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Universal Coefficient Theorems for K-theory

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This paper supplies some universal coefficient theorems for K-theory. In the first two sections, we give as applications calculations of the KU and KO homology of various spaces arising from groups.

~~Much of this paper has been sitting in my files for some time. Recent work of various people on KU_* and KO_* has convinced me that these results are more than curiosities, and deserve publication.~~

There are two proofs of the universal coefficient theorem for KU* given here. The first, in section 3, proceeds by providing the theorem first with Z_p coefficients for p prime (this would work equally well for p any integer). Then a geometric resolution is used to obtain the proper theorem with integral coefficients. A second, algebraic proof is given in second 4. This has three advantages over the simple proof of section 3. First, we do not need the Toda-Araki result on the existence of multiplications on mod p KU-theory. Second, this technique is adequate to handle infinite complexes, and indeed sheds light on KU*(X) when X if infinite. Third, this technique gives us results for KO* which could not be obtained by the first method.

We make applications of the universal coefficient theorem to KU_* in section 1 and to KO_* in section 2. We call the readers attention to the result at the end of section 2. There we show that if G is a simply connected compact Lie group whose fundamental representations are all either real or symplectic, $KO_*(G)$ is a free primitively generated exterior Hopf algebra over $KO_*(pt)$. Examples are $Sp(n)$ and $Spin(n)$ for $n \not\equiv 2 \mod 4$.

1. Results and Applications for KU

If X is a CW-complex, we shall write $\lim^i K^n(X)$ for $\lim^i_{\mathfrak{U}(X)} K^n(X^\alpha)$, where $\mathfrak{U}(X)$ is the set of finite subcomplexes X^α of X, and $\lim^i$ is the i-th derived functor of inverse limit. We show that $\lim^i K^n(X) = 0$ for $i > 1$ if K^* is any cohomology theory for which the groups $K^m(\text{point})$ are all finitely generated, and that there is a natural exact sequence for all n, X:

$$0 \to \lim^1 K^{n-1}(X) \to K^n(X) \to \lim^0 K^n(X) \to 0.$$

(Here, cohomology theory means representable cohomology theory.)

Our universal coefficient theorem (Th. 4.4) tells us that for complex K-theory, KU^*, we have an exact sequence for all n, X:

$$0 \to \lim^1 KU^{n-1}(X) \to \operatorname{Ext}(KU_{n-1}(X), Z) \to$$
$$\lim^0 KU^n(X) \to \operatorname{Hom}(KU_n(X), Z) \to 0$$

Our first application is slightly unusual.

Theorem 1.1 If X is a CW-complex for which the groups $H_i(X;Z)$ are all finite for $i > 0$, then $\lim^1 KU^n(X) = 0$ for all n, and $\widetilde{KU}_n(X)$ is the character group of $\widetilde{KU}^{n+1}(X)$. (The group of continuous homomorphisms of $\widetilde{KU}^{n+1}(X)$ into the circle, where $\widetilde{KU}^n(X)$ is given the inverse limit topology.) Thus, if $X = K(\pi,1)$, the Eilenberg-MacLane complex for a finite group π, $\widetilde{KU}_0(X) = 0$, and $\widetilde{KU}_1(X)$ is the character group of the representation ring of π.

Proof. The groups $H^i(X;Z)$ are all finite for $i > 0$, so that the image $\widetilde{KU}^n(X^{r+1}) \to \widetilde{KU}^n(X^r)$ is finite for all n, r, where X^r is the r-skeleton of X. Thus the $\widetilde{KU}^n(X^r)$ satisfy the Mittag-Leffler condition (see section 3), so $\lim^1 \widetilde{KU}^n(X) = 0$.

The groups $\widetilde{KU}_n(X) = \lim_0 \widetilde{KU}_n(X^r)$ are the direct limit of finite groups. Thus $\mathrm{Hom}(\widetilde{KU}_n(X),Z) = 0$. Thus $\widetilde{KU}^n(X)$ $= \mathrm{Ext}(\widetilde{KU}_n(X),Z)$. Notice that since $\widetilde{KU}_n(X)$ is the limit of finite groups, $\mathrm{Hom}(\widetilde{KU}_n(X),Q) = 0$, so that $\mathrm{Hom}(\widetilde{KU}_n(X),Q/Z) = \mathrm{Ext}(\widetilde{KU}_n(X),Z)$. Since R/Z, the circle, is algebraically the direct sum of Q/Z and an uncountable number of copies of Q, $\widetilde{KU}^{n+1}(X) = \mathrm{Hom}(\widetilde{KU}_n(X),R/Z)$. By Pontrjagin duality, $\widetilde{KU}(X)$ is the character group of $\widetilde{KU}^{n+1}(X)$.

If $X = K(\pi,1)$, Atiyah has shown [5] that $KU^1(X) = 0$, $KU^0(X) = R(\pi)^\wedge$, the completed representation ring of π.

The case $X = K(\pi,1)$ of (1.1) was proved by other techniques by J. Vick [20].

It is part of the folklore that if G is a Lie group with classifying space BG, then $\lim^1 KU^n(BG) = 0$ for all n. Atiyah has proved that $KU^0(BG)$ is the completion of the representation ring of G, and that $KU^1(BG) = 0$. Thus, we see that $\mathrm{Ext}(KU_0(BG),Z) = 0$, and that $\mathrm{Hom}(KU_1(BG),Z) = 0$, and that there is an exact sequence

$$0 \to \mathrm{Ext}(KU_1(BG),Z) \to R(G)^\wedge \to \mathrm{Hom}(KU_0(BG),Z) \to 0.$$

Theorem 1.2. If G is a connected compact Lie group, $KU_1(BG) = 0$, and $KU_0(BG)$ is naturally isomorphic to $\lim_0 \mathrm{Hom}(R(G)/I^n(G),Z)$, the group of continuous homomorphisms of $R(G)$ into Z, where $R(G)$ has the $I(G)$-adic topology, and Z is discrete.

Proof. We know that the $I(G)$-adic topology on $KU^0(BG)$ agrees with the topology induced by restriction to the finite skeletons of BG. Thus, as the natural map $KU_0(BG) \to \mathrm{Hom}(KU^0(BG),Z)$ has its image in the continuous homomorphisms, we have a map defined $KU_0(BG) \to$ Continuous $\mathrm{Hom}(R(G)^\wedge,Z) =$ Continuous $\mathrm{Hom}(R(G),Z)$.

We know that $R(G)$ is the polynomial ring on the fundamental representations $\lambda_1,\ldots,\lambda_m$ of G, and that the ideal $I(G)$ is generated by the elements $\rho_i = \lambda_i - \dim(\lambda_i)$. Thus $R(G)/I^n(G)$ is the free group generated by the monomials in $\rho_1,\ldots,\rho_m$ of degree $< n$. This implies that the sequence $\mathrm{Hom}(R(G)/I^n(G),Z)$ is a sequence of free groups and split monomorphisms. Thus Continuous $\mathrm{Hom}(R(G),Z)$ is free, and the obvious map $R(G)^\wedge \to \mathrm{Hom}(C.\mathrm{Hom}(R(G),Z),Z)$ is an isomorphism, since $R(G)/I^n(G) \to \mathrm{Hom}(\mathrm{Hom}(R(G)/I^n(G),Z),Z)$ is an isomorphism for all n.

It is a result of algebra that if A is an abelian group with $\mathrm{Hom}(A,Z) = 0$, $\mathrm{Ext}(A,Z) = 0$, then $A = 0$. Thus, if we prove that $KU_1(BG) = 0$ and that $KU_0(BG)$ is free, we can apply simple homological algebra to see that $KU_0(BG) \to C.\ \mathrm{Hom}(R(G),Z)$ is an isomorphism. We leave the algebra to the reader, as it is standard.

If T is a maximal torus for G, it is elementary to show that $KU_1(BT) = 0$, and that $KU_0(BT)$ is free. In [4] it is observed that the Todd genus of G/T is 1. If we proceed as in [4], replacing KU^* by KU_*, we see that $KU_*(BT) \to KU_*(BG)$ is a split surjection, with the splitting given by the Gysin map. Thus $KU_1(BG) = 0$, and $KU_0(BG)$ is free.

Remark. The basis of monomials in the ρ_i give us an isomorphism $R(G) \cong C.\mathrm{Hom}(R(G),Z)$; or $R(G) \cong KU_0(BG)$.

The infinite classical groups U, Sp, SO, $Spin$ are defined to be the unions of the corresponding finite groups. Then $KU_*(BU) = \lim KU_*(BU(n))$, etc. Thus $KU_1(BG) = 0$ for G an infinite classical group.

Theorem 1.3. $\lim^1 KU^0(BG(n)) = 0$ and $KU_0(BG)$ is free for $G = U,\ Sp,\ SO$. However, $\lim^1 KU^0(B\,\mathrm{Spin}(n)) \neq 0$, and $KU_0(B\,\mathrm{Spin})$ is not free. close up close up

Proof. The maps $R(G(n+1)) \to R(G(n))$ are split surjections for $G = U,\ Sp$, and the maps $R(SO(2n+1)) \to R(SO(2n-1))$ are split surjections. Furthermore, these maps take fundamental representations either to 0 or to fundamental representations, so that one sees easily that our theorem holds in these cases.

The maps $R(\mathrm{Spin}(2n+1)) \to R(\mathrm{Spin}(2n-1))$ are not surjections. This is due to the fact that the Spin representations correspond under $\Delta_{2n+1} \to 2\Delta_{2n-1}$. Since the group generated by Δ_{2n+1} is a summand in $R(\mathrm{Spin}(2n+1))$, we see that $KU_0(B\,\mathrm{Spin})$ contains as a direct summand a group isomorphic to $Z[\frac{1}{2}]$, the ring of fractions whose denominators are powers of 2. Now $\mathrm{Ext}(Z[\frac{1}{2}],Z) \neq 0$ since $\mathrm{Hom}(Z[\frac{1}{2}],Z) = 0$. Thus $\lim^1 KU^0(B\,\mathrm{Spin}(2n+1)) \neq 0$. Thus $KU^1(B\,\mathrm{Spin}) \neq 0$. close up close up

In [2], it was shown that $\lim^0 \widetilde{KU}^m(K(\pi,n)) = 0$ for any π, any $n > 2$. If π is a finite group, $\lim^1 \widetilde{KU}^m(K(\pi,n)) = 0$, since each group $\widetilde{H}^i(K(\pi,n);Z)$ is finite. Thus, if π is finite, $\widetilde{KU}^*(K(\pi,n)) = 0$ for $n > 1$, since the above result held for $n > 1$ if π was a torsion group. From the universal coefficient theorem above, $\widetilde{KU}_*(K(\pi,n)) = 0$ if $n > 1$, π finite. Since a torsion group is the direct limit of finite groups, $\widetilde{KU}_*(K(\pi,n)) = 0$ for π a torsion group.

