Proper Actions on C^*-algebras

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Proper Actions on Spaces

Definition

We say that G acts properly on a locally compact space T if the map $(t, s) \mapsto (t, t \cdot s)$ is proper as a map from $T \times G \to T \times T$.

This notion underlies a host of important results in the theory. A prototypical example is the following result due to Green.

Theorem (Green '78)

If T is a free and proper G-space then $C_0(T) \rtimes_{\mathrm{rt}} G$ is Morita equivalent to $C_0(T/G)$.

Remark

If G is not finite and T/G is topologically "nice" — say T/G has finite covering dimension — then $C_0(T) \rtimes_{\mathrm{rt}} G$ is stable and $C_0(T) \rtimes_{\mathrm{rt}} G \cong C_0(T/G) \otimes \mathcal{K}$.

Proper Actions on General Algebras

- Fix a free and proper (right) G-space T.
- We restrict here to the collection $\mathtt{C^*act}\big(G, (C_0(T), \mathrm{rt})\big)$ of triples (A, α, ϕ) where $\alpha : G \to \operatorname{Aut} A$ is a dynamical system and $\phi : C_0(T) \to M(A)$ is a nondegenerate homomorphism such that $\alpha_s(\phi(f)a) = \phi(\mathrm{rt}_s(f))\alpha_s(a)$.
- Let $A_0 := \phi\big(C_c(T)\big)A\,\phi\big(C_c(T)\big)$. Then there is a positive linear map $E : A_0 \to M(A)^\alpha$ characterized by $\omega\big(E(a)\big) = \displaystyle\int_G \omega\big(\alpha_s(a)\big)\,ds$ for $\phi \in A^*$.
- The map E is called the Olesen-Pedersen-Quigg expectation.
- The closure of $E(A_0)$ is called Rieffel's generalized fixed point algebra and is denoted A^α.

An Application of Rieffel's Theory

Theorem (Rieffel '90)

Suppose that $(A, \alpha, \phi) \in \mathtt{C^*act}\big(G, (C_0(T), \mathrm{rt})\big)$ *and* $A_0 := \phi\big(C_c(T)\big) A\, \phi\big(C_c(T)\big)$. *Then* A_0 *can be completed to a* $A \rtimes_{\alpha,r} G - A^\alpha$*-imprimitivity bimodule* $\mathsf{Z}(A, \alpha)$. *In particular,* $A \rtimes_{\alpha,r} G$ *and* A^α *are Morita equivalent.*

Example (RW '85)

Suppose that T is a free and proper G-space and that (D, G, α) is *any* dynamical system. Then we get a "diagonal" system $\big(C_0(T, D), G, \mathrm{rt} \otimes \alpha\big)$: $(\mathrm{rt} \otimes \alpha)_s f(t) = \alpha_s\big(f(t \cdot s)\big)$. Then $A_0 = C_c(T, A)$, and Rieffel's generalized fixed point algebra is the induced algebra $\mathrm{Ind}_G^T(D, \alpha)$ consisting of bounded continuous functions $f : T \to D$ such that $f(t \cdot s) = \alpha_s^{-1}\big(f(t)\big)$ and $G \cdot t \mapsto \|f(t)\|$ is in $C_0(T/G)$.

Noncommuntative Duality

Theorem (Rieffel '90)

Suppose that $(A, \alpha, \phi) \in \mathtt{C^*act}\big(G, (C_0(T), \mathrm{rt})\big)$ *and* $A_0 := \phi\big(C_c(T)\big) A\, \phi\big(C_c(T)\big)$. *Then* A_0 *can be completed to a* $A \rtimes_{\alpha,r} G - A^\alpha$*-imprimitivity bimodule* $\mathsf{Z}(A, \alpha)$. *In particular,* $A \rtimes_{\alpha,r} G$ *and* A^α *are Morita equivalent.*

Example (Noncommutative Duality)

The crossed product $A \rtimes_\delta G$ of A by a coaction of G comes equipped with a canonical nondegenerate homomorphism j_G of $C_0(G)$ into its multiplier algebra $M(A \rtimes_\delta G)$. Furthermore, j_G is equivariant with respect to the dual action, $\hat{\delta}$, and right translation. We can restrict $\hat{\delta}$ to a closed subgroup H and then Rieffel's Theorem gives a Morita equivalence between the iterated crossed product $(A \rtimes_\delta G) \rtimes_{\hat{\delta}|_H} H$ and the corresponding fixed point algebra. When G is amenable, this fixed point algebra is the crossed product $A \rtimes_{\delta|} G/H$ by the homogeneous space G/H.

