Twisted Geometric Cycles

This talk is on some results of my collaborator Bai-Ling Wang in

arXiv:0710.1625: Geometric cycles, index theory and twisted K-homology. (Journal of NCG 2008)

It had its origins in:

arXiv:0708.3114: Differential Twisted K-theory and Applications (Alan L. Carey, Jouko Mickelsson, Bai-Ling Wang)

and

arXiv:math/0507414: Thom isomorphism and Push-forward map in twisted K-theory (Alan L. Carey, Bai-Ling Wang)

Overview

There are several ingredients that need explaining from the topological and geometric side (the analytic side is essentially known).

(i) Twisted K theory.

(ii) Twisted Poincaré duality between twisted K-cohomology and twisted K-homology.

(iii) Generalising Baum-Douglas K-homology to the twisted situation.

0. Motivation

Understand Witten's ideas on D-brane charges as taking values in twisted K-theory and to get a twisted version of some of what is in his original article

hep-th/9810188 'D-branes and K-theory'

Other motivating factors are in the work of BMRS, Adv. Theor. Math. Phys. **13** (2009) 497552, 'Non-commutative correspondences, duality and D-branes in bivariant K-theory' which focuses on the analytic version.

The upshot of Wang's approach is that there is a way to think geometrically of D-branes, at least insofar as they relate to topological twisted K-homology, as twisted versions of the Baum-Douglas geometric cycles.

Finally I mention some additional topics to do with D-branes.

1. Twisted K-theory: topological and analytic definitions

Let X be a paracompact Hausdorff topological space, and $\mathcal{H}$ be an infinite dimensional, complex and separable Hilbert space.

$PU(\mathcal{H})$ is the projective unitary group with norm topology. $PU(\mathcal{H})$ can be identified with an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. So the classifying space $BPU(\mathcal{H})$ is a $K(\mathbb{Z}, 3)$.

A twisting is a continuous map $\alpha : X \to K(\mathbb{Z}, 3)$. The associated $PU(\mathcal{H})$ bundle $\mathcal{P}_\alpha$ is given by pulling back the universal $PU(\mathcal{H})$-bundle over $K(\mathbb{Z}, 3)$.

The set of isomorphism classes of principal $PU(\mathcal{H})$-bundles over X is the homotopy classes of maps

$$[X, K(\mathbb{Z}, 3)] \cong H^3(X, \mathbb{Z}).$$

Let $\mathbf{Fred}$ be the space of Fredholm operators with norm topology.

The 'conjugation' action $PU(\mathcal{H}) \times \mathbf{Fred} \longrightarrow \mathbf{Fred}$ defines an associated bundle with fiber the Fredholm operators

$$\mathcal{P}_\alpha(\mathbf{Fred}) = \mathcal{P}_\alpha \times_{PU(\mathcal{H})} \mathbf{Fred}$$

Let $\Omega^n_X \mathcal{P}_\alpha(\mathbf{Fred}) = \mathcal{P}_\alpha \times_{PU(\mathcal{H})} \Omega^n \mathbf{Fred}$ be the fiberwise n-iterated loop spaces.

The (topological) twisted K-groups of (X, α) are defined to be

$$K^{-n}(X, \alpha) := \pi_0\Big(C_c(X, \Omega^n_X \mathcal{P}_\alpha(\mathbf{Fred}))\Big),$$

the set of homotopy classes of compactly supported sections. Due to Bott periodicity, we only have two different twisted K-groups, denoted by $K^0(X, \alpha)$ and $K^1(X, \alpha)$.

Associated with the $PU(\mathcal{H})$ bundle $\mathcal{P}_\alpha$ is a continuous trace C*-algebra and one may define the analytic twisted K-theory of (X, α) as the K-theory (via Kasparov) of this algebra.

2. Twisted K-homology: Analytic and topological definitions

The analytic twisted K-homology of (X, α), denoted by $K^{\mathbf{an}}_{ev/odd}(X, \alpha)$, is defined as the K-homology (via Kasparov) of the continuous trace C*-algebra associated to $\mathcal{P}_\alpha$.

Introduce the space $\mathcal{P}_\alpha(\mathbf{Fred})/X$ obtained by identifying the base points (the identity operator) in the fibers. Then the topological twisted K-homology $K^{\mathbf{top}}_{ev/odd}(X, \alpha)$ is defined to be

$$K^{\mathbf{top}}_{ev}(X, \alpha) = \varinjlim_{k \to \infty} \pi_{2k}\Big(\mathcal{P}_\alpha(\mathbf{Fred})/X\Big)$$

and

$$K^{\mathbf{top}}_{odd}(X, \alpha) = \varinjlim_{k \to \infty} \pi_{2k+1}\Big(\mathcal{P}_\alpha(\mathbf{Fred})/X\Big).$$

The proof that the topological and analytic objects are isomorphic uses twisted Poincare dualities in the topological and analytic settings and the equivalence between topological and analytic twisted K-theory.

