(I)

# Motivation and stratified spaces

## A. Motivation 1. Mflds.

Let $M^n$ be a closed connected oriented mfld

Then $H_n(M) \cong \mathbb{Z} = \langle \Gamma \rangle$

Poincare Duality: $H^i(M) \xrightarrow{\cap \Gamma} H_{n-i}(M)$ is an isomorphism.

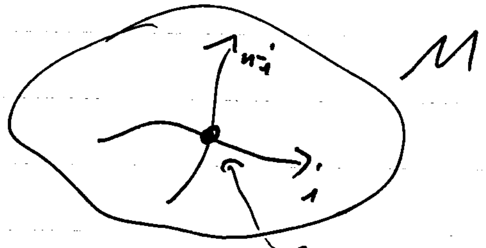
W/ field coefficients, this says

$H_{n-i}(M;F) \cong H^i(M;F) \cong \mathrm{Hom}_F(H_i(M;F),F)$

~~Geometric interpretation:~~ So there is a nonsingular pairing $H_i(M;F) \otimes H_{n-i}(M;F) \xrightarrow{\pitchfork} F$

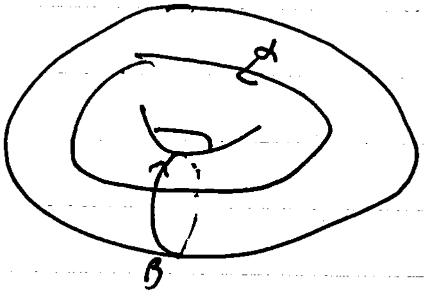
In cohomology this is the cup product pairing

In homology nice geometric interpretation as intersection product!



count w/ sign

Ex. Torus



$H_1(T^2) = \mathbb{Z}^2 = \langle \alpha, \beta \rangle$

$\alpha \pitchfork \alpha = 0 = \beta \pitchfork \beta$

$\alpha \pitchfork \beta = 1 = -\beta \pitchfork \alpha =$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Also $H_0(T) = \mathbb{Z} = \langle v \rangle$

$H_2(T) = \mathbb{Z} = \Gamma$

$v \pitchfork \Gamma = 1$

A resulting invariant for $M^{4k}$

$$H_{2k}(M;\mathbb{Q}) \otimes H_{2k}(M;\mathbb{Q}) \to \mathbb{Q}$$ is a symmetric pairing

$\downarrow \sigma$

signature $\sigma(M) \in \mathbb{Z}$ = # of eigenvalues $>0$

− # of eigenvalues $<0$

~~this leads to L classes ...~~

Fun fact: If $\partial W^{4k+1} = M_1 \amalg -M_2$, then $\sigma(M_1) = \sigma(M_2)$

this leads to L classes ...

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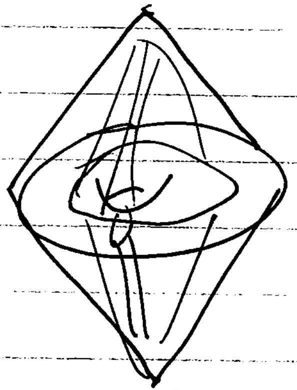
2. Consider suspended torus $X = ST^2$



$H_3(X) = \mathbb{Z}$

$H_2(X) = \mathbb{Z} \oplus \mathbb{Z}$ not dual

$H_1(X) = 0$

$H_0(X) = 0$

Two bad points: no P.D.

Goal of intersection homology: redefine $H_*$ to fix this problem.

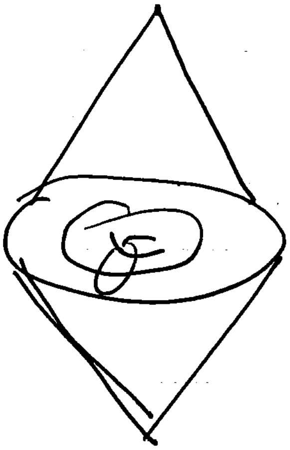
(Goresky-MacPherson)

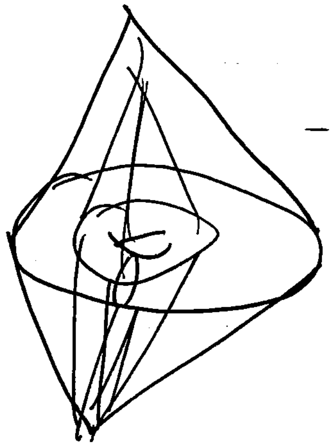
Intersection homology groups $I^pH_*(X)$

perversity parameter - controls how chains intersect the singularities

Ex, Pick two perversities $p, q$

$p$ - can't touch singularities $q$ - sing. ok.





dual!

