homotopy type,
2) there is a map $f: X \longrightarrow Y$ such that $f^{\#}: \pi_q(X,x) \xrightarrow{\approx} \pi_q(Y,f(x))$ for all q, where $x \in X_0$, and
3) X and Y have isomorphic minimal subcomplexes.

The proof is straightforward, using the earlier theorems of the appendix and the fact that every Kan complex has a minimal subcomplex which is a deformation retract of the original complex (1A-14 Theorem 8).

The fact that 1) and 2) in the preceding theorem are equivalent is in the topological case a theorem of J. H. C. Whitehead [1].

Corollary: If X is a connected Kan complex, $x \in X_0$, $\pi_q(X,x) = 0$ for $q < n$, and $E_n(X,x)$ is the n-th Eilenberg subcomplex of X based at x, then the inclusion map

$i: E_n(X,x) \longrightarrow X$ is a homotopy equivalence.

Definitions and Notations: If X is a semi-simplicial complex, then $C_n(X)$, the group of n-chains of X, is the free abelian group generated by the elements of X_n. $C(X) = \Sigma C_n(X)$ is the chain group of X. Let $\partial: C_{n+1}(X) \longrightarrow C_n(X)$ be the homomorphism defined by $\partial x = \Sigma_{i=0}^{n+1} (-1)^i \partial_i x$ for $x \in X_{n+1}$. $C(X)$, together with the endomorphism ∂, is the chain complex of X. Let $Z_n(X)$ be the kernel of $\partial: C_n(X) \longrightarrow C_{n-1}(X)$, $B_n(X)$ the image of $\partial: C_{n+1}(X) \longrightarrow C_n(X)$. The group $Z_n(X)$ is the group of n-cycles of X, and $B_n(X)$ is the group of n-dimensional boundaries of X. The endomorphism ∂ of $C(X)$ has the property that $\partial\partial = 0$. Therefore $B_n(X) \subset Z_n(X)$, and the n-dimensional homology group of X is $H_n(X) = Z_n(X)/B_n(X)$. The homology group of X is $H(X) = \Sigma_{n \geq 0} H_n(X)$.

Theorem: If X and Y are semi-simplicial complexes, and $f,g: X \longrightarrow Y$ are homotopic maps, then $f_* = g_*: H(X) \longrightarrow H(Y)$.

Proof: Let $k_i: X_q \longrightarrow Y_{q+1}$ be functions determining a homotopy between f and g (1A-2, proposition 2), and define $k: C_q(X) \longrightarrow C_{q+1}(Y)$ by $k(x) = \Sigma_{i=0}^{q} (-1)^i k_i(x)$ for $x \in X_q$. Now $\partial k(x) + k\partial(x) = f(x) - g(x)$, and the result follows.

The preceding theorem is the usual statement that homology is an invariant of homotopy type.

Theorem: If X is a Kan complex, then $H_0(X) = Z(\pi_0(X))$, the free abelian group generated by $\pi_0(X)$.

Proof: There is a natural map $X_0 \longrightarrow \pi_0(X)$, which induces a homomorphism $C_0(X) \longrightarrow Z(\pi_0(X))$. Clearly this map is an epimorphism (homomorphism onto). Suppose that $x \epsilon X_1$; then $\partial_0 x$ and $\partial_1 x$ are in the same component of X, so that the above epimorphism induces an epimorphism $\phi : H_0(X) \longrightarrow Z(\pi_0(X))$. For $x \epsilon \pi_0(X)$, let $\hat{x}$ be an element of X_0 which represents x. Suppose that $\hat{y}$ also represents x, then there exists $z \epsilon X$, such that $\partial_0 z = \hat{x}$, $\partial_1 z = \hat{y}$, and $\hat{x} - \hat{y} \epsilon B_0(X)$. Consequently $x \longrightarrow \hat{x}$ induces a homomorphism $\psi : Z(\pi_0(X)) \longrightarrow H_0(X)$. Since $\psi\phi$ and $\phi\psi$ are the respective identities, ϕ is an isomorphism.