From a consideration of the appropriate Serre spectral sequence, one can now observe that if $n > 2$, for any π, the map $\widetilde{KU}_*(K(\pi,n)) \to \widetilde{KU}_*(K(\pi\otimes Q,n))$, where Q = rational numbers, is an isomorphism.

The groups $H_*(K(\pi\otimes Q,n);Z)$ are all rational vector spaces. Thus the spectral sequence for $KU_*(K(\pi\otimes Q,n))$ (analogous to the spectral sequence in [4]) collapses, and all extensions are trivial since Q is injective. This gives us the following result.

Theorem 1.3. If $n > 2$, $\widetilde{KU}_0(K(\pi,n))$ is isomorphic to the direct sum of the groups $\widetilde{H}_{2i}(K(\pi\otimes Q,n))$, and $\widetilde{KU}_1(K(\pi,n))$ is isomorphic to the direct sum of the groups $\widetilde{H}_{2i+1}(K(\pi\otimes Q,n))$.

The previous results are all for infinite complexes, and could therefore be considered to be the esoteric results. For finite complexes, there are much simpler results.

Theorem 1.4. If X is a finite complex, there are natural exact sequences which split for all i:

$$0 \to \mathrm{Ext}(KU_{i-1}(X),Z) \to KU^i(X) \to \mathrm{Hom}(KU^i(X),Z) \to 0$$

$$0 \to \mathrm{Ext}(KU^{i-1}(X),Z) \to KU_i(X) \to \mathrm{Hom}(KU^i(X),Z) \to 0$$

Proof. The existence of the first exact sequence follows from 1.1. The splitting follows from considerations which arise by replacing X by $X \wedge L_p$, as described in section 3. The second sequence follows either algebraically from the first or by Spanier duality from the first, depending upon the reader's preference

To finish this section, we show how to express $KU_*(X)$ in terms of $KU^*(X)$ when X is a countable complex satisfying the Mittag-Leffler condition for KU^*.

First, suppose that A_n is an inverse system of abelian groups. Let B_n be the image of $(\lim^0 A_m) \to A_n$, $C_n = A_n/B_n$. If the A_n satisfy the Mittag-Leffler condition, for each n there is a $k = k(n)$ so that the image of $A_{n+k} \to A_n$ is B_n. Thus, $C_{n+k} \to C_n$ is zero, and $\lim^0 B_n \to \lim^0 A_n$ is an isomorphism. Notice that since $\lim_0$ is exact, we have an exact sequence

$$0 \to \lim_0 \mathrm{Hom}(C_n, Z) \to \lim_0 \mathrm{Hom}(B_n, Z) \to \lim_0 \mathrm{Hom}(A_n, Z)$$

$$\to \lim_0 \mathrm{Ext}(C_n, Z) \to \lim_0 \mathrm{Ext}(B_n, Z) \to \lim_0 \mathrm{Ext}(A_n, Z) \to 0.$$

Thus, since $C_{n+k(n)} \to C_n$ is zero, we see that $\lim_0 \mathrm{Hom}(B_n, Z) \to \lim_0 \mathrm{Hom}(A_n, Z)$ and $\lim_0 \mathrm{Ext}(B_n, Z) \to \lim_0 \mathrm{Hom}(A_n, Z)$ are both isomorphisms.

Given any filtered group $G = \bigcup G_n$, we define Cont. $\mathrm{Hom}(G, Z) = \lim_0 \mathrm{Hom}(G/G_n, Z)$, Cont.$\mathrm{Ext}(G, Z) = \lim_0 \mathrm{Ext}(G/G_n, Z)$. Then Cont.$\mathrm{Hom}(G, Z)$ and Cont.$\mathrm{Ext}(G, Z)$ depend only on the topology which the G_n define on G, as cofinal filtrations can be seen to define the same limits (argue as above).

The following theorem follows immediately from the universal coefficient theorem for finite complexes.

Theorem 1.5. If X is a countable CW-complex which satisfies the Mittag-Leffler condition for KU^*, there is a natural exact sequence for all n.

$$0 \to \mathrm{Cont.Ext}(K^{n+1}(X), Z) \to K_n(X) \to \mathrm{Cont.Hom}(K^n(X), Z) \to 0$$

Corollary 1.6. If G is a compact Lie group,

$$K_0(BG) = \mathrm{Cont.Hom}(R(G), Z)$$

$$K_1(BG) = \mathrm{Cont.Ext}(R(G), Z)$$

2. Applications to KO-theory

There is a theorem which allows us sometimes to calculate $KO^*(X)$ from $KU^*(X)$. A similar result holds for $KO_*(X)$.

Theorem 2.1. If there is a torsion free subgroup $F^* \subset KO^*(X)$ such that the natural map $F^* \otimes KU^*(pt) \to KU^*(X)$ is an isomorphism, then the natural map $F^* \otimes KO^*(pt) \to KO^*(X)$ is an isomorphism.

This theorem follows immediately from the exact sequence of [1], together with the usual 5-lemma algebraic arguments. We refer the reader to p. 257 of [14] for the exact sequences involved. As they are given there, they hold for compact spaces. However, since the sequences are proven by identifying KU^* and KC^* as "KO^*-theory with coefficients", they hold for any complex.

As immediate applications, we observe that if $G = Sp(n)$, $SO(2n+1)$, $SO(4n)$, $Spin(2n+1)$, or $Spin(4n)$, every irreducible (complex) representation of G is either real or symplectic. Thus, we can split $R(G)$ up into two parts, R_1, R_2 so that $R_1 \subset RO(G)$, $R_2 \subset R\,Sp(G)$, $R(G) = R_1 \oplus R_2$. We have natural inclusions $R_1{}^\wedge \to KO^0(BG)$, $R_2{}^\wedge \to KO^4(BG) = K\,Sp^0(BG)$. close up

Theorem 2.2. If we consider $R_1{}^\wedge$ to have degree 0, $R_2{}^\wedge$ to have degree 4, $(R_1{}^\wedge \oplus R_2{}^\wedge) \otimes KO^*(pt) \to KO^*(BG)$ is an isomorphism for any of the groups above. Thus $KO^*(BG)$ is a flat $KO^*(pt)$-module.

In [11], Hodgkin showed that for a compact simply connected Lie group G, $KU^*(G)$ is the exterior algebra generated by the supension of the fundamental representations. This fact, together with the observations above, proves the following theorem.

Theorem 2.3. If $G = Sp(n)$, $Spin(2n+1)$, or $Spin(4n)$, the group $KO^*(G)$ is the free exterior algebra over $KO^*(pt)$ on generators which are the suspensions of the fundamental representations (in $KO^{-1}(G)$ if the representation is real, $KO^{-5}(G)$ if the representation is symplectic).

From considerations of Spanier duality, it is clear that the proof of 2.1 yields a similar theorem with KO^* and KU^* replaced by KO_* and KU_* throughout (the statement of 2.1 will suffice for X a finite complex). To prove theorems of the type above for $KO_*(X)$, we need only produce appropriate subgroups of $KO_*(X)$.

Theorems 4.4 and 4.6 imply the following version of the exact sequence of section 1.

Theorem 2.4. There is a natural exact sequence for all X,n

$$0 \to \lim{}^1 KO^{n-1}(X) \to \operatorname{Ext}(KSp_{n-1}(X), Z)$$

$$\to \lim{}^0 KO^n(X) \to \operatorname{Hom}(KSp_n(X), Z) \to 0$$

The last map is given by the pairing

$$KO^n(X) \otimes KSp_n(X) \to KSp_0(pt).$$

Theorem 2.5. If $KU_1(X) = 0$, $KU_0(X)$ is free, and $KO^0(X) \to KU^0(X)$ is an isomorphism, then $KO_0(X) \to KU_0(X)$ is an isomorphism, and $KO_*(X) = KO_0(X) \otimes KO_*(pt)$.

Proof. Notice that the kernel of $KO_0(X) \to KU_0(X)$ consists of 2-torsion, since the composition with $KU_0(X) \to KO_0(X)$ is multiplication by 2. Let T be this kernel. Since the cokernel of $T \to KO_0(X)$ is free, $\mathrm{Ext}(T,Z) = \mathrm{Ext}(KO_0(X),Z)$. However, since $KO^0(X) = KU^0(X) = \mathrm{Hom}(KU_0(X),Z)$ is torsion free, $KSp^1(X) = KSp^1(pt) \otimes KO^0(X) = 0$. Thus $\mathrm{Ext}(KO_0(X),Z) = 0$, so $\mathrm{Ext}(T,Z) = 0$. Since $\mathrm{Hom}(T,Z) = 0$, $T = 0$, so $KO_0(X) \to KU_0(X)$ is a monomorphism.

Now $KU^0(pt) \to KSp^0(pt)$ is an isomorphism, so that $KU^0(X) \to KSp^0(X)$ is an isomorphism. Thus $\mathrm{Hom}(KU_0(X),Z) \to \mathrm{Hom}(KO_0(X),Z)$ is surjective. Notice that $KU_{-1}(X) = 0$, so that $KO_{-1}(X)$, and thus $\mathrm{Ext}(KO_{-1}(X),Z)$ consists entirely of 2-torsion. Thus $\lim^1 KSp^0(X)$ consists of 2-torsion. Since $KSp^0(X) = KU^0(X)$ is torsion free, $\lim^1 KSp^0(X) = 0$, and $KSp^0(X) \to \mathrm{Hom}(KO_0(X),Z)$ is an isomorphism. Thus, by simple algebra, $KO_0(X) \to KU_0(X)$ is an isomorphism.

Corollary 2.6. If $G = SO(2n+1)$, $\mathrm{Spin}(8n \pm 1)$, $\mathrm{Spin}(8n)$, $KO_*(BG)$ is the free $KO_*(pt)$ module on $KO_0(BG)$. Also, $KO_*(BSO)$ is free on $KO_0(BSO)$ over $KO_*(pt)$.

Theorem 2.7. If X is a finite complex and if $KO^*(X)$ is a free graded $KO^*(pt)$-module, then $KO_*(X)$ is a free graded

$KO_*(pt)$-module.

Proof. Let $x_i \in KO^*(X)$ be homogeneous elements which form a free basis for $KO^*(X)$ as a $KO^*(pt)$-module. In the case of a finite complex, 2.4, together with Spanier duality, imply that for all n, $KO_n(X) \to \mathrm{Hom}(KSp^n(X), Z)$ is onto. Let $y_i = x_i \otimes q \in KSp^*(X)$, where q is the generator of $KSp^0(X)$. Then $KSp^*(X)$/torsion is the free group on the y_i and the x_i, together with their translates under the periodicity (we identify x_i in $KSp^n(X)$ by the isomorphism $KSp^n(X) = KO^{n+4}(X)$).