3. The Twisted Poincaré duality

The twisted version introduces a shift in the twist

$$\alpha \mapsto \alpha + (W_3 \circ \tau)$$

where $\tau : X \to BSO$ is the classifying map of the stable tangent bundle and W_3 is the classifying map for the bundle $\mathbf{BSpin^c} \to \mathbf{BSO}$, and $\alpha + (W_3 \circ \tau)$ denotes the map $X \to K(\mathbb{Z}, 3)$, representing the class $[\alpha] + W_3(X)$ in $H^3(X, \mathbb{Z})$. (There is a tricky point in this definition where we proceed by fixing an isomorphism $\mathcal{H} \otimes \mathcal{H} \cong \mathcal{H}$.)

Theorem Let X be a smooth manifold with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$.

(i) (Wang) There exists an isomorphism

$$K^{\mathbf{top}}_{ev/odd}(X, \alpha) \cong K^{ev/odd}_{\mathbf{top}}(X, \alpha + (W_3 \circ \tau))$$

with the degree shifted by $dim X (mod\ 2)$.

(ii) (Tu, Echterhoff-Emerson-Kim) There exists an isomorphism

$$K^{\mathbf{an}}_{ev/odd}(X, \alpha) \cong K^{ev/odd}_{\mathbf{an}}(X, \alpha + (W_3 \circ \tau))$$

with the degree shifted by $dim X (mod\ 2)$.

4. Twisted geometric cycles

Let (X, α) be a paracompact Hausdorff space with a twisting α.

A geometric cycle for (X, α) is a quintuple

$$(M, \iota, \nu, \eta, [E])$$

where $[E]$ is a K-class in $K^0(M)$, M an oriented smooth closed manifold with a classifying map ν of its stable normal bundle, $\iota : M \to X$ is a continuous map such that there exists a homotopy commutative diagram:

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[d, "\iota"'] & \mathbf{BSO} \arrow[d, "W_3"] \arrow[dl, Rightarrow, dashed, "\eta"'] \\ X \arrow[r, "\alpha"] & K(\mathbb{Z}, 3), \end{tikzcd}$$

with a homotopy η between $W_3 \circ \nu$ and $\alpha \circ \iota$. We refer to this diagram of maps as an 'α-twisted Spin^c structure'.

Remarks.

1. M admits an α-twisted $Spin^c$ structure if and only if
$$\iota^*([\alpha]) + W_3(M) = 0.$$
If ι is an embedding, this is the anomaly cancellation condition introduced by Freed and Witten.

2. If the twists are all trivial this reduces to the Baum-Douglas definition and η corresponds to a choice of Spin^c structure.

Two geometric cycles $(M_1, \iota_1, \nu_1, \eta_1, [E_1])$ and $(M_2, \iota_{,2} \nu_2, \eta_2, [E_2])$ are isomorphic if there is an isomorphism $f : (M_1, \iota_1, \nu_1, \eta_1) \to (M_2, \iota_2, \nu_2, \eta_2)$, as α-twisted $Spin^c$ manifolds over X, such that $f_!([E_1]) = [E_2]$.

Let $\Gamma(X, \alpha)$ be the collection of all geometric cycles for (X, α). We now impose an equivalence relation $\sim$ on $\Gamma(X, \alpha)$, generated by the following three relations:

Direct sum - disjoint union

If $(M, \iota, \nu, \eta, [E_1])$ and $(M, \iota, \nu, \eta, [E_2])$ are two geometric cycles with the same α-twisted $Spin^c$ structure, then

$$(M, \iota, \nu, \eta, [E_1]) \cup (M, \iota, \nu, \eta, [E_2])$$

$$\sim (M, \iota, \nu, \eta, [E_1] + [E_2]).$$

Bordism

If there exists an α-twisted $Spin^c$ manifold (W, ι, ν, η) and $[E] \in K^0(W)$ such that

$$\partial(W, \iota, \nu, \eta) = -(M_1, \iota_1, \nu_1, \eta_1) \cup (M_2, \iota_2, \nu_2, \eta_2)$$

and $\partial([E]) = [E_1] \cup [E_2]$. Here $-(M_1, \iota_1, \nu_1, \eta_1)$ denotes the manifold M_1 with the opposite α-twisted $Spin^c$ structure, then

$$(M_1, \iota_1, \nu_1, \eta_1, [E_1]) \sim (M_2, \iota_2, \nu_2, \eta_2, [E_2]).$$

$Spin^c$ vector bundle modification

Take a geometric cycle $(M, \iota, \nu, \eta, [E])$ and a $Spin^c$ vector bundle V over M with even dimensional fibers. Denote by $\underline{\mathbb{R}}$ the trivial rank one real vector bundle. Choose a Riemannian metric on $V \oplus \underline{\mathbb{R}}$, let

$$\hat{M} = S(V \oplus \underline{\mathbb{R}})$$

be the sphere bundle of $V \oplus \underline{\mathbb{R}}$.