Definition: Let X be a Kan complex, $x \epsilon X_0$, and define a homomorphism

$$\phi : \pi_n(X,x) \longrightarrow H_n(X,x) \quad \text{for } n > 0$$

as follows:

Let $\alpha \epsilon \pi_n(X,x)$ have representative $a \epsilon X_n$ such $\partial_i a = s_0^{n-1} x$ for $i = 0, \ldots, n$. Now if n is odd, $\partial a = 0$, while if n is even, $\partial a = s_0^{n-1} x$. Therefore we may take $\phi(\alpha)$ to have representative a if n is odd, and $a - s_0^n x$ if n is even.

To show that ϕ is single-valued, suppose that $a' \epsilon X_n$ also represents α. Then there exists $w \epsilon X_{n+1}$

such that

$$\partial_n w = a,\ \partial_{n+1} w = a',\ \partial_i w = s_0^n x \text{ for } i < n.$$

Then if n is even, $\partial w = a - a'$, while if n is odd, $\partial w = s_0^n x - a + a'$; and since $s_0^n x$ is a boundary, in either case a' is homologous to a.

To show that ϕ is homomorphism, suppose that $a, b \in X_n$ represent $\alpha, \beta \in \pi_n(X,x)$. There exists $v \in X_{n+1}$ such that

$$\partial_{n+1} v = a,\ \partial_{n-1} v = b, \text{ and } \partial_i v = s_0^n x \text{ for } i < n-1,$$

and $\partial_n v$ then represents $\alpha + \beta$. If n is odd, $\phi(\alpha + \beta)$ is represented by $\partial_n v$; but since $\partial v = b - \partial_n v + a$, $\partial_n v$ is homologous to $a + b$, which represents $\phi(\alpha) + \phi(\beta)$. Similarly if n is even, $\phi(\alpha + \beta)$ is represented by $\partial_n v - s_0^n x$; and since $\partial v = s_0^n x - b + \partial_n v - a$, this is homologous to $a + b$, which represents $\phi(\alpha) + \phi(\beta)$.

<u>Theorem (Poincaré):</u> If X is a connected Kan complex and $x \in X_0$, then $\phi : \pi_1(X,x) \longrightarrow H_1(X)$ induces an isomorphism $\phi' : \pi_1(X,x)/[\pi_1(X,x), \pi_1(X,x)]$.

<u>Proof</u>: We may assume that $X = E_1(X,x)$. Then there is a natural map $\eta : Z_1(X) = C_1(X) \longrightarrow \pi_1/[\pi_1, \pi_1]$, and as natural map $\lambda : Z_1(X) \longrightarrow H_1(X)$. We thus have a diagram

$$\begin{array}{ccc} Z_1(X) & \xrightarrow{\ \lambda\ } & \\ \downarrow \eta & & \searrow \\ \pi_1(X,x)/[\pi_1(X,x), \pi_1(X,x)] & \longrightarrow & H_1(X) \end{array}$$

and we know that $\pi_1 / [\pi_1, \pi_1,] \longrightarrow H_1(X)$ is an epimorphism.

If $a \in B_1(X)$, $a = \partial b$, $b \in X_2$, then a is represented by $\partial_0 b - \partial_1 b + \partial_2 b$, which is already 0 in $\pi_1(X,x)$; hence $\eta(B_1(X)) = 0$, and η induces a homomorphism $\eta' : H_1(X) \longrightarrow \pi_1 / [\pi_1, \pi_1]$. Clearly $\phi' \eta'$ and $\eta' \phi$ are the respective identities, and the result follows.

<u>Definition</u>: A Kan complex X is <u>n-connected</u> if for $x \in X_0$, $\pi_q(X,x) = 0$ for $q \leq n$.

<u>Theorem (Hurewicz)</u>: Let X be a Kan complex, $x \in X_0$. If X is $(n-1)$ connected, $n \geq 2$, then $H_q(X) = 0$ for $0 < q < n$, and $\phi : \pi_n(X,x) \xrightarrow{\approx} H_n(X)$.

The proof of this theorem is similar to that of the preceding theorem. Here it may be assumed that $X = E_{n-1}(X,x)$ so that X has only one simplex in each dimension $< n$.

<u>References</u>

[1] Whitehead, J. H. C., <u>Combinatorial homotopy</u>, Bull. American Math. Soc., 55(1949), 213-145.

[2] Poincaré, Henri, Oeuvres, (Paris, 1928), vol. 6, 142-293.