The EKQR Approach

- Recent work of Echterhoff, Kaliszewski, Quigg and Raeburn has shown that many important results involving Morita equivalence and its attendant representation theory can be profitably described via the formalism of category theory.
- One of the objects of today's talk is to show that $(A, \alpha, \phi) \mapsto A^{\alpha}$ is functorial, and that Rieffel's Morita equivalence of A^{α} with $A \rtimes_{\alpha,r} G$ is "natural" in the technical sense.
- Of course, category theory is just a tool. We need to tailor our choice of morphism so that our results are interesting and significant.
- Our starting point is that the "right" notion of isomorphism is Morita equivalence.

Getting the Morphism Right

Definition

An $A-B$-bimodule X is called a right Hilbert $A-B$-bimodule if X is a right Hilbert B-module and the A-action is given by a nondegenerate homomorphism of A into $\mathcal{L}(\mathsf{X})$. (The term $A-B$ correspondence is also used.)

Example

Let $\kappa_A : A \to M(B)$ be a nondegenerate homomorphism. Then B is naturally a right Hilbert $A-B$-bimodule.

The EKQR Category $\mathtt{C}^*$

The previous example is meant to suggest that we can think of right Hilbert bimodules as generalized $*$-homomorphisms.

- In the EKQR category $\mathtt{C}^*$, the morphisms from A to B are isomorphism classes of right Hilbert $A-B$-bimodules.
- The composition is given by the internal tensor product: $[{}_A\mathsf{X}_B][{}_C\mathsf{Y}_A] = [{}_C(\mathsf{Y} \otimes_A \mathsf{X})_B]$. The identity morphism is $[{}_A A_A]$.
- The invertible morphisms are exactly the imprimitivity bimodules.
- Note that any right Hilbert bimodule factors as an imprimitivity bimodule and a homomorphism: $[{}_A\mathsf{X}_B] = [{}_{\mathcal{K}(\mathsf{X})}\mathsf{X}_B][\kappa_A]$, where $\kappa_A : A \to \mathcal{L}(X) = M\big(\mathcal{K}(X)\big)$.
- This just means that ${}_A\mathsf{X}_B \cong_A (\mathcal{K}(\mathsf{X}) \otimes_{\mathcal{K}(\mathsf{X})} \mathsf{X})_B$ as right Hilbert bimodules.

Getting the Morphism Right — The General Case

- We make $\texttt{C*act}\big(G, \big(C_0(T), \mathrm{rt}\big)\big)$ into a category by defining the morphisms analogously to those in the EKQR Category $\texttt{C*}$: the morphisms in $\texttt{C*act}\big(G, \big(C_0(T), \mathrm{rt}\big)\big)$ will be isomorphism classes of equivariant right Hilbert modules:

Definition

If (A, α) and (B, β) are G-systems, then a right Hilbert $A-B$-bimodule is called equivariant if there is a strictly continuous homomorphism $u : G \to \mathrm{GL}(\mathrm{X})$ such that

1. $u_s(a \cdot x) = \alpha_s(a) \cdot u_s(x)$
2. $u_s(x \cdot b) = u_s(x) \cdot \beta_s(b)$
3. $\big\langle u_s(x)\ ,\ u_s(y)\big\rangle_B = \beta_s\big(\langle x\ ,\ y\rangle_B\big)$.

Note $(3) \implies (2)$.

- The isomorphisms in $\texttt{C*act}\big(G, \big(C_0(T), \mathrm{rt}\big)\big)$ are equivariant imprimitivity bimodules.

The Object of this Exercise

- Recall that we aim to show that there is a functor $\mathtt{Fix}$ from $\mathtt{C^*act}\big(G, \big(C_0(T), \mathrm{rt}\big)\big)$ to $\mathtt{C^*}$ taking (A, α, ϕ) to the generalized fixed point algebra A^α.
- To make sense of this, we also have to specify what $\mathtt{Fix}$ does to morphisms. Thus given and equivariant right Hilbert A – B-bimodule (X, u), I need to specify an appropriate right Hilbert A^α – B^β-bimodule $\mathtt{Fix}(\mathsf{X}, u)$.
- The EKQR philosophy stresses that this formalism neatly summarizes a good deal of nontrivial and interesting information. This will imply results which not only apply to the relationships, such as Morita equivalence, between the fixed point algebras and crossed products involved, but equally importantly, this will imply results about the structure of the imprimitivity bimodules involved.
- I hope that this will begin to become apparent as we see how $\mathtt{Fix}(\mathsf{X}, u)$ is constructed.