Denote by $\rho : \hat{M} \to M$ the projection which is K-oriented. The vertical tangent bundle $T^v(\hat{M})$ of $\hat{M}$ admits a natural $Spin^c$ structure with an associated $\mathbb{Z}_2$-graded spinor bundle $S_V^+ \oplus S_V^-$. Then

$$(M, \iota, \nu, \eta, [E]) \sim (\hat{M}, \iota \circ \rho, \nu \circ \rho, \eta \circ \rho, [\rho^* E \otimes S_V^+]).$$

Definition. The geometric twisted K-homology $K^{\mathbf{geo}}_{ev/odd}(X, \alpha)$ is defined to be $\Gamma(X, \alpha)/\sim$ with the grading given by even or odd dimension of α-twisted $Spin^c$ manifolds . Addition is given by the disjoint union - direct sum relation.

4. Twisted assembly map

There exists a natural homomorphism

$$\mu : K^{\mathbf{geo}}_{ev/odd}(X, \alpha) \to K^{\mathbf{an}}_{ev/odd}(X, \alpha)$$

where $\mu(M, \iota, \nu, \eta, [E])$ is defined by composition of a sequence of maps:

$$\begin{tikzcd}
{[E] \in K^0(M)} \arrow[r, "PD"] & K^{\mathbf{an}}_{ev/odd}(M, W_3 \circ \tau) \arrow[d, "I_*"] \\
K^{\mathbf{an}}_{ev/odd}(M, \alpha \circ \iota) \arrow[d, "\iota_*", "\cong"'] & K^{\mathbf{an}}_{ev/odd}(M, W_3 \circ \nu) \arrow[l, "\cong", "\eta_*"'] \\
K^{\mathbf{an}}_{ev/odd}(X, \alpha). &
\end{tikzcd}$$

Here $PD : K^0(M) \cong K^{\mathbf{an}}_{ev/odd}(M, W_3 \circ \tau)$ is the Kasparov's Poincaré duality with the degree shift by $dimM(mod\ 2)$, ι_* is the natural push-forward map in twisted K-homology, η_* is the isomorphism induced by the homotopy η, and I_* is the isomorphism induced by the trivial $Spin^c$ structure on the trivial bundle $\tau \oplus \nu$.

Theorem (Wang) The twisted assembly map

$$\mu : K^{\mathbf{geo}}_{ev/odd}(X, \alpha) \to K^{\mathbf{an}}_{ev/odd}(X, \alpha)$$

is an isomorphism for any **smooth** closed manifold X with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$.

The proof of this theorem is via establishing that there is a map $\Psi : K^{\mathbf{top}}_{ev}(X, \alpha) \to K^{\mathbf{geo}}_{0}(X, \alpha)$ such that the following diagram

$$\begin{tikzcd} & K^{\mathbf{top}}_{ev/odd}(X, \alpha) \arrow[dl, "\Psi"'] \arrow[dr, "\Phi", "\cong"'] & \\ K^{\mathbf{geo}}_{ev/odd}(X, \alpha) \arrow[rr, "\mu"] & & K^{\mathbf{an}}_{ev/odd}(X, \alpha) \end{tikzcd}$$

commutes and Ψ is surjective.

5. The twisted index theorem

One of the applications of geometric cycles is to express an index pairing between twisted K-theory and twisted K-homology in terms of an index pairing on geometric cycles.

Theorem Let X be a smooth manifold with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$. The index pairing

$$K_0(X, \alpha) \times K^0(X, \alpha) \longrightarrow \mathbb{Z}$$

is given by

$$\begin{aligned} & < (M, \iota, \nu, \eta, [E]), \xi > \\ = \ & \int_M ch_{w_2(M)}\Big(\eta_*(\iota^*\xi \otimes E)\Big)\widehat{A}(M) \end{aligned}$$

where $\xi \in K^0(X, \alpha)$, and the geometric cycle

$$(M, \iota, \nu, \eta, [E])$$

defines a twisted K-homology class on (X, α). Here

$$\eta_* : K^*(M, \iota^*\alpha) \cong K^*(M, W_3(M))$$

is an isomorphism, and $ch_{w_2(M)}$ is the Chern character on $K^0(M, W_3(M))$ which we now explain.