[3] Hurewicz, W., Beitrage zur Topologie der Deformationen I - IV, K. Akademie van Wetenschappen, Amsterdam, Proceedings, 38(1935), 112-19, 521-8; 39(1936), 117-26, 215-24.

<u>errata:</u>

1A-7 line 7, should be $i > \nu_j$ instead of $i > \mu_j$.

1A-8 line 2, should start $s_i F_{(\mu, \nu)} = F_{(\bar{\mu}, \bar{\nu})} s_{i-r}$ instead of $s_i F_{(\mu, \nu)} = F_{(\bar{\mu}, \bar{\nu})}$

The geometric realization of a semi-simplicial complex

John Milnor

Corresponding to each (complete) semi-simplicial complex K, a topological space $|K|$ will be defined. This construction will be different from that used by Giever [4] and Hu [5] in that the degeneracy operations of K are used. This difference is important when dealing with product complexes.

If K and K' are countable it is shown that $|K \times K'|$ is canonically homeomorphic to $|K| \times |K'|$. It follows that if K is a countable group complex then $|K|$ is a topological group. In particular $|K(\pi,n)|$ is an abelian group.

The terminology for semi-simplicial complexes will follow John Moore [7].

1. The definition

As standard n-simplex Δ_n take the set of all (n+2)-tuples $(t_0, \ldots, t_{n+1})$ satisfying $0 = t_0 \leq t_1 \leq \cdots \leq t_{n+1} = 1$. The face and degeneracy maps $\partial_i : \Delta_{n-1} \longrightarrow \Delta_n$ and $s_i : \Delta_{n+1} \longrightarrow \Delta_n$ are defined by

$$\partial_i(t_0, \ldots, t_n) = (t_0, \ldots, t_i, t_i, \ldots, t_n)$$

$$s_i(t_0, \ldots, t_{n+2}) = (t_0, \ldots, t_i, t_{i+2}, \ldots, t_n).$$

Let $K = \bigcup_{i \geq 0} K_i$ be a semi-simplicial complex. Giving K the discrete topology, form the topological sum

$$\bar{K} = (K_0 \times \Delta_0) + (K_1 \times \Delta_1) + \cdots + (K_n \times \Delta_n) + \cdots .$$

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Thus $\bar{K}$ is a disjoint union of open sets $k_i \times \Delta_i$. An equivalence relation in $\bar{K}$ is generated by the relations

$$(\partial_i k_n, \delta_{n-1}) \sim (k_n, \partial_i \delta_{n-1})$$

$$(s_i k_n, \delta_{n+1}) \sim (k_n, s_i \delta_{n+1}),$$

for $i = 0,1,\ldots,n$. The identification space $|K| = \bar{K}/(\sim)$ will be called the geometric realization of K. The equivalence class of (k_n, δ_n) will be denoted by $|k_n, \delta_n|$.

Theorem 1. $|K|$ is a CW-complex having one n-cell corresponding to each non-degenerate n-simplex of K.

For the definition of CW-complex see Whitehead [8].

lemma 1. Every simplex $k_n \in K_n$ can be expressed in one and only one way as $k_n = s_{j_p} \ldots s_{j_1} k_{n-p}$ where k_{n-p} is non-degenerate and $0 \leq j_1 < \ldots < j_p < n$. The indices j_α which occur are precisely those j for which $k_n \in s_j K_{n-1}$

The proof is not difficult. See [3] 8.3. Similarly it can be shown that every $\delta_n \in \Delta_n$ can be written in exactly one way as $\delta_n = \partial_{i_q} \ldots \partial_{i_1} \delta_{n-q}$ where δ_{n-q} is an interior point (that is $t_0 < t_1 < \ldots < t_{n-q+1}$) and $0 \leq i_1 < \ldots < i_q \leq n$.

By a non-degenerate point of $\overline{K}$ will be meant a point (k_n, δ_n) with k_n non-degenerate and δ_n interior.

<u>lemma 2.</u> Each $(k_n, \delta_n) \in \overline{K}$ is equivalent to a unique non-degenerate point.