Building $\mathtt{Fix}(\mathsf{X}, u)$

We start by considering objects (K, μ, ϕ_K) and (B, β, ϕ_B) in $\mathtt{C^*act}\big(G, \big(C_0(T), \mathrm{rt}\big)\big)$ which are Morita equivalent via ${}_{(K,\mu)}(\mathsf{X}, u)_{(B,\beta)}$. We can form the linking algebra

$$L(\mathsf{X}) := \begin{pmatrix} K & \mathsf{X} \\ \widetilde{\mathsf{X}} & B \end{pmatrix} \cong \mathcal{K}(X \oplus B).$$

There is a G-action, $L(u)$, on $L(\mathsf{X})$:

$$L(u)_s \begin{pmatrix} k & x \\ \tilde{y} & b \end{pmatrix} = \begin{pmatrix} \mu_s(k) & u_s(x) \\ u_s(y)^\sim & \beta_s(b) \end{pmatrix},$$

and there is a compatible nondegenerate homomorphism $\phi : C_0(T) \to M\big(L(\mathsf{X})\big)$: $\phi = \begin{pmatrix} \phi_K & 0 \\ 0 & \phi_B \end{pmatrix}$. Hence $\big(L(\mathsf{X}), L(u), \phi\big) \in \mathtt{C^*act}\big(G, \big(C_0(T), \mathrm{rt}\big)\big)$.

Building $\mathtt{Fix}(\mathsf{X}, u)$

Since $\big(L(\mathsf{X}), L(u), \phi\big) \in \mathtt{C^*act}\big(G, \big(C_0(T), \mathrm{rt}\big)\big)$, we can form its generalized fixed point algebra $L(\mathsf{X})^{L(u)}$. Using the Olesen-Pedersen-Quigg expectation, it is not so hard to see that

$$L(\mathsf{X})^{L(u)} = \begin{pmatrix} K^{\mu} & \mathsf{X}^{u} \\ * & B^{\beta} \end{pmatrix}$$

where the "$(1,2)$" corner X^u is a $K^{\mu} - B^{\beta}$-bimodule with K^{μ}- and B^{β}-valued inner products coming from the "matrix" operations in $L(\mathsf{X})^{L(u)}$. Showing that X^u is actually an imprimitivity bimodule requires that we see that the span of the inner products is dense. This requires some subtle analysis using properties of the Olesen-Pedersen-Quigg expectation.

Building $\texttt{Fix}(\mathsf{X}, u)$ — the General Case

In general, given an equivariant right Hilbert A–B-bimodule ${}_{(A,\alpha)}(\mathsf{X}, u)_{(B,\beta)}$, we get an imprimitivity bimodule ${}_{(\mathcal{K}(\mathsf{X}),\mu)}(\mathsf{X}, u)_{(B,\beta)}$, where μ is the action "induced" from β via X and u. Furthermore, $\mathcal{K}(\mathsf{X})$ inherits a homomorphism of $C_0(T)$ into its multiplier algebra via the given map $\kappa_A : A \to \mathcal{L}(\mathsf{X}) = M\big(\mathcal{K}(\mathsf{X})\big)$. Therefore we can "factor" $[(\mathsf{X}, u)]$, as

$$[{}_{\mathcal{K}(X)}\mathsf{X}_B, u][\kappa_A].$$

Since κ_A is nondegenerate, it extends to a homomorphism $\kappa_A : M(A) \to M\big(\mathcal{K}(\mathsf{X})\big)$. Recent work of [*KQR*] implies that κ_A restricts to a nondegenerate homomorphism

$$\kappa_A| : A^\alpha \to M\big(\mathcal{K}(\mathsf{X})^\mu\big).$$

Since $\mathcal{K}(\mathsf{X})^\mu \cong \mathcal{K}(\mathsf{X}^u)$, we get our morphism — that is, a right Hilbert A^α–B^β-bimodule — via composition:

$$[\texttt{Fix}(\mathsf{X}, u)] := [\mathsf{X}^u][\kappa_A|] = [{}_{A^\alpha}(\mathcal{K}(\mathsf{X}^u) \otimes_{\mathcal{K}(\mathsf{X}^u)} \mathsf{X}^u)_{B^\beta}].$$

The Main Theorem

Theorem

The assignments $\mathtt{Fix}(A,\alpha,\phi) = A^\alpha$ *and* $\mathtt{Fix}\big([{}_A(\mathsf{X},u)_B]\big) = [{}_{A^\alpha}\mathsf{X}^u_{B^\beta}]$ *define a covariant functor from* $\mathtt{C^*act}\big(G,(C_0(T),\mathrm{rt})\big)$ *to* $\mathtt{C^*}$.