Choose $z_i \in KO_*(X)$ so that $\langle z_i, y_j \rangle = \delta_{ij}$, $\langle z_i, x_i \rangle = 0$. Then $KO_*(X)$ will be the free $KO_*(pt)$-module generated by the z_i if $KU_*(X)$ is the free $KU_*(pt)$-module generated by the $c(z_i)$, where $c\colon KO_n(X) \to KU_n(X)$ is complexification. Since the $c(x_i)$ are a free basis for $KU^*(X)$, we need to know that $\langle c(z_i), c(x_j) \rangle = \delta_{ij}$. However, if $s\colon KU_n(X) \to KSp_n(X)$ is "symplectification", for any z, x, $\langle c(z), x \rangle = \langle z, s(x) \rangle$. However, $sc(x_i) = x_i \otimes q = y_i$.

Corollary 2.8. If G is a simply connected Lie group for which all of the fundamental representations are either real of symplectic, then $KO_*(G)$ is a free $KO_*(pt)$-module. As an algebra, it is generated by elements $\tau_i \in KO_{-1}(G)$, $\sigma_i \in KO_{-5}(G)$, one for each real (resp. symplectic) fundamental representation of G. If $\tau^i \in KO^{-1}(G)$, $\sigma^i \in KO^{-5}(G)$ are the classes associated to τ_i, σ_i respectively, the τ_i, σ_i

can be chosen with $\langle \tau_i, \tau_j \rangle = \delta_{ij}$ in $KO_0(pt)$, $\langle \sigma_i, \sigma^j \rangle = 4\delta_{ij}$ in $KO_0(pt)$.

Proof. The first part has been proven already. The second part follows from the observations that $KSp_0(pt) \to KU_0(pt)$ is $Z \to Z$ by multiplication by 2, and that if $\langle z, qx \rangle = a$ in $KSp_0(pt)$, then $\langle z, x \rangle = a$ in $KO_0(pt)$, since $KO_0(pt) \xrightarrow{xq} KSp_0(pt)$ is an isomorphism.

We end this section with the observation that 2.3 and 2.7 can be sharpened. If $KO^*(G)$ is the free exterior algebra on representations of G, those representations are necessarily primitive. Since 2.7 says that in this case, $KO_*(G) = \mathrm{Hom}_{KO^*(pt)}(KO^*(G),\ KO^*(pt))$, we see that $KO_*(G)$ is also a primitively generated exterior algebra as a Hopf algebra.

If G is a compact Lie group, the groups $KO^*(BG)$ can be described in terms of $R(G)$, $RO(G)$, and $RSp(G)$ (see [1], [6], [14]). If we follow the reasoning leading up to Corollary 1.6, we obtain the following result.

Theorem 2.9. If G is a compact Lie group

$$KO_0(BG) = \text{Cont Hom}(RSp(G), Z)$$
$$KO_4(BG) = \text{Cont Hom}(RO(G), Z)$$
$$KO_7(BG) = \text{Cont Ext}(RSp(G), Z)$$
$$KO_3(BG) = \text{Cont Ext}(RO(G), Z)$$
$$KO_5(BG) = \text{Cont Ext}(R(G)/RO(G), Z)$$
$$KO_1(BG) = \text{Cont Ext}(R(G)/RSp(G), Z)$$

and there are short exact sequences

$$0 \to \text{Cont Ext}(RSp(G)/R(G), Z) \to KO_6(BG)$$
$$\to \text{Cont Hom}(R(G)/RO(G), Z) \to 0$$
$$0 \to \text{Cont Ext}(RO(G)/R(G), Z) \to KO_2(BG)$$
$$\to \text{Cont Hom}(R(G)/RO(G), Z) \to 0$$

Proof. The only observation needed is that $RO(G)/R(G)$ and $RSp(G)/R(G)$ consist of 2-torsion, so Cont $\text{Hom}(RO(G)/R(G), Z) = 0$, etc.

3. Universal Coefficient Theorems for K-theory

There are several theorems in ordinary cohomology theory which are sometimes given the name "universal coefficient theorem". These theorems are all special cases of the same theorem in homological algebra, which relates the homology of $T(X,Y)$ to the homology of X and Y, where T is a functor, and X and Y are projective chain complexes. In K-theory, we do not have chain complexes to work with, but we still can prove the same theorems.

In [21], G. Whitehead showed that for every representable cohomology theory, there exists a corresponding homology theory. Specifically, if $\underline{L} = \{L^n, f^n\colon SL^n \to L^{n+1}\}$ is a spectrum, the representable cohomology theory with coefficients in $\underline{L}$ is defined by $\tilde{H}^r(X;\underline{L}) = \lim\operatorname{dir}[S^nX, L^{n+r}]$, and the homology theory by $\tilde{H}_r(X;\underline{L}) = \lim\operatorname{dir}[S^{n+r}, X\wedge L^n]$ (all maps and homotopies respect base-points). If $\underline{L}$ is an Ω-spectrum (that is, if the maps $g^n\colon L^n \to \Omega L^{n+1}$ determined by the maps f^n are homotopy equivalences), then we can drop the direct limit, and we have $\tilde{H}^r(X;\underline{L}) = [X, L^r]$. In this case, the cohomology theory with coefficients in L can clearly be extended to the category of all pairs of spaces with the homotopy extension property. We then can define canonical elements $\iota_n \in \tilde{H}^n(L^n;\underline{L})$ to be the equivalence class of the identity map. In order to define a multiplication in the cohomology theory with coefficients in the spectrum $\underset{\sim}{L}$, it suffices to define elements $\mu(\iota_r \otimes \iota_s) \in \tilde{K}^{r+s}(L^r \wedge L^s; \underset{\sim}{L})$ such that $\sigma(\mu(\iota_r \otimes \iota_s)) = (f^r \wedge 1)^*(\mu(\iota_{r+1} \otimes \iota_s)) = (-1)^r (1 \wedge f^s)^*(\mu(\iota_r \otimes \iota_{s+1}))$ for all r, s. It is easy to see that specifying such a set of elements is equivalent to defining a pairing of the spectrum $\underset{\sim}{L}$ with itself into itself in the sense of Whitehead (p. 254, [21]).

It may happen that for some spectrum $\underset{\sim}{L}$, a multiplication has been defined on the cohomology theory with coefficients in $\underset{\sim}{L}$ restricted to the category of finite CW-complexes. If each L^n is a countable CW-complex, one might hope to define the elements $\mu(\iota_r \otimes \iota_s)$ by some sort of inverse limit process. The homotopy extension property for CW-pairs immediately shows that if X is a CW-complex, and if $X^0 \subset X^1 \subset \ldots \subset X$ are a sequence of subcomplexes, such that $X = \bigcup_n X^n$, the map $H^*(X;\underset{\sim}{L}) \to \lim \operatorname{inv}(H^*(X^n;\underset{\sim}{L}))$ is onto. Under certain conditions, it will also be an injection.

Since we shall have occasion to use the derived functors of the inverse and direct limits, we shall denote lim inv by $\lim^0$ and lim dir by $\lim_0$. From [16] we see that $\lim^0$ is a left exact functor, and $\lim_0$ is a right exact functor. The right derived functors of $\lim^0$ will be denoted by $\lim^p$, and the left derived functors of $\lim_0$ by $\lim_p$. Let $\underset{\sim}{\mathfrak{U}}$ be a partially ordered set, and let $\{G_\alpha : \alpha \in \underset{\sim}{\mathfrak{U}}\}$ be a system of groups ordered by $\underset{\sim}{\mathfrak{U}}$ with homomorphisms $g^\alpha_\beta : G_\alpha \to G_\beta$ for all $\alpha \geq \beta$. These homomorphisms are assumed to satisfy the relation $g^{\beta}_{\gamma} g^{\alpha}_{\beta} = g^{\alpha}_{\gamma}$ whenever $\alpha \geq \beta \geq \gamma$. We take note of two results of [16]: i) if $\mathfrak{U}$ is a directed set (that is, for $\alpha, \beta \in \mathfrak{U}$ there exists $\gamma \in \mathfrak{U}$ with $\alpha \geq \gamma$ and $\beta \geq \gamma$), then $\lim_0$ is exact, and $\lim_p = 0$ for all $p > 0$; ii) if $\mathfrak{U}$ is countable, $\lim^p = 0$ for $p > 1$. Since we shall always be working with directed sets, we shall not have to worry about the functors $\lim_p$.

Grothendieck has introduced a condition for countable inverse systems of groups which insures the vanishing of $\lim^1$; he calls this condition the Mittag-Leffler condition [10]:

(ML) For all $\alpha \in \mathfrak{U}$ there exists $\beta \geq \alpha$ such that for all $\gamma \geq \beta$,

$$g^{\gamma}_{\alpha}(G_{\gamma}) = g^{\beta}_{\alpha}(G_{\beta}).$$

By the usual sort of homological algebra, Grothendieck's proposition 13.2.1 implies that if an inverse directed countable system $\{G_{\alpha}\}$ of groups satisfies the Mittag-Leffler condition, then $\lim^1(G_{\alpha}) = 0$.

Nöbeling has proven a duality theorem which is of great help in proving that the functors $\lim^p$ vanish for $p \neq 0$. His duality theorem (Satz 9 of [16]) states that if M_{α} is an inverse system of modules over a ring R, and if I is an injective R-module, then $\lim^p(\mathrm{Hom}_R(M_{\alpha}, I) \cong \mathrm{Hom}_R(\lim_p(M_{\alpha}), I)$. One example of the usefulness of this is the case when each M_α has the structure of a compact (Hausdorff) topological group, and the homomorphisms g^α_β are all continuous. If $\hat{M}_\alpha$ is the group of continuous homomorphisms of M_α into the circle S^1, then by Pontrjagin duality, we know that there is a natural isomorphism $M_\alpha \cong \mathrm{Hom}(\hat{M}_\alpha, S^1)$. Since S^1 is injective, we see that if the M_α form a directed system, $\lim^p(M_\alpha) = 0$ for $p \neq 0$, and $\lim^0(M_\alpha) = \mathrm{Hom}(\lim_0(\hat{M}_\alpha), S^1)$. One simple subcase of this is the case when each M_α is a finite group in the discrete topology.