Define the map $\lambda : \overline{K} \longrightarrow \overline{K}$ as follows. Given k_n choose $j_1, \ldots, j_p, k_{n-p}$ as in lemma 1 and set

$$\lambda(k_n, \delta_n) = (k_{n-p}, s_{j_1} \ldots s_{j_p} \delta n).$$

Define the discontinuous function $\rho : \overline{K} \longrightarrow \overline{K}$ by choosing $i_1 \ldots i_q, \delta_{n-q}$ as above and setting

$$\rho(k_n, \delta_n) = (\partial_{i_1} \ldots \partial_{i_q} k_n, \delta_{n-q})$$

Now the composition $\lambda\rho : \overline{K} \longrightarrow \overline{K}$ carries each point into an equivalent, non-degenerate point. It can be verified that if $x \sim x'$ then $\lambda\rho(x) = \lambda\rho(x')$; which proves lemma 2.

Take as n-cells of $|K|$ the images of the non-degenerate simplexes of $\overline{K}$. By lemma 2 the interiors of these cells partition $|K|$. Since the remaining conditions for a CW-complex are easily verified, this proves theorem 1.

<u>lemma 3.</u> A semi-simplical map $f : K \longrightarrow K'$ induces a continuous map $|K| \longrightarrow |K'|$

In fact the map $|f|$ defined by $|k_n, \delta_n| \longrightarrow |f(k_n), \delta_n|$ is clearly well defined and continous.

As an example of the geometric realization, let C be an ordered simplicial complex with space $|C|$. (See [2] pg. 56 and 67). From C we can define a semi-simplicial complex K, where K_n is the set of all $(n+1)$-tuples $(a_0,\ldots,a_n)$ of vertices of C which (1) all lie in a common simplex, and (2) satisfy $a_0 \leq a_1 \leq \ldots \leq a_n$. The operations ∂_i, s_i are defined in the usual way.

Assertion The space $|C|$ is homeomorphic to the geometric realization $|K|$. In fact the point $|(a_0,\ldots,a_n);(t_0,\ldots,t_{n+1})|$ of $|K|$ corresponds to the point of $|C|$ whose a-th barycentric coordinate, a being a vertex of C, is the sum, over all i for which $a_i = a$, of $t_{i+1}-t_i$. The proof is easily given.

2. Product complexes

Let $K \times K'$ be the cartesian product of two semi-simplicial complexes (that is $(K \times K')_n = K_n \times K'_n$). The projection maps $\rho : K \times K' \longrightarrow K$ and $\rho' : K \times K' \longrightarrow K'$ induce maps $|\rho|$ and $|\rho'|$ of the geometric realizations. A map $\eta : |K \times K'| \longrightarrow |K| \times |K'|$ is defined by $\eta = |\rho| \times |\rho'|$.

Theorem 2. η is a one-one map of $|K \times K'|$ onto $|K| \times |K'|$. If either (a) K and K' are countable, or (b) one of the two CW-complexes $|K|, |K'|$ is locally finite; then η is a homeomorphism.

The restrictions (a) or (b) are necessary in order to prove that $|K|x|K'|$ is a CW-complex. For the proof in case (b) see [8] and for case (a) see [6].

Proof (Compare [2] pg.68). If x'' is a point of $|K \times K'|$ with non-degenerate representative $(k_n \times k'_n, \delta_n)$ we will first determine the non-degenerate representative of $|\rho|(x'') = |k_n, \delta_n|$. Since δ_n is an interior point of Δ_n, this representative has the form

$$(k_{n-p}, s_{i_1} \dots s_{i_p} \delta_n) \quad \text{where} \quad k_n = s_{i_p} \dots s_{i_1} k_{n-p}$$

(See proof of lemma 2). Similarly $|\rho'|(x'')$ is represented by $(k'_{n-q}, s_{j_1} \dots s_{j_q} \delta_n)$ where $k'_n = s_{j_q} \dots s_{j_1} k'_{n-q}$. The induces i_α and j_β must be distinct; for if $i_\alpha = j_\beta$ for some α, β then $k_n \times k'_n$ would be an element of $s_{i_\alpha}(K_{n-1} \times K'_{n-1})$.