Even though we had to work a bit just to be able to state the theorem, we still have some work left for the proof. The crucial thing — and the point of all this — is to see that Fix preserves composition of morphisms. Thus if (X,u) is an equivariant right Hilbert (A,α) – (B,β)-bimodule and if (Y,v) is an equivariant right Hilbert (B,β) – (C,γ)-bimodule, then we are claiming that

$$(\mathsf{X}\otimes_B \mathsf{Y})^{u\otimes v} \cong \mathsf{X}^u \otimes_{B^\beta} \mathsf{Y}^v$$

as right Hilbert A^α – C^γ-bimodules.

I'll sketch the argument when X and Y are imprimitivity bimodules.

A 3×3 Matrix Trick

We realize $F := \mathcal{K}\big((\mathsf{X} \otimes_B \mathsf{Y}) \oplus \mathsf{Y} \oplus C\big)$ as 3×3 matrices of the form $\begin{pmatrix} A & \mathsf{X} & \mathsf{X} \otimes_B \mathsf{Y} \\ \widetilde{\mathsf{X}} & B & \mathsf{Y} \\ (\mathsf{X} \otimes_B \mathsf{Y})^{\sim} & \widetilde{\mathsf{Y}} & C \end{pmatrix}$. Then there is a natural G-action on F given by

$$\eta := \begin{pmatrix} \alpha & u & u \otimes v \\ \tilde{u} & \beta & v \\ (u \otimes v)^{\sim} & \tilde{v} & \gamma \end{pmatrix},$$

and there is a nondegenerate homomorphism $\phi : C_0(T) \to M(F)$ given by

$$\phi = \begin{pmatrix} \phi_A & 0 & 0 \\ 0 & \phi_B & 0 \\ 0 & 0 & \phi_C \end{pmatrix}.$$

In other words, $(F, \eta, \phi) \in \mathtt{C^*act}\big(G, (C_0(T), \mathrm{rt})\big)$.

The Fixed Point Algebra F^η

Recall

$$F = \begin{pmatrix} A & \mathsf{X} & \mathsf{X} \otimes_B \mathsf{Y} \\ \widetilde{\mathsf{X}} & B & \mathsf{Y} \\ (\mathsf{X} \otimes_B \mathsf{Y})^\sim & \widetilde{\mathsf{Y}} & C \end{pmatrix}.$$

Notice that the $(1,2)$, $(2,3)$ and $(1,3)$ minors are linking algebras. Therefore we can apply the same analysis we used when constructing X^u to conclude that

$$F^\eta = \begin{pmatrix} A^\alpha & \mathsf{X}^u & (\mathsf{X} \otimes_B \mathsf{Y})^{u \otimes v} \\ * & B^\beta & \mathsf{Y}^v \\ * & * & C^\gamma \end{pmatrix}.$$

Now we invoke a general "linking algebra type" result:

A (Generalized) Linking Algebra Lemma

Lemma

Suppose that p_i are full projections in $M(D)$ such that $p_1 + p_2 + p_3 = 1$. Then p_iDp_j is a $p_iDp_i - p_jDp_j$-imprimitivity bimodule, and $p_1dp_2 \otimes p_2d'p_3 \mapsto p_1dp_2d'p_3$ extends to a $p_1Dp_1 - p_3Dp_3$-imprimitivity bimodule isomorphism of $p_1Dp_2 \otimes_{p_2Dp_2} p_2Dp_3$ onto p_1Dp_3.

Applying this to $F^\eta = \begin{pmatrix} A^\alpha & \mathsf{X}^u & (\mathsf{X} \otimes_B \mathsf{Y})^{u \otimes v} \\ * & B^\beta & \mathsf{Y}^v \\ * & * & C^\gamma \end{pmatrix}$ implies that

$$(\mathsf{X} \otimes_B \mathsf{Y})^{u \otimes v} \cong \mathsf{X}^u \otimes_{B^\beta} \mathsf{Y}^v$$

Which is what we wanted to show.
The general result is obtained by factoring our right Hilbert bimodules as a product of an imprimitivity bimodule and a nondegenerate homomorphism.