If we are given an arbitrary cohomology theory, it is hard to relate the cohomology of a CW-complex to the cohomology of either its skeletons or its finite subcomplexes. Milnor has introduced axioms for homology and cohomology theories which give a precise relationship between the homology or cohomology of a CW-complex and that of its skeletons - or any other countable collection of subcomplexes whose union is the whole complex. These conditions are satisfied by representable cohomology theories, and by ordinary singular cohomology, but not by ordinary Čech cohomology. Milnor's axioms for additive theories are as follows [15]:

(A_*) If $\{X^\alpha \mid \alpha \in I\}$ is an admissible family of spaces for a homology theory K_*, then their topological sum X is admissible; further, if $i^\alpha\colon X^\alpha \to X$ is the natural inclusion map for each α, then the direct sum i_* of the $(i^\alpha)_*$ induces an isomorphism:

$$i_*: \sum K_r(X^\alpha) \cong K_r(X) \quad \text{for all} \quad r.$$

(A*) If $\{X^\alpha \mid \alpha \in I\}$ is an admissible family of spaces for a cohomology theory K^*, then their topological sum X is admissible; further, if $i^\alpha: X^\alpha \to X$ is the natural inclusion map for each α, then the direct product i^* of the $(i^\alpha)^*$ induces an isomorphism:

$$i^*: \prod K^r(X^\alpha) \cong K^r(X) \quad \text{for all} \quad r.$$

Suppose that X is a CW-complex, and that there is an increasing sequence of subcomplexes $X^0 \subset X^1 \subset \ldots \subset X$ whose union is X. In this case, Milnor has proven the following theorems:

(M_*) If K_* is an additive homology theory, there is a natural isomorphism

$$K_r(X) \cong \lim{}_0 K_r(X^n) \quad \text{for all } r.$$

(M^*) If K^* is an additive cohomology theory, there is a natural exact sequence $0 \to \lim^1(K^{r-1}(X^n)) \to K^r(X) \to \lim^0(K^r(X^n)) \to 0$ for all r.

In proposition 4.2, we show that Milnor's first theorem easily implies a corresponding theorem for additive homology theories for a CW-complex and all of its finite subcomplexes. In lemma 4.3, we prove a corresponding theorem for additive cohomology under slight restrictions.

It is clear that if L is an Ω-spectrum, the cohomology theory with coefficients in L is additive on the category of all CW-complexes. The complex K-theory KU^* can be defined as the cohomology theory with coefficients in the unitary spectrum $\underline{U}$, where $U^{2r+1} = U$, the infinite unitary group, and $U^{2r} = Z \times B_U$.

Let $U(n)$ be the n-dimensional unitary group, and let $G_{m,n} = U(m+n)/U(m) \times U(n)$ be the Grassman manifold of complex m-planes in complex $m+n$-dimensional affine space. Then $U = \bigcup_n U(n)$, and $B_U = \bigcup_n G_{n,n}$. Let $G_n \subset Z \times B_U$ be those components of $Z \times G_{n,n}$ whose Z component has absolute value less than or equal to n. Then $Z \times B_U = \bigcup_n G_n$. Let r and s be fixed integers, and let $\{X_n\}$ be the following sequences of CW-complexes:

i) if r is odd and s is odd, $X_n = U(n) \wedge U(n)$

ii) if r is odd and s is even, $X_n = U(n) \wedge G_n$

iii) if r is even and s is odd, $X_n = G_n \wedge U(n)$

iv) if r is even and s is even, $X_n = G_n \wedge G_n$.

Then, for all r and all s, $H^*(X_n;Z)$ is torsion free, and the restriction maps $H^*(X_{n+1};Z) \to H^*(X_n;Z)$ are onto. By the usual sort of spectral sequence argument, we see that $KU^*(X_n)$ is torsion free, and that the restriction maps $KU^*(X_{n+1}) \to KU^*(X_n)$ are all onto. Thus for all t we see that $\lim^1(KU^t(X_n)) = 0$. Thus, by Milnor's theorem, we see that for all t, $KU^t(U^r \wedge U^s) = \lim^0(KU^t(X_n))$. Thus, if $i_{n,2r+1}\colon U(n) \to U^{2r+1}$ and $i_{n,2r}\colon G_n \to U^{2r}$ are the usual inclusion maps, the elements $\mu(i_{n,r}^*(\iota_r) \otimes i_{n,s}^*(\iota_s))$ define an inverse system of elements of the groups $\widetilde{KU}^{r+s}(X_n)$, and thus define uniquely an element $\mu(\iota_r \otimes \iota_s)$ of $\widetilde{KU}^{r+s}(U^r \wedge U^s)$. It is clear that these elements, thought of as maps of $U^r \wedge U^s$ into U^{r+s}, have the correct properties to define on $\underset{\sim}{U}$ the structure of a ring spectrum in the sense of G. Whitehead [21]. Since for every integer p, the theory $KU^*(\ ;Z_p)$ is the representable cohomology theory with coefficients in an Ω-spectrum, and since that spectrum can be chosen so that every element of it is a CW-complex, our earlier construction of μ_p defines on this spectrum the structure of a ring spectrum. Further, the reduction mod p clearly can be defined by a transformation of ring spectra.

As a direct consequence of Proposition 1.1, we have the following:

Proposition 3.1. If X is a CW-complex, and if Y is a finite CW-complex, then for every prime p, there is an isomorphism:

$$\mu_r\colon KU^*(X;Z_p) \otimes_{\Lambda_p} KU^*(Y;Z_p) \cong KU^*(X\times Y;Z_p),$$

where $\Lambda_p = KU^*(\text{point};Z_p)$.

Proof. For each r, $KU^r(X;Z_p)$ is a vector space over Z_p. Since $KU^*(X;Z_p) = \Lambda_p \otimes (KU^0(X;Z_p) \oplus KU^1(X;Z_p))$, $KU^*(X;Z_p)$ is free over Λ_p.

In [3], Atiyah showed that if $\Lambda = KU^*(\text{point})$, then for any two finite CW-complexes X and Y, there was a short exact sequence of the form:

$$0 \to KU^*(X) \otimes_\Lambda KU^*(Y) \xrightarrow{\mu} KU^*(X\times Y) \to \mathrm{Tor}^\Lambda(KU^*(X),KU^*(Y)) \to 0$$

Corollary 3.1. The exact sequence of Atiyah's Künneth theorem always splits (unnaturally), so that for every pair of finite CW-complexes,

$$KU^*(X\times Y) \cong (KU^*(X)\otimes_\Lambda KU^*(Y) \oplus \mathrm{Tor}^\Lambda(KU^*(X),KU^*(Y))$$

Proof. We have a commutative diagram

$$\begin{array}{ccc}
KU^*(X) \otimes_\Lambda KU^*(Y) & \xrightarrow{\mu} & KU^*(X\times Y) \\
\downarrow{\scriptstyle \rho_p\otimes\rho_p} & & \downarrow{\scriptstyle \rho_p} \\
KU^*(X;Z_p) \otimes_{\Lambda_p} KU^*(Y;Z_p) & \xrightarrow{\mu_p} & KU^*(X\times Y;Z_p)
\end{array}$$

Since μ_p is an isomorphism, $\mu \otimes 1$: $(KU^*(X) \otimes_\Lambda KU^*(Y)) \otimes Z_p \to KU^*(X \times Y) \otimes Z_p$ is an injection, as the universal coefficient theorem says that $\rho_p \otimes \rho_p \otimes 1$: $(KU^*(X) \otimes_\Lambda KU^*(Y)) \otimes Z_p \to KU^*(X;Z_p) \otimes_{\Lambda_p} KU^*(Y;Z_p)$ is an injection. Since $\mu \otimes 1$ is an injection for all primes p, μ must embed $KU^*(X) \otimes_\Lambda KU^*(Y)$ as a direct summand of $KU^*(X \times Y)$.

Since $\underset{\sim}{U}$ is a ring spectrum, following Whitehead, we can define slant product pairings

$$KU^r(X \times Y) \otimes KU_s(Y) \to KU^{r-s}(X) \quad \text{and}$$
$$KU_r(X \times Y) \otimes KU^s(Y) \to KU_{r-s}(X)$$

whenever X and Y are finite CW-complexes. If $X = Y =$ point, these are the usual multiplication in $KU^*(\text{point})$ under the natural identification $KU_s(\text{point}) = KU^{-s}(\text{point})$. In particular, this says that the map $KU^r(\text{point}) \otimes KU_r(\text{point}) \to KU^0(\text{point})$ is an isomorphism when r is even.

The slant product pairing defines maps

$$KU^*(X \times Y) \to \mathrm{Hom}_\Lambda(KU_*(Y), KU^*(X)) \quad \text{and} \quad KU_*(X \times Y) \to \mathrm{Hom}_\Lambda(KU^*(Y), KU_*(X))$$

which are isomorphisms when $X = Y =$ point. If we take $X =$ point, we have homomorphisms: $KU^*(Y) \to \mathrm{Hom}_\Lambda(KU_*(Y), \Lambda)$ and $KU_*(Y) \to \mathrm{Hom}_\Lambda(KU^*(Y), \Lambda)$. Similarly, we obtain maps

$$KU^*(Y;Z_p) \to \mathrm{Hom}_{\Lambda_p}(KU^*(Y;Z_p), \Lambda_p) \quad \text{and}$$
$$KU_*(Y;Z_p) \to \mathrm{Hom}_{\Lambda_p}(KU_*(Y;Z_p), \Lambda_p \quad \text{for all } p.$$

Proposition 3.2. If p is a prime, the maps

$$KU^r(Y;Z_p) \to \mathrm{Hom}(KU_r(Y;Z_p),Z_p) \quad \text{and}$$
$$KU_r(Y;Z_p) \to \mathrm{Hom}(KU^r(Y;Z_p),Z_p)$$

are isomorphisms for all r and all finite CW-complexes Y.

Proof. $KU^r(Y;Z_p)$ is a vector space over Z_p for all primes p, so that $\mathrm{Hom}(KU^r(Y;Z_p),Z_p)$ defines a homology theory. Since the map $KU_r(Y;Z_p) \to \mathrm{Hom}(KU^r(Y;Z_p),Z_p)$ is a natural transformation of homology theories, and since it induces an isomorphism when Y = point, it induces an isomorphism for all Y. The other statement follows similarly.

Proposition 3.3. If Y is a finite complex, $KU^*(Y)$ is torsion free if and only if $KU_*(Y)$ is torsion free. If $KU^*(Y)$ is torsion free, the maps

$$KU^r(Y) \to \mathrm{Hom}(KU_r(Y),Z) \quad \text{and}$$
$$KU_r(Y) \to \mathrm{Hom}(KU^r(Y),Z)$$

are isomorphisms for all r.

Proof. Suppose that $KU^*(Y)$ is torsion free. Then, for each prime p, $KU^*(Y;Z_p) = KU^*(Y) \otimes Z_p$. Thus $KU_r(Y;Z_p)$ has the same rank for every prime p. Since $KU_r(Y)$ is finitely generated, for all r, the universal coefficient theorem implies that $KU_r(Y)$ must be torsion free for all r. The converse can be shown in the same manner.