6. Twisted Chern character

Under the identification between $K^0(M, W_3(M))$ and the K-theory of Clifford modules over M,

$$ch_{w_2(M)} : K^0(M, W_3(M)) \longrightarrow H^{ev}(M, \mathbb{R})$$

is given by the relative Chern character on Clifford modules as described for example in Berline-Getzler-Vergne.

The general twisted Chern character on $K^0(X, \alpha)$ requires a choice of gerbe connection and curving. A geometric definition was given in *Differential Twisted K-theory and its Applications*, C-Mickelsson-Wang. An analytical definition using the Chern-Connes character in noncommutative geometry was given Mathai-Stevenson. A topological definition was given by Atiyah-Segal.

7. Twisted Riemann-Roch

There is a Riemann-Roch theorem in C-Mickelsson-Wang *op cit*, which implies that the above index formula can be written as

$$
\begin{aligned}
& < (M, \iota, \nu, \eta, [E]), \xi > \\
= & \int_M ch_{w_2(M)}\Big(\eta_*(\iota^*\xi \otimes E)\Big)\widehat{A}(M) \\
= & \int_X ch_{w_2(X)}\Big(\ \iota_!(E) \otimes \xi\Big)\widehat{A}(X)
\end{aligned}
$$

where $\iota_!$ is the push-forward map on twisted K-theory defined by

$$
\begin{aligned}
K^0(M) & \cong K_0(M, W_3(M)) \\
& \cong K_0(M, -\iota^*\alpha) \\
& \cong K_0(X, -\alpha) \\
& \cong K^0(X, -\alpha + W_3(X))
\end{aligned}
$$

and $ch_{w_2(X)}$ is the canonical twisted Chern character on

$$K^0(X, -\alpha + W_3(X)) \otimes K^0(X, \alpha) \to K^0(X, W_3(X)).$$

8. D-branes

Theorem. (Wang) Given a twisting $\alpha : X \to K(\mathbb{Z}, 3)$ on a smooth manifold X, every twisted K-class in

$$K^{ev/odd}(X, \alpha)$$

is represented by a geometric cycle supported on an $(\alpha + (W_3 \circ \tau))$-twisted closed $Spin^c$-manifold M and an ordinary K-class $[E] \in K^0(M)$.

Thus there are three definitions of twisted K-theory $K^*(X, \alpha)$ for a smooth manifold X:

1. A topological definition in terms of homotopy equivalence classes of sections of a bundle of K-theory spectra associated to (X, α).

2. An analytical definition in terms of the continuous trace C^*-algebra associated to (X, α).

3. A geometric definition in terms of a geometric cycle (M, ι, ν, η, E) with ν the classifying map for the map $\iota : M \to X$.

We propose that this geometric cycle is the so-called Type II D-brane for a class in $K^*(X,\alpha)$. The equivalence of these three definitions gives a candidate for the D-brane charge map on the category of D-branes:

$$\{\text{D-branes over } (X,\alpha)\} \longrightarrow K^*(X,\alpha).$$

There is a version of Type I D-branes using twisted $Spin$-manifolds over (X,α) with $\alpha : X \to K(\mathbb{Z}_2, 2)$.

Remark on T-duality

Given a principal T^n-bundle $p : Y \to X$ with a twisting α on Y satisfying $p_!\alpha = 0 \in H^1(X, \mathbb{Z}^{\frac{n(n-1)}{2}})$, there is a classical T-dual $(Y^\#, \alpha^\#)$ such that

$$K^*(Y,\delta) \cong K^{*+n}(Y^\#, \delta^\#).$$

The dependence of twisted Chern character

$$ch_{\breve{\alpha}} : K^*(Y,\alpha) \longrightarrow H^*(Y, curv(\breve{\alpha}))$$

on $\breve{\alpha}$ (a gerbe connection and curving) makes the geometric formulation of classical T-duality, in terms of geometric cycles with connection

$$(M, \iota, \nu, \eta, E, \nabla_E),$$

more subtle. More work is needed in this direction.

9. Remark on String structures

One may think of the obstruction to the existence of a string structure on the loop space LM as an analogue of the class $W_3(M)$ except that the string class lies in $H^4(M,\mathbb{Z})$.

In Wang's paper he draws on this analogy with the view to making a connection with elliptic cohomology. This leads to some interesting conjectures which are under investigation.