Naturality

The EKQR machinery provides a funtor, RCP, from $\mathtt{C^*act}\big(G, (C_0(T), \mathrm{rt})\big)$ to $\mathtt{C^*}$ which takes (A, α, ϕ) to the reduced crossed product $A \rtimes_{\alpha,r} G$, and the morphism $[\mathsf{X}, u]$ to $[\mathsf{X} \rtimes_{u,r} G]$. The Rieffel theory for proper actions gives us an isomorphism in the category $\mathtt{C^*}$ in the form of an imprimitivity bimodule $\mathsf{Z}(A, \alpha, \phi)$ implementing the equivalence between $A \rtimes_{\alpha,r} G$ and A^α.

Theorem

The equivalences $\mathsf{Z}(A, \alpha, \phi) : A \rtimes_{\alpha,r} G \to A^\alpha$ form a natural isomorphism between the functors RCP *and* Fix*.*

And that means . . .

Simply put, the last theorem just says that for each morphism $(\mathsf{X}, u) : (A, \alpha, \phi_A) \to (B, \beta, \phi_B)$ in $\mathtt{C^*act}\big(G, \big(C_0(T), \mathrm{rt}\big)\big)$, we get a commutative diagram

$$\begin{tikzcd}
A \rtimes_{\alpha,r} G \arrow[r, "\mathsf{Z}(A,\alpha,\phi_A)"] \arrow[d, "\mathsf{X} \rtimes G"'] & A^{\alpha} \arrow[d, "\mathtt{Fix}(\mathsf{X},u)"] \\
B \rtimes_{\beta,r} G \arrow[r, "\mathsf{Z}(B,\beta,\phi_B)"'] & B^{\beta}.
\end{tikzcd} \tag{‡}$$

This is just an elegant way of saying that

$$\mathsf{Z}(A, \alpha, \phi_A) \otimes_{A^{\alpha}} \mathtt{Fix}(X, u) \cong \mathsf{X} \rtimes_{u,r} G \otimes_{B \rtimes_r G} \mathsf{Z}(B, \beta, \phi_B)$$

as right Hilbert $A \rtimes_{\alpha,r} G - B^{\beta}$-modules.

A Remark

Remark

Since imprimitivity bimodules are isomorphisms in our category, Rieffel's imprimitivity bimodules $\mathsf{Z}(\cdots)$ are invertible — the inverse is given by the dual or opposite module: $\mathsf{Z}(\cdots)^{\mathrm{op}}$. Consequently, the naturality diagram (‡) not only gives the isomorphism

$$\mathsf{Z}(A, \alpha, \phi_A) \otimes_{A^\alpha} \mathtt{Fix}(X, u) \cong \mathsf{X} \rtimes_{u,r} G \otimes_{B \rtimes_r G} \mathsf{Z}(B, \beta, \phi_B),$$

but it also implies statements like

$$\mathtt{Fix}(X, u) \cong \mathsf{Z}(A, \alpha, \phi_A)^{\mathrm{op}} \otimes_{A \rtimes_r G} \mathsf{X} \rtimes_{u,r} G \otimes_{B \rtimes_r G} \mathsf{Z}(B, \beta, \phi_B).$$

The Proof of Naturality

To show that the naturality diagram (‡) commutes, we consider the following picture:

$$\begin{tikzcd}
A \rtimes_r G \arrow[rrr, "Z(A,\alpha,\phi_A)"] \arrow[dr, "\kappa \rtimes_r G"] \arrow[dd, "X \rtimes_r G"'] & & & A^{\alpha} \arrow[dl, "\kappa|"'] \arrow[dd, "\mathtt{Fix}(X,u)"] \\
& \mathcal{K} \rtimes_r G \arrow[r, "Z(\mathcal{K},\mu,\phi_{\mathcal{K}})"'] \arrow[dl, "X_B \rtimes_r G"] & \mathcal{K}^{\mu} \arrow[dr, "({}_{\mathcal{K}}X_B)^u"'] & \\
B \rtimes_r G \arrow[rrr, "Z(B,\beta,\phi_B)"'] & & & B^{\beta}
\end{tikzcd}$$

The triangles commute as they are just the standard factorizations for a right Hilbert bimodule. The upper quadrangle involves only modules built from homomorphisms and commutes by work of KQR. The lower quadrangle involves only imprimitivity bimodules. It commutes using the general EKQR theory.