$I^pH_2(X) = 0$ $I^qH_2(X) = \mathbb{Z}^2$

$I^pH_1(X) = \mathbb{Z}^2$ $I^qH_1(X) = 0$

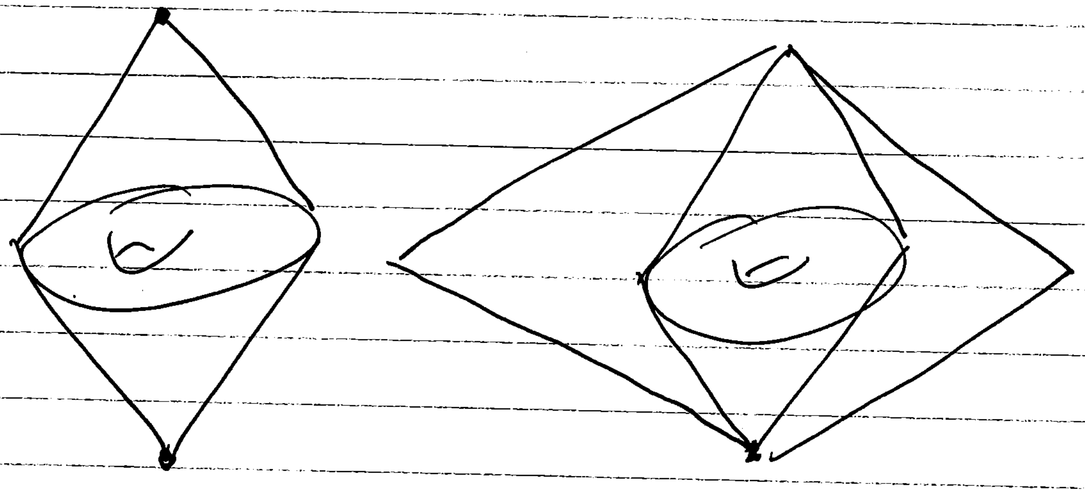
Theorem: To every perversity $p$, there is a dual perversity $Dp$ so that w/ field coeff: $I^pH_i(X;F) \otimes I^{Dp}H_{?-i}(X) \xrightarrow{\cap} F$ is nonsing if $X$ is a "pseudo manifold"

thm (Siegel): There are some special psmanifolds called "Witt spaces" such that there is a perversity $\underline{m}$ w/ $I^mH_*(X) \cong I^{Dm}H_*(X)$.

For these, if $\dim(X) = 4k$ we get a $\sigma$, ~~L classes,~~ bordism invariance, L classes, ...

## B. Pseudomanifolds

Ex.



discuss strata

~~Pseudomanifold (inductive def):~~

~~A 0 dim pm is a discrete set of points~~

~~An n-dim pm is a space X w/ a filtration $X^0 \subset X^1 \subset \dots \subset X^n$ s.t.~~

Filtered space: $X^0 \subset X^1 \subset \dots \subset X^n$

$X^i$ closed subsets

$X^i$ - skeleton

$X_i = X^i - X^{i-1}$ - connected components are called strata

Can have $X^i = X^{i+1}$

Ex. Simplicial complex w/ simp. skel.

CW complexes w/ CW skeleta

Submanifold $\subset$ Manifold $N^n \subset M^m$

Stratified Pseudomanifold: inductive def.

A 0-dim pm is a discrete set of pts.

An n-dim pm is a filtered space $X^0 \subset \dots \subset X^n$ such that:

1. Each ~~pt~~ stratum in $X^i - X^{i-1}$ is an $i$-mfld.
2. $X^n - X^{n-1}$ is dense in $X$
3. If $x \in X_i$, then $x$ has a distinguished nghd $\cong \mathbb{R}^i \times cL$ where $L$ is an compact $n-i-1$ dim pm and $(\mathbb{R}^i \times cL)^j \cong \mathbb{R}^i \times cL^{j-i-1}$,

(some require $X^{n-2} = X^{n-1}$ - we don't)

$X^{n-1} = \Sigma_X$

Ex.
1. $ST^2$
2. $SST^2$
3. Mfld w/ trivial filt.
4. Smooth submanifold
5. Cone on mfld.
6. If $X$ is a pm: $cX$ and $\mathbb{R}^i \times cX$ and any $M \times X$.
7. irred. complex variety
8. orbit space of compact Lie group acting smoothly on a mfld.

If time: simplicial pms:

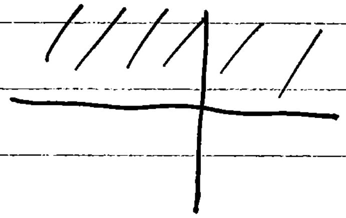
If we take a pile of n simplices and glue so that each n-1 face is a face of exactly two n-simplices, we get a pm.

Pseudomanifold: a space that can be stratified as a strat. pm

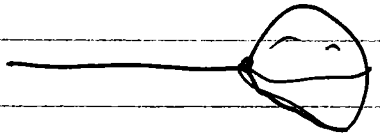
More
Extra time stuff:

1. Frontie condition: if $S$ and $T$ are strata and $S \wedge \overline{T} \neq \emptyset$ then $S \subset \overline{T}$

Bad example:



2. CS sets: like pms but w/out density condition and links can be any compact filtered space



3. PL pseudomanifolds

4. Links are not unique - Double Susp. thm. but in PL links are unique

# II. Intersection Homology

## A. Perversities

$X^n$ - strat pm.

Call n dim strata "regular"

other strata "singular"

Def. a perversity is a function

$$\bar{p} : \{ \text{singular strata of } X \} \to \mathbb{Z}$$

Most important examples: define codim

$\bar{0}$ · zero perv. $\bar{0}(S) = 0$

$\bar{t}$ - top perv. : $\bar{t}(S) = \mathrm{codim}(S) - 2$

$\bar{m}$ - lower middle $\bar{m}(S) = \left\lfloor \frac{\mathrm{codim}(S)-2}{2} \right\rfloor$

$\bar{n}$ - upper middle $\bar{n}(S) = \left\lceil \frac{\mathrm{codim}(S)-2}{2} \right\rceil$

If $p$ is a perv, $Dp = t - p$

i.e. $D\bar{p}(S) = \bar{t}(S) - p(S)$

Note: $D(Dp) = \bar{p}$

$D\bar{0} = \bar{t}$

$D\, m = n$ ← closest dual perv,

also $m = n$ if all strata have even codim

Common assumption: $p$ depends only on codim

$p : \mathbb{Z}_{\geq 1 \text{ or } 2} \to \mathbb{Z}$

$t$ : $[-1, 0, 1, 2, \ldots$

$m$ : $[-1, 0, 0, 1, 1, \ldots$

$n$ $[0, 0, 1, 1, 2, \ldots$

## B. Intersection Homology (Simplicial / PL)

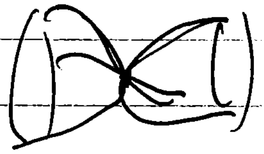
Motivation: Suppose $M^m$ is a smooth mfld and $N^n \subset M$ is a smooth submfld.