However the point x'' can be completely determined by its image.

$$|k_{n-p}, s_{i_1} \dots s_{i_p} \delta_n| \times |k'_{n-q}, s_{j_1} \dots s_{j_q} \delta_n|.$$

In fact given any pair $(x, x') \in |K|x|K'|$ define $\bar{\eta}(x, x') \in |K \times K'|$ as follows. Let (k_a, δ_a) and (k'_b, δ'_b) be the non-degenerate representatives; where $\delta_a = (t_0, \dots, t_{a+1})$, $\delta'_b = (u_0, \dots, u_{b+1})$. Let $0 = w_0 < \dots < w_{n+1} = 1$ be the distant numbers t_i and u_j arranged in order. Set $\delta''_n = (w_0, \dots, w_{n+1})$. Then if $\mu_1 < \dots < \mu_{n-a}$ are the indices μ such that $w_{\mu+1}$ is not one of the t_i, we have $\delta''_n = s_{\mu_1} \dots s_{\mu_{n-a}} \delta_a$. Similarly $\delta''_n = s_{\nu_1} \dots s_{\nu_{n-b}} \delta'_b$ where the sets $\{\mu_i\}$ and $\{\nu_j\}$ are disjoint.

Now define

$$\bar{\eta}(x,x') = |(s_{\mu_{n-a}}\dots s_{\mu_1}k_a)x(s_{\nu_{n-b}}\dots s_{\nu_1}k'_b), \delta_n|.$$

Clearly

$$|\rho|\bar{\eta}(x,x') = |s_{\mu_{n-a}}\dots s_{\mu_1}k_a, \delta_n| = |k_a, s_{\mu_1}\dots s_{\mu_{n-a}}\delta_n|$$

$$= |k_a, \delta_a| = x$$

and $|\rho'|\bar{\eta}(x,x') = x'$, which proves that $\eta\bar{\eta}$ is the identity map of $|K|x|K'|$. On the other hand, taking x'' as above we have $\eta\bar{\eta}(x'') = \bar{\eta}(|k_{n-p}, s_{i_1}\dots s_{i_p}\delta n|, |k'_{n-q}, s_{j_1}\dots s_{j_q}\delta n|)$

$$= |(s_{i_p}\dots s_{i_1}k_{n-p})x(s_{j_q}\dots s_{j_1}k'_{n-q}), \delta_n| = x''.$$

To complete the proof it is only necessary to show that $\bar{\eta}$ is continuous. However it is easily verified that $\bar{\eta}$ is continous on each product cell of $|K|x|K'|$. Since we are assuming that this product is a CW-complex, this completes the proof.

An important special case is the following. Let I denote the semi-simplicial complex consisting of a 1-simplex and its faces and degeneracies.

<u>Corollary.</u> A semi-simplicial homotopy $h: K \times I \longrightarrow K'$ induces an ordinary homotopy $|K| \times [0,1] \longrightarrow |K'|$.

In fact the interval $[0,1]$ may be identified with $|I|$. The homotopy is now given by the composition

$$|K| \times |I| \xrightarrow{\bar{\eta}} |K \times I| \xrightarrow{|h|} |K'|.$$

3. Product operations

Now let K be a countable complex. Any semi-simplicial map $p: K \times K \longrightarrow K$ induces by lemma 3 and theorem 2 a continuous product

$$|p|\bar{\eta} : |K| \times |K| \longrightarrow |K|.$$

If there is an element e_0 in K_0 such that $s_0^n e_0$ is a two-sided identity in each K_n, then it follows that $|e_0, \delta_0|$ is a two-sided identity in $|K|$; so that $|K|$ is an H-space. If the product operation p is associative or commutative then it is easily verified that $|p|\bar{\eta}$ is associative or commutative. Hence we have the following.

<u>Theorem 3</u>. If K is a countable group complex (countable abelian group complex), then $|K|$ is a topological group (abelian topological group).

Let $K(\pi,n)$ denote the Eilenberg MacLane semi-simplicial complex (see [1]).

<u>Corollary</u>. If π is a countable abelian group, then for $n \geq 0$ the geometric realization $|K(\pi,n)|$ is an abelian topological group.

It will be shown in the next section that $|K(\pi,n)|$ actually is a space with one non-vanishing homotopy group.

The above construction can also be applied to other algebraic operations. For example a pairing $K \times K' \longrightarrow K''$ between countable group complexes induces a pairing between their realizations. If K is a semi-simplicial complex of Λ-modules, where Λ is a discrete ring, then $|K|$ is a topological Λ-module.