The commuting diagram

$$\begin{tikzcd} KU^r(Y) \arrow[r] \arrow[d, "\rho_p"] & \mathrm{Hom}(KU_r(Y),Z) \arrow[d] \\ KU^r(Y;Z_p) \arrow[r] & \mathrm{Hom}(KU_r(Y;Z_p),Z_p) \end{tikzcd}$$

shows that if $KU^*(Y)$ is torsion free, the map $KU^r(Y) \otimes Z_p \to \mathrm{Hom}(KU_r(Y),Z) \otimes Z_p$ is an isomorphism for all r and all primes p. Thus, the map $KU^r(Y) \to \mathrm{Hom}(KU_r(Y),Z)$ is an isomorphism for all r. The other statement is similar.

Remark: One might be tempted to try to prove the above theorem by using the Chern character and comparing with rational homology and cohomology. However, there is a danger here - it is quite possible that $KU^*(Y)$ is torsion free, while $H^*(Y;Z)$ is not. One simple example is given by the symmetric square of the three-sphere [13].

We are now in a position to prove a universal coefficient theorem relating KU^* and KU_*. In [3], Atiyah showed how to produce, topologically, a free resolution of $KU^*(X)$ for any finite complex X. He showed that for any finite complex X, there was another finite complex Y such that $KU^*(Y)$ was torsion free, and a map $f\colon X \to Y$ such that $f^*\colon KU^*(Y) \to KU^*(X)$ was onto. If T_f is the mapping cone of f, we then have a free resolution

$$0 \to \widetilde{KU}^*(T_f) \to KU^*(Y) \xrightarrow{f^*} KU(X) \to 0.$$

Theorem 3.3. If X is a finite CW-complex, there are short exact sequences:

$$0 \to \mathrm{Ext}(KU_{r-1}(X), Z) \to KU^r(X) \to \mathrm{Hom}(KU_r(X), Z) \to 0$$

$$0 \to \mathrm{Ext}(KU^{r+1}(X), Z) \to KU_r(X) \to \mathrm{Hom}(KU^r(X), Z) \to 0$$

Proof. Let $f\colon X \to Y$ be as described above, and let T_f be the mapping cone of f. By our last proposition, $KU_*(Y)$ and $\widetilde{KU}_*(T_f)$ are torsion free, and the maps $KU_r(Y) \to \mathrm{Hom}(KU^r(Y), Z)$ and $\widetilde{KU}_r(T_f) \to \mathrm{Hom}(\widetilde{KU}^r(T_f), Z)$ are isomorphisms. From the exact sequence $\ldots \to \widetilde{KU}_{r+1}(T_f) \xrightarrow{\partial} KU_r(X) \xrightarrow{f_*} KU_r(Y) \to \widetilde{KU}_r(T_f) \xrightarrow{\partial} \ldots$, we obtain the second exact sequence. The first one follows from it by simple homological algebra.

Whitehead has also defined pairings $KU_r(X) \otimes KU_s(Y) \to KU_{r+s}(X \times Y)$. It is natural to expect that there would be a Künneth theorem for KU_*.

Theorem 3.4. If p is a prime, then the pairing

$$KU_*(X; Z_p) \otimes_{\Lambda_p} KU_*(Y; Z_p) \to KU_*(X \times Y; Z_p)$$

is an isomorphism for all finite CW-complexes X and Y.

Proof. $KU_*(X; Z_p)$ is a free module over Λ_p. Since the theorem holds when Y is a sphere, by induction it holds for any finite complex Y.

Theorem 3.5. For any two finite CW-complexes X and Y, there is a natural short exact sequence, which splits unnaturally:

$$0 \to KU_*(X) \otimes_\Lambda KU_*(Y) \to KU_*(X \times Y) \to \mathrm{Tor}^\Lambda(KU_*(X), KU_*(Y)) \to 0.$$

(The right hand map is understood to decrease degrees by one.)

Proof. If $KU_*(X)$ is torsion free, it is a free Λ module. By the usual induction argument, the theorem holds if $KU_*(X)$ is torsion free. If X is arbitrary, let $f: X \to X'$ be as constructed by Atiyah so that $KU^*(X')$ is torsion free and such that f^* is onto. Let T_f be the cofiber of f. By our earlier theorem, $KU_*(X')$ and $\widetilde{KU}_*(T_f)$ are torsion free. Thus, $KU_*(X' \times Y) = KU_*(X') \otimes_\Lambda KU_*(Y)$ and $KU_*(T_f \times Y) = KU_*(T_f) \otimes_\Lambda KU_*(Y)$. By the obvious exact sequence argument, we see that $KU_*(T_f \times Y, Y) = \widetilde{KU}_*(T_f) \otimes_\Lambda KU_*(Y)$. The cofibration $X^+ \wedge Y^+ \to (X')^+ \wedge Y^+ \to T_f \wedge Y^+$ gives us an exact sequence: $\ldots \to KU_{r+1}(T_f \times Y, Y) \xrightarrow{\partial} KU_r(X \times Y) \to KU_r(X' \times Y) \to KU_r(T_f \times Y, Y) \xrightarrow{\partial} \ldots$ Since the theorem holds for X' and T_f, it holds for X. Naturality is proved in the same way as in [3]. The existence of an unnatural splitting is proved in the same way as in corollary 3.1.

§4 Extensions to Infinite Complexes

The theorems of the last section were all proved under the restriction that all of the complexes involved were finite. If we extend KU_* to infinite complexes, the same theorems can at least be formulated. It is not hard to show that, in general, Atiyah's Künneth theorem for KU^* does not hold for arbitrary complexes. However, the Künneth theorem for KU_* holds without any restriction. The universal coefficient theorem relating KU^* and KU_* can be generalized, but it looks slightly different from the form encountered for finite complexes, due to the fact that the functor $\lim^1$ occurs. Also, the fact that we are using KU^* is not entirely relevant - we shall show that if K^* is a cohomology theory such that $K^*(\text{point})$ is of finite type, then there is a dual theory DK^* such that K^* and DK_* are related by a universal coefficient theorem. The operation D has the property that $D^2K^* = K^*$, and is an exact contravariant functor from the category of cohomology theories to itself. D seeems to behave on cohomology theories in much the same way that Spanier's S-duality behaves on finite complexes. It is not clear that D is of any particular interest in its own right, but it does simplify some of our proofs.

We now give a method for extending a homology theory to the category of all CW-complexes. If X is a CW-complex, let $\mathfrak{U}(X)$ be the set of all finite subcomplexes of X, ordered under inclusion. Then $\mathfrak{U}(X)$ is closed under finite union and arbitrary intersection. Since it is closed under finite union, $\mathfrak{U}(X)$ is a direct set. We shall consider $\mathfrak{U}(X)$ abstractly as an indexing set, and denote by X^α the finite complex indexed by the element α of $\mathfrak{U}(X)$. If $f: X \to Y$ is a continuous map of CW-complexes, since the compact subsets of CW-complexes are the closed subsets of the finite subcomplexes, if X^α is a finite subcomplex of X, there exists a finite subcomplex Y^β of Y such that $f(X^\alpha) \subset Y^\beta$. Since $\underset{\sim}{\mathfrak{U}}(Y)$ is closed under arbitrary intersection, there is a minimal such Y^β. Thus f induces a homomorphism $f_*: \underset{\sim}{\mathfrak{U}}(X) \to \underset{\sim}{\mathfrak{U}}(Y)$ of partially ordered sets.

Definition 4.1. If (X,A) is a CW-pair, K_* is a homology theory defined on the category of finite CW-complexes, the direct limit extension of K_* to all CW-complexes is defined by:

$$K_r(X,A) = \lim_0{}^{\underset{\sim}{\mathfrak{U}}(X)} K_r(X^\alpha, X^\alpha \cap A).$$

We notice that the direct limit extension of K_* is a homology theory:

If (X,A) is a CW-pair, we have an exact sequence (since $\underset{\sim}{\mathfrak{U}}(X)$ is directed):

$$\cdots \xrightarrow{\partial} \lim_0{}^{\underset{\sim}{\mathfrak{U}}(X)} K_r(X^\alpha \cap A) \to K_r(X) \to K_r(X,A) \xrightarrow{\partial} \cdots$$

The inclusion map gives us a homomorphism $K_r(A) \to \lim_0{}^{\underset{\sim}{\mathfrak{U}}(X)} K_r(X^\alpha \cap A)$. Since every finite subcomplex of A is of the form $X^\alpha \cap A$, we see that this map is an isomorphism. Thus the exact sequence for a pair holds. A strong form of the excision axiom holds - since CW-complexes are closure finite, every finite subcomplex of X/A is of the form $X^\alpha/(X^\alpha \cap A)$ for some finite subcomplex X^α of X. Thus, we have an isomorphism $\widetilde{K}_r(X/A) \cong K_r(X,A)$. Finally, if f_0 and f_1 are two maps of X into Y, and $F: X \times [0,1] \to Y$

is a homotopy such that $F|X \times [i] = f_i$ for $i = 0,1$, then for each finite subcomplex X^α of X, there is some finite subcomplex Y^β of Y such that $F(X^\alpha \times [0,1]) \subset Y^\beta$. Thus, $(f_0)_* = (f_1)_*$, and the homotopy axiom is satisfied.

Before we begin to extend Künneth theorems to infinite complexes, we should pay some attention to the topology that we put on the product of two CW-complexes. In general, the product of two CW-complexes is not a CW-complex in the product topology. However, it does have the homotopy type of a CW-complex - in fact, if the product of two CW-complexes is given the weak topology, the identity map gives a homotopy equivalence between this complex and the product in the product topology (see lemma 2.1 of [21]). In order that we should not leave the category, we shall give all cartesian products and all smash products of CW-complexes the weak topology.

We see from theorem V, 9.4* of [8] that $\otimes$ and Tor are both functors of type $L\Sigma^*$; that is, they both commute with direct limits, when we are working over directed sets. Thus, we obtain the following theorem:

Theorem 4.1. If X and Y are CW-complexes, then there is a natural exact sequence:

$$0 \to KU_*(X) \otimes_\Lambda KU_*(Y) \to KU_*(X \times Y) \to \mathrm{Tor}^\Lambda(KU_*(X), KU_*(Y)) \to 0.$$

(The right hand map is understood to decrease dimension by one.) If p is a prime, then we have an isomorphism:

$$KU_*(X;Z_p) \otimes_{\Lambda_p} KU_*(Y;Z_p) \cong KU_*(X\times Y;Z_p)$$

Proof. These statements follow immediately from the corresponding statements about finite complexes, together with the observation that $\{X^\alpha \times Y^\beta \mid \alpha \in \mathfrak{U}(X),\ \beta \in \mathfrak{U}(Y)\}$ is a cofinal subset of $\mathfrak{U}(X\times Y)$.

Since the inverse limit functor is not exact and does not commute with tensor products, we might expect that the Künneth theorem would not generalize to infinite complexes for KU^*. However, if we work mod p for a prime p, it is quite possible that $KU^*(X;Z_p)$ is of finite type even though $KU^*(X)$ is not. Since examples of such spaces are fairly plentiful, a theorem to cover this situation would be useful. First we must prove a general theorem about additive cohomology theories.