Let $x$ be an $i$ chain. By general position, we can perturb $x$ to get

$\dim(x \cap N) = i + n - m = i - \operatorname{codim}(N)$

If $x$ is a cycle can perturb w/in homology class.

Not possible w/ sing.



~~Let $C_*(X)$ be~~

So let $X$ be a strat pm and $C_*(X)$ the simplicial chain complex.

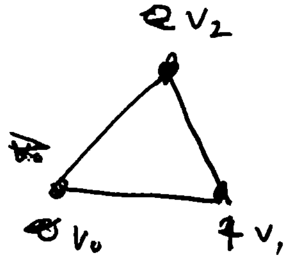
We say $x \in C_i(X)$ is p-allowable if $\forall$ singular strata $S$,

$\dim(x \cap S) \le i - \operatorname{codim}(S) + p(S)$ ~~and~~

~~$\dim(\partial x \cap S) \le i-1-\operatorname{codim}(S)+p(S)$~~

Let $I^pC_*(X) \subset C_*(X)$ be the complex of p-allowable chains whose boundaries are also p-allowable. Also called intersection chains

$I^pH_*(X) = H_*(I^pC_*(X))$

Ex. 

Filt. $v_0 \subset X^1 = X$

Only sing stratum $= v_0$
$p(v_0) \in \mathbb{Z}$,

So as simplexes, $v_1, v_2$ are always allowable (not near sing strat).

$v_0$ is allowable iff

$$\dim(v_0 \wedge v_0) \leq 0 - \operatorname{codim}(v_0) + p(v_0)$$
$$\overset{\shortparallel}{0} \qquad = 0 - 1 + p(v_0)$$

i.e. if $p(v_0) \geq 1$

The edge $[v_1, v_2]$ is allowable.
But $e = [v_0, v_1]$ or $[v_0, v_2]$ is allowable iff
$\underset{\overset{\shortparallel}{0}}{\dim(e \wedge v_0)} \leq 1 - 1 + p(v_0) \iff p(v_0) \geq 0.$

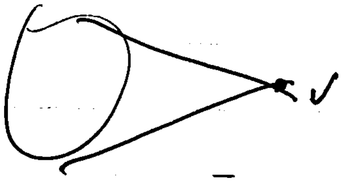
Cases: 1. $p(v_0) < 0$ : $I^pC_*(X) = C_*($  $)$ $\quad I^pH_0 = \mathbb{Z}$, $I^pH_1 = 0$

2. $p(v_0) \geq 1$ : $I^pC_*(X) = C_*(X)$ $\quad I_pH_0 = I_pH_1 = \mathbb{Z}$

3. $p(v_0) = 0$. All edges are allowable but $v_0$ is not.
Still get $I^pH_0(X) = \mathbb{Z}$, gen. by $v_1$ or $v_2$
$I^pH_1(X) = \mathbb{Z}$ because $\sum e_i$ is allowable even though $\partial[v_0, v_1]$ isn't.

Ex. $M^{n-1}$ - mfld (connected, compact)

$X = cM$ (open cone on any pm)



only sing strat $= v$

Let $p(v) = p$

An $i$ chain in $cM - v$ is allowable

If $x \in C_i(X)$ and $x \cap v \neq \emptyset$ then $\dim(x \cap v) = 0$

So $x$ is allowable if

$0 \leq i - n + p \iff i \geq n - p$

So $x \in I^pC_i(X)$ if:

if $i \geq n-p+1$ all

$i = n-p$ $\partial x \cap v = \emptyset$

$i < n-p$ $x \cap v = \emptyset$

If $i \geq n-p-1$ and $x$ is a cycle, then $cx$ is allowable so $IH_i(cX) = 0$

If $i < n-p-1$, then $cx$ is not allowable and $x \cap v = \emptyset$.

So $IH_i(cM) = H_i(cM - v) \cong H_i(M)$

(Special case: $IH_0(X) = \mathbb{Z}$ no matter what!)

even if $M$ not connected

thm. $I^pH_i(cM) = 0$ $i \geq n-p-1$ $(i \neq 0)$

$= H_i(M)$ $i < n-p-1$

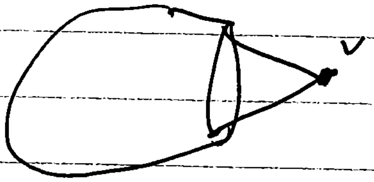
$= \mathbb{Z}$ $i = 0$ and $0 \geq n-p-1$

filtered by $(cX)^i = c(X^{i-1})$
$(cX)^0 = v$

Thm. If $X$ is a compact strat pM, then

$$I^pH_i(cX) = \begin{cases} 0 & i \geq n-p^{(v)}-1 \quad i \neq 0 \\ I^pH_i(X) & i < n-p^{(v)}-1 \\ \mathbb{Z} & i = 0 \geq n-p-1 \end{cases}$$

Ex. $M^n$ a $\partial$-mfld w/ $\partial M \neq \emptyset$.
Let $X = M \cup c\partial M \hat{\cong} M/\partial M$



sing strat $v$

Similar to prev. comp.! $x \in IC_i(X)$ iff:

~~x(i) all~~

$i \geq n-p+1$ all
$i = n-p$ $\partial x \cap v = \emptyset$
$i < n-p$ $x \cap v = \emptyset$

$\exists$

$$I^pH_i(X) = \begin{cases} H_i(X) \cong H_i(M, \partial M) & i \geq n-p \\ \operatorname{im}(H_i(M) \to H_i(M, \partial M)) & i = n-p-1 \\ H_i(X-v) \cong H_i(M) & i < n-p-1 \end{cases}$$

Observation $D_p(v) = t(v) - p(v) = n-2-p(v)$
So $n-p-1 = D_p(v)+1 = n-(n-Dp-1)$
So $n-p-1$ and $n-Dp-1$ are dual dimensions

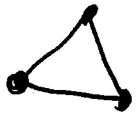
So
$$I^{Dp}H_i(X) = \begin{cases} H_i(M, \partial M) & i \geq n-Dp \\ \operatorname{im}(H_i(M) \to H_i(M, \partial M)) & i = n-Dp-1 \\ H_i(M) & i < n-Dp-1 \end{cases}$$

IH duality = Lefschetz duality!