4. The topology of $|K|$.

For any space X let $S(X)$ be the total singular complex. For any semi-simplicial complex K a one-one semi-simplicial map $i : K \longrightarrow S(|K|)$ is defined by

$$i(k_n)(\delta_n) = |k_n, \delta_n|.$$

Let $H_*(K)$ denote homology with integer coefficients.

lemma 4. The inclusion $K \longrightarrow S(|K|)$ induces an isomorphism $H_*(K) \approx H_*(S|K|)$ of homology groups.

By the n-skeleton $K^{(n)}$ of K is meant the subcomplex consisting of all $K_i, i \leq n$ and their degeneracies. Thus $|K^{(n)}|$ is just the n-skeleton of $|K|$ considered as a CW-complex. The filtration

$$K^{(0)} \subset K^{(1)} \subset \dots$$

gives rise to a spectral sequence $\{E^r_{pq}\}$; where E^∞ is the graded group corresponding to $H_*(K)$ under the induced filtration; and

$$E^1_{pq} = H_{p+q}(K^{(p)} \bmod K^{(p-1)}).$$

It is easily verified that $E^1_{pq} = 0$ for $q \neq 0$, and that

E^1_{po} is the free abelian generated by the non-degenerate p-simplexes of K. From the first assertion it follows that $E^2_{po} = E^{\infty}_{po} = H_p(K)$.

On the other hand the filtration

$$S(|K^{(o)}|) \subset S(|K^{(1)}|) \subset \ldots.$$

gives rise to a spectral sequence $\{\bar{E}^r_{pq}\}$ where $\bar{E}^{\infty}$ is the graded group corresponding to $H_*(S(|K|))$. Since it is easily verified that the induced map $E^1_{pq} \longrightarrow \bar{E}^1_{pq}$ is an isomorphism, it follows that the rest of the spectral sequence is also mapped isomorphicly; which completes the proof.

Now suppose that K satisfies the Kan extension condition, so that $\pi_1(K,k_0)$ can be defined.

<u>lemma 5</u>. If K is a Kan complex then the inclusion i induces an isomorphism of $\pi_1(,k_0)$ onto $\pi_1(S(|K|),i(k_0)) = \pi_1(|K|,|k_0,S_0|)$

Let K' be the subcomplex consisting of all simplices of K whose vertices are all at k_0. Then $\pi_1(K,k_0)$ can be considered as a group with one generator for each element of K'_1 and one relation for each element of K'_2.

The space $|K'|$ is a CW-complex with one vertex. For such a space the group π_1 is known to have one generator for each edge and one relation for each face. Thus the homomorphism $\pi_1(K) = \pi_1(K') \longrightarrow \pi_1(|K'|)$ is an isomorphism.

We may assume that K is connected. Then it is known (see [7]) that there is a semi-simplicial deformation retraction $r: K \times I \longrightarrow K$ of K onto K'. By the corollary to theorem 2 this proves that $|K'|$ is a deformation retract of $|K|$ which completes the proof.

Remark 1. From lemmas 4 and 5 it can be proved, using a relative Hurewicz theorem, that the homomorphisms

$$\pi_n(K,k_0) \longrightarrow \pi_n(|K|, |k_0,\delta_0|)$$

are isomorphisms for all n.

Remark 2. The space $|K(\pi,n)|$ has n-th homotopy group π, and other homotopy groups trivial. This clearly follows from the preceding remark. Alternatively the proof given by Hu ([5]) may be used without essential change.

Now let X be any topological space. There is a canonical map

$$j : |S(x)| \longrightarrow X$$

defined by $j(|k_n,\delta_n|) = k_n(\delta_n)$.

<u>Theorem 4.</u> The map $j : |S(x)| \longrightarrow X$ induces isomorphisms of the singular homology and homotopy groups.

(This result is essentially due to Giever [4]).

The map j induces a semi-simplicial map $j_{\#}: S(|S(x)|) \longrightarrow S(X)$. A map i in the opposite direction was defined at the beginning of this section. The composition $j_{\#}i: S(X) \longrightarrow S(X)$ is the identity map. Together with lemma 4 this implies that $j_{\#}$ induces isomorphisms of the singular homology groups of $|S(x)|$ onto those of X. By lemma 5, the fundamental group is also mapped isomorphicly. Using the relative Hurewicz theorem, this completes the proof.

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