Proposition 4.2. Let H^* and K^* be additive cohomology theories which are defined on the category of CW-complexes, and let φ be a natural transformation of cohomology theories from H^* to K^*. Then, if φ: $H^*(\text{point}) \to K^*(\text{point})$ is an isomorphism, φ: $H^*(X) \to K^*(X)$ is an isomorphism for all CW-complexes X.

Proof. It is clear that φ induces an isomorphism whenever X is either a disjoint union of spheres or a disjoint union of points. Thus, φ induces an isomorphism whenever X is a one-point union of spheres. If X is an arbitrary complex, and X^n is the n-skeleton of X, X^n/X^{n-1} is a one-point union of spheres. Thus, by induction, $\varphi: H^*(X^n) \to K^*(X^n)$ is an isomorphism for all n. From Milnor's theorem, we see that $\varphi: H^*(X) \to K^*(X)$ is an isomorphism.

There is an analogous result for additive homology theories. One result of this is that the extension of a homology theory from the category of finite complexes to the category of all complexes is the only additive extension.

Proposition 4.3. Let H_* and K_* be additive homology theories, and let $\varphi: H_*(X) \to K_*(X)$ be a natural transformation of homology theories. If φ induces an isomorphism when X is a point, then φ induces an isomorphism when X is any CW-complex. Thus, if X is a CW-complex, there is a natural isomorphism for any additive homology theory K_*:

$$K_r(X) \cong \lim{}_0^{\mathfrak{A}(X)} K_r(X^\alpha).$$

Proof. The proof proceeds just as in the last proposition. The last statement follows from the fact that the right hand side defines an additive homology theory, since direct limit and direct sum commute.

We now can make two extensions of earlier theorems.

Theorem 4.4. If X and Y are CW-complexes, and if p is a prime, then if $KU^*(X;Z_p)$ is of finite type, we have an isomorphism:

$$KU^*(X;Z_p) \otimes_{\Lambda_p} KU^*(Y;Z_p) \cong KU^*(X \times Y;Z_p)$$

Proof. Since $KU^*(X;Z_p)$ is a finitely generated free module over Λ_p, both sides of the isomorphism are additive cohomology theories in the variable Y. Since they are isomorphic when Y is a point, by proposition 4.1, they are isomorphic for all Y.

Theorem 4.5. If X is a CW-complex, there is a natural isomorphism:

$$KU^r(X;Z_p) \cong \mathrm{Hom}(KU_r(X;Z_p),Z_p).$$

Proof. Since $KU_*(-;Z_p)$ is an additive homology theory, $\mathrm{Hom}(KU_r(-;Z_p),Z_p)$ is an additive cohomology theory. The theorem therefore follows from proposition 4.2.

In order to prove a universal coefficient theorem of this type for $KU^*(X)$ and $KU_*(X)$, we must separate the free and the torsion parts of $KU_*(X)$. We do this by means of using the coefficient groups Q (the rational numbers) and Q/Z, both of which are injective groups. Since most of what we will be doing can be done more generally with no more effort, we shall not restrict ourselves to KU^*. From now on, K^* will be a cohomology theory such that $K^*(\text{point})$ is of finite type.

According to E. Brown [7], if K^* is a cohomology theory defined on the category of finite CW-complexes such that K^*(point) is countable, there is an Ω-spectrum $\underline{A} = \{A^n\}$ such that $K^*(X) = H^*(X;\underline{A})$. We shall simply call H^* countable if H^*(point) is.

Proposition 4.6. If H^*, K^* are two countable cohomology theories, defined for finite complexes, and if $\Phi: H^*(X) \longrightarrow K^*(X)$ is a stable cohomology operation defined for X a finite complex, then there exist Ω-spectra $\underline{A}, \underline{B}$, together with maps $\varphi^n: A^n \longrightarrow B^n$, such that H^* is naturally isomorphic to $H^*(-;\underline{A})$, K^* is naturally isomorphic to $H^*(-;\underline{B})$, the φ^n induce Φ, and for each n, φ^n is homotopic to $\Omega\varphi^{n+1}$.

Proof. Since H^* and K^* are representable, one can choose spectra $\underline{C}$ and $\underline{B}$ respectively which represent them, such that each C^n and each B^n is countable. In view of Milnor's theorem, there are maps $\psi^n: C^n \longrightarrow B^n$ such that for each n, ψ^n is homotopic to $\Omega\psi^{n+1}$ on all finite subcomplexes of C^n. Because each C^n is countable, we can choose finite subcomplexes D^n of each C^n which form a subspectrum, and such that the infinite mapping cylinders associated to the sequences $D^n \longrightarrow \Omega D^{n+1} \longrightarrow \Omega^2 D^{n+2} \longrightarrow \ldots$ are homotopy equivalent to the C^n. Let A^n be the infinite mapping cylinder of the sequence of maps above. Then $\underline{A} = \{A^n\}$ is an Ω-spectrum which represents H^*. Once we have chosen, for $m \geq n$, homotopies between the restriction of ψ^m to D^m and the restriction of $\Omega\psi^{m+1}$, we have defined the desired maps $\varphi^n: A^n \longrightarrow B^n$.

If we have a map $\varphi: \underline{A} \longrightarrow \underline{B}$ of spectra as above, we can define a new spectrum $\underline{C}$ by letting C^n be the mapping fiber of φ^{n+1}. This defines a long sequence of maps of spaces, in which the composition of any two successive maps is homotopy equivalent to a Hurewicz fibration (see [12]):

$$\cdots \longrightarrow B^{n-1} \xrightarrow{\rho^{n-1}} C^{n-1} \xrightarrow{\lambda^n} A^n \xrightarrow{\varphi^n} B^n \xrightarrow{\rho^n} C^n \longrightarrow \cdots$$

In general, the cohomology theory $H^*(-;\underset{\sim}{C})$ depends upon the choice of $\underset{\sim}{A}, \underset{\sim}{B}$, and φ, and not just upon the associated cohomology theories and the associated cohomology operation. The following result will allow us to overcome this difficulty in one important case. No doubt stronger results can be proven, but we shall not need them.

Proposition 4.7. Suppose that K^* is a countable cohomology theory, that $\psi: \underset{\sim}{A} \longrightarrow \underset{\sim}{B}$ is a map of Ω-spectra, and that $\Phi: K^*(-) \longrightarrow H^*(-;\underset{\sim}{A})$ is a stable cohomology operation defined for finite complexes. Then, if $\underset{\sim}{C}$ is the mapping fiber of ψ, Φ can be factored through a cohomology operation $K^*(-) \longrightarrow H^*(-;\underset{\sim}{C})$, defined for all complexes, if the following conditions hold:

a) The composition of Φ with the cohomology operation defined by ψ is zero for all finite complexes.

b) There exists a homology theory L_*, together with natural isomorphisms of functors for all finite complexes $H^*(X;A) = \mathrm{Hom}(L_*(X),Q)$, $H^*(X;B) = \mathrm{Hom}(L_*(X),Q/Z)$, compatible with ψ, where each $L_n(\text{point})$ is finitely generated.

Proof. Let $\underset{\sim}{D} = \{D^n\}$ represent K^*, and let $\varphi: \underset{\sim}{D} \longrightarrow \underset{\sim}{A}$ represent Φ. Because of Nöbeling's duality theorem [16] relating direct and inverse limits, and because of Milnor's theorem (M^*), for each n, the image of φ^n in $H^n(D^n;\underset{\sim}{B})$ is zero. Thus, each φ^n can be lifted to a map of D^n to C^{n-1}. To show that there is a compatible collection of liftings, we must show that the sequence $\lim^0 H^{n-1}(D^n;\underset{\sim}{C}) \longrightarrow \lim^0 H^n(D^n;\underset{\sim}{A}) \longrightarrow \lim^0 H^n(D^n;\underset{\sim}{B})$ is exact. A little algebra will convince the reader that this will be true if $\lim^1 H^n(D^n;\underset{\sim}{A}) = 0 = \lim^1 H^n(D^n;\underset{\sim}{B})$. This follows from Nöbeling's duality theorem directly.

We next need universal coefficient theorems for

additive homology and cohomology theories with coefficients other than the integers. Our presentation here follows F. Peterson. We recall that a Moore space for a group G was a space $Y(G,n)$ such that $\tilde{H}_r(Y(G,n);Z) = 0$ for $r \neq n$, $\tilde{H}_n(Y(G,n);Z) = G$ (singular homology). These exist and have unique homotopy type for all abelian groups G and all $n \geq 2$. A co-Moore space $Y'(G,n)$ has the property that $\tilde{H}^r(Y'(G,n);Z) = 0$ for $r \neq 0$, and $\tilde{H}^n(Y'(G,n);Z) = G$. If F is a free group, $Y(F,n) = Y'(\mathrm{Hom}(F,Z),n)$ can be taken to be a one point union of n-spheres, one for each generator of F.

Definition 4.8. If K_* is a homology theory, and G is a group, $K_r(X;G) = \tilde{K}_{r+n}(Y(G,n) \wedge (X^+))$. If G is a group such that there exists a space $Y'(G,n)$ for some n, then for a cohomology theory K^*, $K^r(X;G) = \tilde{K}^{r+n}(Y'(G,n) \wedge (X^+))$.

Proposition 4.9. Let K_* be an additive homology theory, and let G be an abelian group. Then, there is a natural short exact sequence:

$$0 \to K_r(X) \otimes G \to K_r(X;G) \to \mathrm{Tor}(K_{r-1}(X),G) \to 0.$$

Let K^* be an additive cohomology theory, and let G be a group such that $\mathrm{Hom}(G,Z) = 0$. Then $Y(G,n) = Y'(\mathrm{Ext}(G,Z),n+1)$. We have a natural exact sequence:

$$0 \to \mathrm{Ext}(G,K^r(X)) \to K^r(X;\mathrm{Ext}(G,Z)) \to \mathrm{Hom}(G,K^{r+1}(X)) \to 0.$$

If $K^*(X)$ is of finite type, there are natural isomorphisms $\mathrm{Ext}(G,K^r(X)) \cong \mathrm{Ext}(G,Z)\otimes K^r(X)$, $\mathrm{Hom}(G,K^{r+1}(X)) \cong \mathrm{Tor}(\mathrm{Ext}(G,Z),K^{r+1}(X)$

Proof. Let F be a free group. By additivity, it is clear that there are natural isomorphisms $K_r(X;F) \cong K_r(X) \otimes F$, and $K^r(X,\mathrm{Hom}(F,Z)) \cong \mathrm{Hom}(F,K^r(X))$. If $K^r(X)$ is finitely generated, there is a natural isomorphism $\mathrm{Hom}(F,K^r(X)) \cong \mathrm{Hom}(F,Z) \otimes K^r(X)$. If G is any group, there exists a free resolution of G of the form $0 \to F'' \to F' \to G \to 0$. To this exact sequence, there corresponds a cofibration sequence $Y(F'',n) \to Y(F',n) \to Y(G,n)$. If $\mathrm{Hom}(G,Z) = 0$, this is also a cofibration sequence $Y'(\mathrm{Ext}(G,Z),n) \to Y'(\mathrm{Hom}(F'',Z),n) \to Y'(\mathrm{Hom}(F',Z),n)$. This gives us exact sequences:

$$\ldots \to K_r(X;F'') \to K_r(X;F') \to K_r(X;G) \to K_{r-1}(X;F'') \to \ldots$$

and, in the case when $\mathrm{Hom}(G,Z) = 0$,

$$\ldots \to K^{r-1}(X,\mathrm{Ext}(G,Z)) \to K^r(X,\mathrm{Hom}(F'',Z)) \to K^r(X,\mathrm{Hom}(F',Z)) \to \ldots$$

The proposition now follows from knowing the corresponding statements for the coefficient groups F', F'', $\mathrm{Hom}(F',Z)$, and $\mathrm{Hom}(F'',Z)$.