## III Properties of IH

1. Simplicial IH not indep. of subdivision!

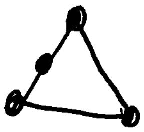
Ex.



$X^0$ = all vertices $\quad \bar{p}(v_i) = -1$

$I^{\bar p}C_0(X) = 0$ !

vs



same $X^0$ $\quad I^{\bar p}C_0(X) = \mathbb{Z}.$

But indep. if triang is fine enough. If $K$ is a triang. and $K'$ is bary sub. and $K''$ any further subdivision, then $I^{\bar p}H_*(K') \cong I^{\bar p}H_*(K'')$.

triang

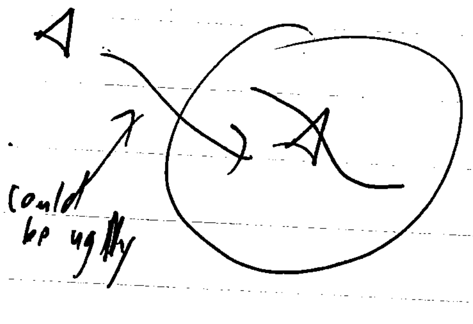
Nice idea: Let $I^{\bar p}C_*^{PL}(X) = \varinjlim_{T} I^{\bar p}C_*^{T}(X)$

Then $I^{\bar p}H_*^{PL}(X) = H_*(\lim I^{\bar p}C_*^T(X)) = \varinjlim H_*(I^{\bar p}C_*^T(X))$

Idea of PL homology - can ~~we~~ pick chains from any triangulation you want w/out having to talk about specific subdivision. More like singular homology

2. Singular IH

Let $S_*(X)$ = singular chains.



If $\sigma \in S_i(X)$, say $\sigma$ is $p$-allowable if $\forall$ sing. strata, $\sigma^{-1}(S) \subset i - \mathrm{codim}(S) + p(S)$ skeleton of $\Delta^i$

$x \in I^p S_*(X)$ if every simplex of $x$ and $\partial x$ is allowable.

~~get~~

Thm. On PL psmflds, singular $IH \approx PL\ IH$.

3. Maps - sort of unpleasant

~~Call $f: X \to Y$~~ ~~w/ perv~~

Suppose $X, Y$ psmflds w/ perv $p, q$.

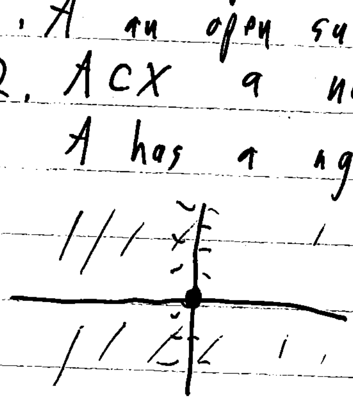
Call $f: X \to Y$ stratified if for each stratum $S \subset X$ $\exists$ stratum $T \subset Y$ s.t. $f(S) \subset T$.

$f$ is $(p,q)$ strat if
$$p(S) - \mathrm{codim}_X(S) \leq q(T) - \mathrm{codim}_Y(T).$$

Then get $f: I^pH_*(X) \to I^qH_*(Y)$.

Most important examples: ~~$A \subset X$ w/ $A' = A \cap X'$~~

1. $A$ an open subset of $X$ w/ $q = p|A$
2. $A \subset X$ a normally nonsingular subset: $A$ has a nghd $\cong A \times \mathbb{R}^i$ $\quad q = p|A$



note codims agree

Can also have more general inclusions if we don't care about spaces being psmflds.

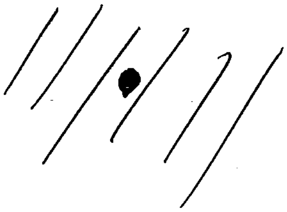
4. Long exact sequences.

w/ $A \subset X$ as above:

$$\cdots \to I^{p}H_i(A) \to I^{p}H_i(X) \to I^{p}H_i(X,A) \to I^{p}H_{i-1}(A) \to \cdots$$

5. $IH$ is not a homotopy invariant!

Ex.



$v \in \mathbb{R}^2$

$p(v) = -1$

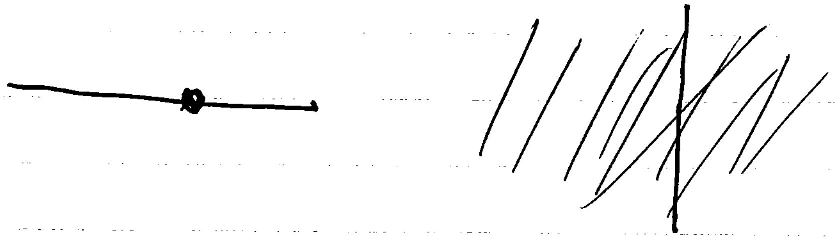
$I^{p}H_1(\mathbb{R}^2) = \mathbb{Z}$

But $IH$ <u>is</u> a stratified homotopy invariant.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

each stratum of $f, g$ mapped to one stratum of $g$.
homotopies preserve strata.