We now have prepared all of the machinery which we need for generalizing the universal coefficient theorem. Suppose that K^* is a cohomology theory such that $K^*(\text{point})$ is of finite type. Then $K^*(X)$ is of finite type for all finite CW-complexes. Let Q = rationals. Then $\mathrm{Ext}(Q,Z)$ is a rational vector space, and so is torsion free. The exact sequence $0 \to \mathrm{Hom}(Z,Z) \to \mathrm{Ext}(Q/Z,Z) \to \mathrm{Ext}(Q,Z) \to 0$ shows us that $\mathrm{Ext}(Q/Z,Z)$ is also torsion free. Thus, if X is a finite complex, we see that $K^r(X;\mathrm{Ext}(Q/Z,Z)) = K^r(X) \otimes \mathrm{Ext}(Q/Z,Z)$ and $K^r(X;\mathrm{Ext}(Q,Z)) = K^r(X) \otimes \mathrm{Ext}(Q,Z)$. ~~This gives us a resolution which separates the free and torsion parts of $K^r(X)$~~.

Lemma 4.10. Let K^* be an additive cohomology theory such that $K^*(\text{point})$ is of finite type. Then, if X is a finite CW-complex, the exact coefficient sequence for X corresponding to the exact sequence

$$0 \to Z \to \mathrm{Ext}(Q/Z,Z) \to \mathrm{Ext}(Q,Z) \to 0$$

breaks up into short exact sequences:

$$0 \to K^r(X) \to K^r(X;\mathrm{Ext}(Q/Z,Z)) \to K^r(X;\mathrm{Ext}(Q,Z)) \to 0.$$

Proof. This follows immediately from the discussion above, since $\mathrm{Tor}(K^r(X),\mathrm{Ext}(Q,Z)) = 0$, because $\mathrm{Ext}(Q,Z)$ is torsion free.

Definition 4.11. Let K^* be a cohomology theory such that $K^*(\text{point})$ is of finite type. Define homology theories $h_r(X) = \mathrm{Hom}(K^r(X),Q)$, $k_r(X) = \mathrm{Hom}(K^r(X),Q/Z)$ for X a finite complex, and extend them to all CW-complexes by the usual direct limit construction. Let $\varphi\colon h_r(X) \to k_r(X)$ be the homology operation induced by the natural map $Q \to Q/Z$. Since both $h_*(\text{point})$ and $k_*(\text{point})$ are countable, they are the homology theories associated to representable cohomology theories h^* and k^*, and φ is a representable cohomology operation. We define the dual cohomology theory

DK^* of K^* to be the cohomology theory derived from the cohomology operation φ.

Lemma 4.12. If K^* is as above, then there are, for all finite CW-complexes X, natural isomorphisms:

$$K^r(X;\mathrm{Ext}(Q/Z,Z)) \cong \mathrm{Hom}(k_r(X),Q/Z) \cong \mathrm{Ext}(k_r(X),Z)$$
$$K^r(X;\mathrm{Ext}(Q,Z)) \cong \mathrm{Ext}(h_r(X),Z)$$

Proof. If A, B, C are groups, there is a natural homomorphism $A \otimes \mathrm{Hom}(B,C) \to \mathrm{Hom}(\mathrm{Hom}(A,B),C)$ which is an isomorphism when A is finitely generated and C is injective. Thus, $K^r(X;\mathrm{Ext}(Q/Z,Z)) = K^r(X) \otimes \mathrm{Ext}(Q/Z,Z) =$ $K^r(X) \otimes \mathrm{Hom}(Q/Z,Q/Z) = \mathrm{Hom}(\mathrm{Hom}(K^r(X),Q/Z),Q/Z) = \mathrm{Hom}(k_r(X),Q/Z)$. Since $k_r(X)$ is a torsion group, $\mathrm{Hom}(k_r(X),Q) = 0$, so $\mathrm{Hom}(k_r(X),Q/Z) = \mathrm{Ext}(k_r(X),Z)$. From the exact sequences

$$0 \to \mathrm{Hom}(h_r(X),Q) \to \mathrm{Hom}(h_r(X),Q/Z) \to \mathrm{Ext}(h_r(X),Z) \to 0$$

$$0 \to K^r(X) \otimes \mathrm{Hom}(Q,Q) \to K^r(X)\otimes\mathrm{Hom}(Q,Q/Z) \to K^r(X)\otimes\mathrm{Ext}(Q,Z) \to 0,$$

we see that $\mathrm{Ext}(h_r(X),Z) = K^r(X) \otimes \mathrm{Ext}(Q,Z) = K^r(X;\mathrm{Ext}(Q,Z))$.

From this last lemma, we obtain a generalization of Milnor's theorem on inverse limits for additive theories.

Lemma 4.13. Let K^* be an additive cohomology theory, and let X be a CW-complex, If $K^*(\text{point})$ has finite type, then there is a natural exact sequence:

$$0 \to \lim{}^1_{\mathfrak{U}(X)} K^{r-1}(X^\alpha) \to K^r(X) \to \lim{}^0_{\mathfrak{U}(X)} K^r(X^\alpha) \to 0.$$

Furthermore, $\lim^p_{\mathfrak{U}(X)} K^r(X^\alpha) = 0$ for $p \geq 2$.

Proof. For each finite subcomplex X^α of X, we have an exact sequence

$$0 \to K^r(X^\alpha) \to K^r(X^\alpha; \mathrm{Ext}(Q/Z,Z)) \to K^r(X^\alpha; \mathrm{Ext}(Q,Z)) \to 0.$$

Since $K^r(X^\alpha; \mathrm{Ext}(Q/Z,Z)) = \mathrm{Hom}(k_r(X^\alpha), Q/Z)$, we see that $\lim^p_{\mathfrak{U}(X)} K^r(X^\alpha; \mathrm{Ext}(Q/Z,Z)) = 0$ for $p \neq 0$. From the exact sequences

$$0 \to \mathrm{Hom}(h_r(X^\alpha), Z) \to \mathrm{Hom}(h_r(X^\alpha), Q/Z) \to K^r(X^\alpha; \mathrm{Ext}(Q,Z)) \to 0,$$

we see that $\lim^p_{\mathfrak{U}(X)} K^r(X^\alpha; \mathrm{Ext}(Q,Z)) = 0$ for $p \neq 0$. From the exact sequences above, we obtain an exact sequence

$$0 \to \lim^0_{\mathfrak{U}(X)} K^r(X^\alpha) \to \lim^0_{\mathfrak{U}(X)} K^r(X^\alpha; \mathrm{Ext}(Q/Z,Z)) \to$$
$$\lim^0_{\mathfrak{U}(X)} K^r(X^\alpha; \mathrm{Ext}(Q,Z)) \to \lim^1_{\mathfrak{U}(X)} K^r(X^\alpha) \to 0.$$

Also, we obtain the fact that $\lim^p_{\mathfrak{U}(X)} K^r(X^\alpha) = 0$ for $p \geq 2$. Since $\lim^p_{\mathfrak{U}(X)} K^r(X^\alpha; \mathrm{Ext}(Q/Z,Z)) = 0$ and $\lim^p_{\mathfrak{U}(X)} K^r(X^\alpha; \mathrm{Ext}(Q,Z)) = 0$ for $p \neq 0$, we see that $\lim^0_{\mathfrak{U}(X)} K^r(X^\alpha; \mathrm{Ext}(Q/Z,Z))$ and $\lim^0_{\mathfrak{U}(X)} K^r(X^\alpha; \mathrm{Ext}(Q,Z))$ define additive cohomology theories (since $\lim^0$ commutes with direct product, they are clearly additive - since $\lim^p = 0$ for $p \neq 0$, they satisfy the exactness axiom, which was the only one about which there would be concern). Thus, by proposition 4.1, we see that there are natural isomorphisms $K^r(X; \mathrm{Ext}(Q/Z,Z)) \cong \lim^0_{\mathfrak{U}(X)} K^r(X^\alpha; \mathrm{Ext}(Q/Z,Z))$ and $K^r(X) \cong \lim^0_{\mathfrak{U}(X)} K^r(X; \mathrm{Ext}(Q,Z)))$. These isomorphisms give us exact sequences

$$0 \to \lim{}^0_{\mathfrak{U}(X)} K^r(X^\alpha) \to K^r(X;\mathrm{Ext}(Q/Z,Z)) \to$$
$$K^r(X;\mathrm{Ext}(Q,Z)) \to \lim{}^1_{\mathfrak{U}(X)} K^r(X^\alpha) \to 0.$$

The lemma now follows from the coefficient exact sequence

$$\ldots \to K^r(X) \to K^r(X;\mathrm{Ext}(Q/Z,Z)) \to K^r(X;\mathrm{Ext}(Q,Z)) \to K^{r+1}(X) \to \ldots$$

We are now in a position to state a form of the universal coefficient theorem which related $K^*(X)$ and $DK_*(X)$.

Theorem 4.14. Let K^* be an additive cohomology theory such that $K^*(\text{point})$ has finite type. Then, if X is any CW-complex, there is a natural exact sequence:

$$0 \to \lim{}^1_{\mathfrak{U}(X)} K^{r-1}(X^\alpha) \to \mathrm{Ext}(DK_{r-1}(X),Z) \to$$
$$\lim{}^0_{\mathfrak{U}(X)} K^r(X^\alpha) \to \mathrm{Hom}(DK_r(X),Z) \to 0$$

If $DK_{r-1}(X)$ is torsion free, then the homomorphism in the center is zero, giving us $\lim{}^1_{\mathfrak{U}(X)} K^{r-1}(X^\alpha) = \mathrm{Ext}(DK_{r-1}(X),Z)$ and $\lim{}^0_{\mathfrak{U}(X)} K^r(X^\alpha) = \mathrm{Hom}(DK_r(X),Z)$.