Ex.



6. Excision: if $B \subset A \subset X$ w/ $\bar{B} \subset A^{\circ}$ then $I^{p}H_*(X,A) \cong I^{p}H_*(X-B, A-B)$

- Not easy because of new $\partial$s

7. Mayer-Vietoris: if $A, B$ open w/ $A \cup B = X$

$$\to I^{p}H_i(A \cap B) \to I^{p}H_i(A) \oplus I^{p}H_i(B) \to I^{p}H_i(X) \to \cdots$$

8. Künneth w/ mfld factor

if $M^m$ is a fld, $X^n$ $p^m$

and $(M\times X)^i = M^m \times X^{i-n}$

~~a~~ and $p(M\times S) = p(S)$

Then $I^pH_*(M\times X) \cong H_*(S_*(M) \otimes I^pS_*(X))$

$\cong \bigoplus_{j+k=i} H_j(M) \otimes IH_k(X) \oplus$ torsion

Note: gen. Künneth can't be this simple.

9. Universal coefficients fails!.

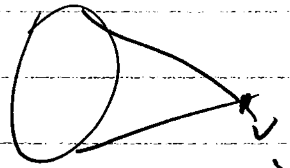
Can define $I^pS_*(X;G)$ analogously as for $I^pS_*(X)$. Not same as $I^pS_*(X)\otimes G$!

Recall for ordinary homology have SES

$$0 \to H_i(X)\otimes G \to H_i(X;G) \to \overset{\text{torsion}}{H_{i-1}(X) * G} \to 0$$

If $\partial x = 2y$ and $y$ is not allowable, this makes a difference!

Consider $X = c\mathbb{R}P^2$ w/ $p(v)=0$



$$I^pH_i(X) = \begin{cases} 0, & i \geq 2 \\ H_i(\mathbb{R}P^2) & i<2 \end{cases} = \begin{cases} 0 & i\geq 2 \\ \mathbb{Z}_2 & i=1 \\ \mathbb{Z} & i=0 \end{cases}$$

by cone formula

Similarly

$$IH_i(c\mathbb{R}P^2;\mathbb{Z}_2) = \begin{matrix} 0 & i\geq 2 \\ H_i(\mathbb{R}P^2;\mathbb{Z}) & i<2 \end{matrix} = \begin{cases} 0 & i\geq 2 \\ \mathbb{Z}_2 & i=1 \\ \mathbb{Z}_2 & i=0 \end{cases}$$

But if UCT holds, $IH_2(c\mathbb{R}P^2;\mathbb{Z}_2) = IH_1(c\mathbb{R}P^2) * \mathbb{Z}_2 = \mathbb{Z}_2$

Problem: truncation messes w/ part of UCT that looks in wrong dim!!

partial
Solution: $X$ is $p$-locally torsion free if for every $x \in X$ w/ link $L_x$ in strat $S$
$I^pH_{\dim(L)-p(S)-1}(L)$ is torsion free.

Avoids problem and cone formula works out correctly

Thm: If $F_*$, $G_*$ are two functors $\{\text{open subsets of } X\} \to \{\text{graded groups}\}$ ~~+ Mayer-Vietoris sequences~~ and $\Phi: F_* \to G_*$ a nat. trans.

1. $\exists$ MV sequences for $F_*, G_*$ and commute w/ $\Phi$
2. $\Phi$ is an iso on open subsets of reg strata
3. if $\Phi: F_*(L) \to G_*(L)$ is iso then so is $\Phi: F_*(\mathbb{R}^i \times cL) \to G_*(\mathbb{R}^i \times cL)$
4. Condition on increasing collections of sets (omit)

then $F_* \cong G_*$

Moral if two "homologies" behave the same on cones, then $F = G$. → foreshadows sheaves.

Ex. if $X$ is locally torsion free,
$F_* := H_*(I^pS_*(X) \otimes G)$
$G_* = I^pH(X;G)$
Then $F_* = G_*$

## 10. Other applications of MV theorem

a. Kunneth thm w/ mfld factor
b. PL = sing IH
c. topological invariance

Thm (GM). If $p$ depends only on codim, $p(1) \geq 0$ and $p(k-1) \leq p(k) \leq p(k-1)+1$, then $I^pH_*(X)$ is indep. of choice of strat, as a pm.

holds for $\bar{0}, \bar{t}, \bar{m}, \bar{n}$ if no codim strat

Ex. If $p$ is any such perv, and $M$ is a manifold w/ **any** strat., then
$I^pH_*(M) \cong H_*(M)$
~~because $I^pH_*(M) \cong I^{\bar{t}}H_*(M) \cong H_*(M)$~~
- use trivial filt, then all chains allowed.

~~(mention knot thry?)~~
If extra time talk about local coeffs and knot thry.

# IV. Goal: Duality

GM Duality $I^{p}H_i(X;\mathbb{Q}) \otimes I^{q}H_{n-i}(X;\mathbb{Q}) \to \mathbb{Q}$
nonsingular pairing by intersection prod.
Proof is combinatorial.

Is there a more modern cup/cap product approach.

Problem: can't use front face / back face construction because won't preserve allowability.

Another classical construction of products
use $\alpha \cup \beta = d^*(\alpha \times \beta)$

$$\begin{tikzcd} S_*(X) \arrow[r, "diag"] \arrow[rr, bend right, "\bar{d}"'] & S_*(X \times X) & S_*(X) \otimes S_*(X) \arrow[l, "\sim_{h.e.}"'] \end{tikzcd}$$

Then if $\alpha, \beta \in S^*(X)$ we have
$(\alpha \cup \beta)(z) = (\alpha \otimes \beta)(\bar{d}(z))$
$\Leftrightarrow \alpha \cup \beta = d^*(\alpha \times \beta)$

For cap product, $\alpha \cap z = (\alpha \otimes \mathrm{id})(\bar{d}(z))$

So for IH, need h.e. $I^{p}S_*(X) \otimes I^{q}S_*(Y)$
$\to I^{Q}S_*(X \times Y)$

For what $Q$?

Can apply MV theorem to local models that look like $(\mathbb{R}^i \times cK) \times (\mathbb{R}^j \times cL) \cong \mathbb{R}^{i+j} \times c(K * L)$

Tricky because cones in different ways - need cone formulas to work out, including torsion Kunneth terms.

Turns out to work if

$Q(S\times T) = p(S)$ if $T$ reg.

$q(T)$ if $S$ reg

$p(S)+q(T)$ or $p(S)+q(T)+1$ if $S, T$ sing

or $p(S)+q(T)+2$ if also $X$ or $Y$ locally torsion free

Then get homology Kunneth thm ⇒ c.h.e.

When is $d: I^r H_*(X) \to I^Q H_*(X\times X)$ defined:

Need $r(S) \leq Q(S\times S) - \text{codim}(S)$

$= p(S) + q(T) - \text{codim}(S)$ (or $+1$ or $+2$)

If $\bar{p}+\bar{q}=\bar{t}$ $(q = Dp)$ then this becomes

$r(S) \leq -2$ (or $-1$ or $0$)

For duality want $\cap\Gamma$. Where is $\Gamma$?

Fact: if $X-\Sigma$ is oriented and $X^n$ is a compact pm, there is an orientation class in $I^0H_n(X)$. If $X$ is connected this is a generator.

(why! Need $\underset{\dim(S)''}{\dim(\Gamma\cap S)} \leq \underbrace{n - \mathrm{codim}(S)}_{\dim(S)} + r(S)$

So $\Gamma$ exist for $\bar{r} \geq 0$).

So just barely works taking
$Q(S \times S) = p(S) + q(S) + \underline{\underline{2}}$ need torsion free!!

This is why original GM proof needed $\mathbb{Q}$ coeffs.

Summary: Let $I_p S^*(X) = \mathrm{Hom}(I^p S_*(X), \mathbb{Z})$
Suppose $X$ is locally $p$-torsion free.
Then we have,
$$\Gamma \in I^0 S_n(X) \xrightarrow{d} I^{\mathbb{Q}} S_n(X \times X) \xleftarrow{\cong \text{che}} I^p S_*(X) \otimes I^{Dp} S_*(X)$$
And ~~$\alpha \cap \Gamma$ is~~ for $\alpha \in I_p S^i(X)$,
$\alpha \cap \Gamma$ is defind by $(\alpha \otimes \mathrm{id})(d(\Gamma))$
$\in I^{Dp} S_{n-i}(X)$

Thm. This induces an iso on homology:
$$I_p H^i(X) \xrightarrow[\cap \Gamma]{\cong} I^{Dp} H_{n-i}(X)$$ ($X$ locally $p$-torsion free)

Proof: MV argument + other stuff

If we use field coeffs $X$ can be any compact oriented pm.

Lefschetz duality ⇒

$$I^{q}_{p}H^{i}(X,\partial X) \xrightarrow{\cong} I^{Dp}H_{n-i}(X)$$

$$I_{p}H^{i}(X) \xrightarrow{\cong} I^{Dp}H_{n-i}(X,\partial X).$$

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Also get cup pairings (nonsing),

$$I^{p}H^{i}(X) \otimes I^{Dp}H^{n-i}(X) \longrightarrow I^{\bar 0}_{0}H^{n}(X)$$

$$\cong I^{\bar t}H_{0}(X) \xrightarrow{aug} \mathbb{Z}$$

(dual to all int pairing)

And linking pairings on torsion.

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For signatures, want for $X^{4k}$ that $I_{\bar p}H^{2k}(X;\mathbb{Q}) \cong I_{Dp}H^{2k}(X;\mathbb{Q})$

From definitions,

$$p(S) + Dp(S) = \operatorname{codim}(S) - 2$$

So $p = Dp \Rightarrow p(S) = \frac{\operatorname{codim}(S)-2}{2}$

problem unless all strata are odd.

(This is enough for complex varieties!!)

Otherwise, consider

$$\bar{m} = \left\lfloor \frac{\operatorname{codim}(S)-2}{2} \right\rfloor$$

$$D\bar{m} = \bar{n} = \left\lceil \frac{\operatorname{codim}(S)-2}{2} \right\rceil$$

There is a map $I^{\bar m}H_*(X;\mathbb{Q}) \to I^{n}H_*(X;\mathbb{Q})$.
When is it an iso?

all $\mathbb{Q}$ coeft now ↙

Look at cones!

$$I^{m}H_i(cL^{k-1}) = \begin{cases} 0 & i \geq k - m(v) - 1 \\ I^{m}H_i(L) & i < k - m(v) - 1 \end{cases}$$

$$I^{n}H_i(cL^{k-1}) = \begin{cases} 0 & i \geq k - n(v) - 1 \\ I^{n}H_i(L) & i < k - n(v) - 1 \end{cases}$$

~~Odd codim stratum ⇒ link dim even.~~

Suppose $I^{m}H_*(L) \cong I^{n}H_*(L)$.
Then these are different only when $k$ is odd
then $m(v) = \left\lfloor \frac{k-2}{2} \right\rfloor = \frac{k-3}{2}$ $\quad n(v) = \left\lceil \frac{k-2}{2} \right\rceil = \frac{k-1}{2}$

So suppose $X$ has property that
for all links, $I^{\bar n}H_{\frac{k-1}{2}}(L^{k-1}) \cong I^{m}H_{\frac{k-1}{2}}(L^{k-1}) = 0$

Since links of links are links, can show by induction and MV argument
$I^{m}H(X) \cong I^{n}H_*(X)$

Witt spaces

Segel:

Witt spaces ⇒ witt signatures

$\mathrm{sig}\left( I_mH^{2k}(X^{4k};\mathbb{Q}) \otimes I_mH^{2k}(X^{4k};\mathbb{Q}) \to \mathbb{Q} \right)$

Examples

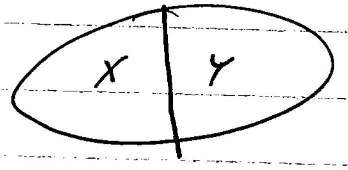
1. $M$ a mfld
2. $M^{4k}$ w/ $\partial$ $\Rightarrow$ $X = M/\partial M = M \cup_{\partial M} c\partial M$
3. Non-example - nothing w/ link a $CP^n$.