Proof. First, suppose that X is a finite complex. Then $h_r(X)$ is a finitely generated vector space over Q, and $k_r(X)$ is a torsion group. Applying direct limits, we see that for all X, $h_r(X)$ is a vector space over Q, and $k_r(X)$ is a torsion group. If we take the exact sequence $\ldots \to DK_r(X) \to h_r(X) \to k_r(X) \to DK_{r-1}(X) \to \ldots$, and apply the functor $\otimes Q$, we see that there is a natural isomorphism

$DK_r(X) \otimes Q \cong h_r(X)$. Thus, we see that $h_r(X) = DK_r(X;Q)$. Since k_* is the theory associated to the homology operation $DK_r(X) \to h_r(X)$, we see that $k_r(X) = DK_r(X;Q/Z)$, so we have an exact sequence for all r and all X: $0 \to DK_r(X) \otimes Q/Z \to k_r(X) \to k_r(X) \to \mathrm{Tor}(DK_{r-1}(X);Q/Z) \to 0$ (this also follows from applying $\otimes Q/Z$ to the earlier exact sequence). Combining Lemma 4.3 with the result in the proof of lemma 4.4, we obtain exact sequences:

$$0 \to \lim{}^0_{\underset{\sim}{\mathfrak{U}}(X)} K^r(X^\alpha) \to \mathrm{Ext}(k_r(X),Z) \to$$
$$\mathrm{Ext}(h_r(X),Z) \to \lim{}^1_{\underset{\sim}{\mathfrak{U}}(X)} K^r(X^\alpha) \to 0.$$

Now, if we apply $\mathrm{Ext}(\ ,Z)$ to the exact sequence relating $DK_*(X)$, $h_*(X)$, and $k_*(X)$, by elementary homological algebra we obtain exact sequences:

$$0 \to \lim{}^1_{\underset{\sim}{\mathfrak{U}}(X)} K^r(X^\alpha) \to \mathrm{Ext}(DK_r(X),Z) \to \mathrm{Ext}(\mathrm{Tor}(DK_r(X),Q/Z),Z) \to 0$$
$$0 \to \mathrm{Ext}(\mathrm{Tor}(DK_{r-1}(X),Q/Z),Z) \to \lim{}^0_{\underset{\sim}{\mathfrak{U}}(X)} K^r(X^\alpha) \to \mathrm{Hom}(DK_r(X),Z) \to 0.$$

The theorem follows immediately.

We now are almost finished - except that we still have to show that $DKU_* = KU_*$. Again, there is nothing special about KU^*, so we shall look at a more general situation. Suppose that we have three spectra $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ all with homotopy groups of finite type, and suppose that there is a pairing $\underset{\sim}{A} \wedge \underset{\sim}{B} \to \underset{\sim}{C}$, in the sense of Whitehead. Let X

and Y be arbitrary CW-complexes. Then, in the usual way, we have slant product pairings:

$$\tilde{H}_n(X \wedge Y;\underset{\sim}{A}) \otimes \tilde{H}^q(X;\underset{\sim}{B}) \to \tilde{H}_{n-q}(Y;\underset{\sim}{C})$$

(we assume that $\underset{\sim}{B}$ is an Ω-spectrum). In particular, if we take Y to be a Moore space for a group G, we obtain a pairing for all Y:

$$H_n(X;\underset{\sim}{A},G) \otimes H^q(X;\underset{\sim}{B}) \to H_{n-q}(\text{point};\underset{\sim}{C},G)$$

(here, the double coefficients mean the homology theory with coefficients in the first named spectrum given coefficients in the second named group). This gives us a natural homomorphism

$$H_n(X;\underset{\sim}{A},G) \to \operatorname{Hom}(H^q(X;\underset{\sim}{B}),H_{n-q}(\text{point};\underset{\sim}{C},G))$$

If we take the two cases $G = Q$ and $G = Q/Z$, then if $H_r(\text{point};\underset{\sim}{C},G)$ has as a direct summand a copy of G, by projecting into that summand, we obtain homomorphisms

$$H_n(X;\underset{\sim}{A},G) \to \operatorname{Hom}(H^{n+r}(X;\underset{\sim}{B}),G) = DH_{n+r}(X;\underset{\sim}{B},G)$$

We now can apply 4.7. to obtain a suitable cohomology operation

$H_n(X;\underset{\sim}{A}) \to DH_{n+r}(X;\underset{\sim}{B})$.

Theorem 4.15. There is a natural isomorphism for all CW-complexes

$$KU_n(X) \cong DKU_n(X)$$

Proof. We have the usual pairing of the unitary spectrum with itself into itself. Since $KU^0(\text{point}) = Z$, we can take $r = 0$ above. Since the map $KU_n(\text{point}) \to \text{Hom}(KU^n(\text{point}), Z)$ is an isomorphism, by the naturality of the constructions involved, we see that the maps $KU_n(\text{point}; G) \to \text{Hom}(KU^n(\text{point}), G)$ are isomorphisms when $G = Q$ or $G = Q/Z$. Thus, the maps $KU_n(X;G) \to DKU_n(X;G)$ are isomorphisms for all CW-complexes X when $G = Q$ or $G = Q/Z$. Thus they are isomorphisms when $G = Z$.

This gives us a new proof of the first half of theorem 3.1, and gives us an extension of this theorem to arbitrary CW-complexes. This new proof does not require the existence of the resolutions as constructed by Atiyah.

For the real and symplectic K-theories we obtain the following curious result:

Theorem 4.16. There is a natural isomorphism for all CW-complexes X:

$$KO_n(X) \cong DKSp_n(X)$$

Proof. There is a well known pairing between the orthogonal spectrum and the symplectic spectrum into the symplectic spectrum. Since $KSp^0(\text{point}) = Z$, there is a well defined map $KO_n(X;G) \to DKSp_n(X;G)$ for all groups G. By looking at the multiplication in $KO^*(\text{point})$, we see easily that the map $KO_n(\text{point}) \to DKSp_n(\text{point})$ is an isomorphism whenever $n \not\equiv 1, 2 \bmod(8)$, so the map $KO_n(X;Q) \to DKSp_n(X;Q)$ is an isomorphism for all n and all X. The symplectification of complex bundles gives us a natural transformation of $KO^*(X)$-modules $KU^*(X) \to KSp^*(X)$. Since the decomplexification $KU^*(X) \to KO^*(X)$ is also a map of $KO^*(X)$-modules, we see that we have a commutative diagram:

$$\begin{array}{ccc} KU_n(X) & \longrightarrow & DKU_n(X) \\ \downarrow & & \downarrow \\ KO_n(X) & \longrightarrow & DKSp_n(X). \end{array}$$

Taking X to be a point, we see that when $n = 2$, we obtain an isomorphism $KO_2(\text{point}) \to DKSp_2(\text{point})$, since the two maps $KU^{-2}(\text{point}) \to KO^{-2}(\text{point})$ and $KU^2(\text{point}) \to KSp^2(\text{point})$ are both surjections of Z onto Z_2. Thus, we are left only with the case when $n \equiv 1 \bmod(8)$.

Let $t \in KO^{-1}(\text{point})$ be the non-zero element. Then t^2 is the non-zero element of $KO^{-2}(\text{point})$. Define $\varphi\colon KO^*(X) \to KO^*(X)$ and $\varphi\colon KSp^*(X) \to KSp^*(X)$ by $\varphi(x) = xt$. Then φ is clearly a map of $KO^*(X)$-modules in both cases. It is well known that $\varphi\colon KO^{-1}(\text{point}) \to KO^{-2}(\text{point})$ and $\varphi\colon KSp^3(\text{point}) \to KSp^2(\text{point})$ are isomorphisms. It is then clear that the induced maps $\varphi\colon KO_2(\text{point}) \to KO_1(\text{point})$ and $\varphi\colon DKSp_2(\text{point}) \to DKSp_1(\text{point})$ are both isomorphisms. Since φ is a map of KO^*-modules, these isomorphisms commute with the maps $KO_n(\text{point}) \to DKSp_n(\text{point})$, so we see that we obtain isomorphisms $KO_*(\text{point}) \cong DKSp_*(\text{point})$. Thus the result follows for all spaces.

Since the usual forms of K-theory are one another's dual theories, we might expect that the dual of a familiar theory would be a familiar theory. Certainly, ordinary cohomology is self-dual, just by looking at the universal coefficient theorem for a point. However, the dual theories of the various cobordism theories seem rather mysterious - one need only look at the groups of a point to be convinced of this. Perhaps this duality could be used to study questions about cohomology theories in general.

We finish with a comment as to why D^2 is the identity. When K^* is a cohomology theory such that $K^*(\text{point})$ is of finite type, and G is a countable group, one can define $K^*(X;G)$ by taking the dual theory to $K_*(X;G)$ in the sense of Whitehead. This can be done even if there are no co-Moore spaces $Y'(G,n)$. We would then obtain natural isomorphisms for all finite CW-complexes X (using Whitehead's generalized Alexander duality) $DK^r(X;Q) \cong \mathrm{Hom}(K_r(X),Q)$. Clearly, $DK^r(X;Q/Z)$ is a torsion group, so $D^2K_r(X;Q) \cong \mathrm{Hom}(\mathrm{Hom}(K_r(X),Q),Q)$. However, this is clearly isomorphic to $K_r(X;Q)$. Likewise, if n is an integer, it is not hard to see from the universal coefficient theorem that there is a natural isomorphism $DK^r(X;Z_n) \cong \mathrm{Ext}(K_r(X;Z_n),Z)$. Similarly, there is a natural isomorphism $D^2K_r(X;Z_n) \cong \mathrm{Ext}(DK^r(X;Z_n),Z)$. This gives us a natural isomorphism $D^2K_r(X;Z_n) \cong \mathrm{Ext}(\mathrm{Ext}(K_r(X;Z_n),Z),Z)$. Since $K_r(X;Z_n)$ is a finite group, we obtain $D^2K_r(X;Z_n) \cong K_r(X;Z_n)$ for all n. It is easy to see that these isomorphisms are natural with respect to the coefficient groups Z_n. Since $K_r(X;Q/Z) = \lim_0 K_r(X;Z_{n!})$, we obtain isomorphism $D^2K_r(X;Q/Z) \cong K_r(X;Q/Z)$ for all finite CW-complexes X. We leave to the reader the task of checking that these isomorphisms are compatible with the coefficient homomorphisms associated to the map $Q \to Q/Z$. Thus, by proposition 4.7 we obtain isomorphisms $D^2K_r(X) = K_r(X)$.

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