Properties

1. $\sigma(X \sqcup Y) = \sigma(X) + \sigma(Y)$
2. $\sigma(-X) = -\sigma(X)$
3. $\sigma(X \times Y) = \sigma(X) \times \sigma(Y)$
4. $\sigma(\partial X) = 0 \Rightarrow$ if $\partial W = X \sqcup -Y$
   Then $\sigma(X) = \sigma(Y)$

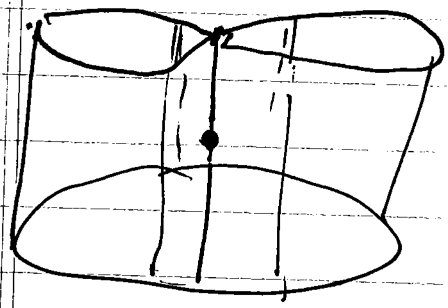
Nice proof of Novikov additivity:

Suppose $Z = X \cup_{\partial X = \partial Y} Y$



Then $\sigma(Z) = \sigma(X) + \sigma(Y)$

pinch bordism



Let $W(\mathbb{Q})$ = Witt group of $\mathbb{Q}$

$$= \frac{\langle \text{symm bilinear forms} \rangle}{\langle \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \rangle}$$

$$\underbrace{W(\mathbb{Q}) \cong \mathbb{Z}}_{\text{signature}} \oplus \text{torsion}$$

Given $X^{4h}$ Witt, it gives an element of $W(\mathbb{Q})$ by its cap pairing.

$$\underset{\text{bordism group}}{\Omega^{witt}_{4h}} \xrightarrow{\cong} W(\mathbb{Q}) \qquad h>0$$

$$\Omega^{witt}_0 \xrightarrow{\cong} \mathbb{Z}$$

$$\Omega^{witt}_i = 0 \qquad i \not\equiv 0 \bmod 4$$

Can define a generalize bordism thry $\Omega^{witt}_*(\cdot)$ by

$$\Omega^{witt}_j(Z) = [\text{maps } X^{witt} \to Z] \Big/ \begin{smallmatrix} W \to Z \\ X \nearrow \end{smallmatrix}$$

Siegel - Sullivan

$$\Omega^{witt}_*(Z) \otimes \mathbb{Z}[\tfrac{1}{2}] \cong KO_*(Z) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

IP-spaces - Witt + torsion free

For these we get integer isos $I^{m}H_*(X) \cong I^{n}H_*(X)$ and duality $I_{\bar{m}}H^{i}(X) \cong I^{\bar{m}}H_{n-i}(X)$.

Bordism groups

$$\Omega_*^{IP} = \begin{cases} \mathbb{Z} & n=4k \\ \mathbb{Z}_2 & n=4k+1, \quad k>0 \\ 0 & \text{o.w.} \end{cases}$$

As a homology theory very close to $\mathbb{L}$-homology

More ...?

# V. Sheaves

$S$ on $X$

A sheaf is a contravariant functor
{open sets of $X$} $\to$ groups
inclusion $U \hookrightarrow V \Rightarrow$ restriction $S(V) \to S(U)$
such that: 1. If $U = \cup U_i$ and $s, t \in S(U)$
w/ $s|_{U_i} = t|_{U_i}$ for all $i \Rightarrow s = t$
2. If $U = \cup U_i$ and $s_i \in S(U_i)$
and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall\, i, j$, $\exists\, s \in S(U)$
s.t. $s|_{U_i} = s_i$

Ex. Sheaf of continuous fns. on a space.

Can also create a "sheaf space"
like a vector bundle $p: E \to X$
but stalks are discrete! Given $S$ on $X$
let $E_x = \varinjlim_{x \in U} S(U)$. Topologize by images of
each $s \in S(U)$ is an open set.

In this case $S(U) = \Gamma(U; E)$.

Sheaf cohomology $H^i(X; S)$ constructed as follows: Take an injective resolution of
$S$: $S \to I^\bullet \leftarrow$ complex of sheaves (maps act pointwise & : injective

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[dl, dashed] \\
 & I &
\end{tikzcd}$$

Then: $H^i(X; S) = H^i(\Gamma(X, I^*))$

More generally if $S^\bullet$ is a complex of sheaves, $\exists$ an injective complex $I^*$ and a map $S^\bullet \to I^\bullet$ inj. res. that induces homology isos over each stalk (called a quasi-iso.). Then define hypercohomology to be

$$\mathbb{H}^*(X; S^*) = H^*(\Gamma(X; I^*))$$

1, This process gives same homology if $I^\bullet$ is not injective but satisfies one of a list of other possible properties (soft, flabby, fine, etc)

2. Example! Let $M$ be a smooth mfld, Let $\mathbb{R}$ be const. sheaf, res. $\underline{\mathbb{R}} \to \Omega^*_{DR}(M)$. ← soft res.

so $\mathbb{H}^*(M; \underline{\underline{\mathbb{R}}}) \cong H^*_{DR}(M)$.

Ex. $X$ a pm,

$\underline{\underline{I^pS}}^i(U) = I^pS_{n-i}(X, X-U)$ - sheaf of int chains.

this is a homotopically fine sheaf complex

$\Rightarrow \mathbb{H}^*(X) = I^pH_{n-*}(X)$ (if $X$ is compact.

otherwise get $I^pH^\infty_{n-*}(X)$

or need to use $\Gamma_c(X; \underline{IS})$)

Sidenote: why resolutions? · Otherwise doesn't play well w/ functors. Note $H^i(X;S)$ is really the right derived functor of $\Gamma$. functor.

Two other important functors:

1. pushforward

$f_*$, if $f: X \to Y$ and $S \in Sh(X)$, define $f_* S \in Sh(Y)$ by $(f_* S)(U) = S(f^{-1}(U))$ this is a sheaf.

Define $Rf_* S$ by $Rf_*(S) = I^*(f^{-1}(U))$ where $S^* \to I^*$ is a res.

Note $H^*(Y; Rf_* S) = H^*(X; S^*)$

2. trunc. Given a sheaf complex $S^*$

$$\begin{tikzcd} \cdots \\ S^1 \arrow[u] \\ S^0 \arrow[u] \end{tikzcd}$$

Let $(\tau_{\leq k} S)^* =$

$$\begin{tikzcd} \ker(d: S^k \to S^{k+1}) \\ S^{k-1} \arrow[u] \\ \vdots \arrow[u] \\ S^0 \arrow[u] \end{tikzcd}$$

Truncation

For each $x$, $H^*(S^*_x) = \begin{cases} 0 & * > k \\ H^*(S^*_x) & * \leq k. \end{cases}$

$\tau_{\leq k}$ is an exact functor

Consider the following "Delign sheaf" complex on a pm. $X$.

Let $U_j = X - X^{n-j}$, and let $p$ be a GM perv. (so $p(S) = p(\mathrm{codim}(S))$ and $p \geq 0$ and increasing).

Let $\underline{\mathbb{R}}_1$ be const sheaf on $U_1$,

let $i_j : U_j \hookrightarrow U_{j+1}$

Let $\mathcal{P}_p^* = \tau_{\leq p(n)} R i_{n*} \ldots \tau_{\leq p(1)} R i_{1*} \mathbb{R}_1$

Let $x \in X^{n-k} - X^{n-k-1}$, and Let $\mathbb{R}^{n-k} \times cL$ be dist nbhd.

Then $H^*(\mathcal{P}_x^*) = \begin{cases} 0 & * > p(k) \\ H^*(\mathbb{R}^{n-k} \times cL - \{pt\}; \mathcal{P}^*|_{uu}) & * \leq p(k) \end{cases}$

$$\qquad\qquad\quad \| \\ \qquad\qquad H^*(L; \mathcal{P}^*|_L)$$

looks like cone formula (sort of).

Fact $\mathcal{P}_p^* \cong_{q.i.} I^p S^*$

Fact. $\mathcal{P}_p^*$, and hence $\underline{I}^p S^*$, are characterised by the following axioms,

1, $S^*$ is bounded (cohomologically), $S^i = 0$ for $i < 0$ and $S^*|_{U_1} \cong \underline{\mathbb{R}}$

2. if $x \in X^{n-k} - X^{n-k+1}$, Then $H^i(S_x^*) = 0$ for $i > p(k)$

3, The attaching map $S^*|_{U_{k+1}} \to \mathbb{R} i_{k*} S|_{U_k}$ is a coho iso in deg $\leq p(k)$

1. for GM perv.

Application: There is an equiv. set of axioms w/out explicit reference to strat.
⟹ strat. indep.

2. There is a fancy machine called Verdier duality,

$$\mathcal{D}S^* = \underline{\mathrm{Hom}}(S^*, \mathbb{D}^*)$$

w/ $\mathbb{H}^i(X;\mathcal{D}S^*) \cong \mathrm{Hom}(\mathbb{H}^{-i}_c(X;S^*,\mathbb{R}))$
(assuming field coeffs).

Can check using the axioms that

$$\mathcal{D}P^*_{\bar p}\underset{\substack{\uparrow\\ \text{shift}}}{[-n]} \cong_{q.i.} P^*_{D\bar p} \implies \text{Poincaré duality}$$

Can extend these ideas to more general $\bar p$, coeffs. Still need locally torsion free because local computation looks like cohomology UCT, so the Ext term messes w/ cone formula unless we have extra cond.

Banagl's thesis:

If $X$ is a Witt space
$$\begin{tikzcd} \underline{\underline{I}}^{\bar m}S^* \arrow[r, "q.i."] & \underline{\underline{I}}^{\bar n}S^* \arrow[d, phantom, "\shortparallel"] \\ & \mathcal{D}(\underline{\underline{I}}^{\bar m}S^*)[-n] \end{tikzcd}$$

What if not Witt? Is there a sheaf $\mathcal{B}$ w/ $I^{\bar m}S^* \to B^* \to I^{\bar n}S^*$ and $\mathcal{D}(B)_{[-n]} \simeq_{q.i.} B$.

A. Yes if certain conditions are satisfied:
if $S$ is an odd codim strata and

1. $L^{2n}$ is the link, need $H^*(L^n; B^*)$ to have a Lagrangian subspace
2. Lagrangian subspaces need to be ~~globally compa~~ a section of sheaf $\underline{\underline{H}}^r(S, B^*)$ (global compatibility).

If so can define $\beta$ inductively.

On such "L spaces" one gets sigs, L-classes, etc.
And there is a bordism theory w/ $\Omega_n \cong \begin{cases} \mathbb{Z} & n \equiv 0 \bmod 4 \\ 0 & \text{o.w.} \end{cases}$