The Geometry behind Nongeometric Fluxes

Peter Bouwknegt [1,2]

[1] Department of Theoretical Physics
Research School of Physics and Engineering
[2] Department of Mathematics
Mathematical Sciences Institute

The Australian National University
Canberra, AUSTRALIA

NSF/CBMS Regional Conference in the Mathematical Sciences
"Topology, C^*-algebras, and String Duality"
TCU, Fort Worth, May 18-22, 2009

Abstract

By considering T-duality for strings moving in a geometric background, i.e. in the presence of curvature and H-fluxes, one arises at situations in which the string is coupled to, what is known in the literature as, non-geometric fluxes. In this talk we will consider T-duality in the context of generalized geometry and unravel the geometry behind these non-geometric fluxes.

- Topological T-duality, as developed in Rosenberg's lectures, is a mere shadow of the equivalence of certain string theories under T-duality. The full picture involves geometry.
- What is the 'geometry' behind the 'missing T-duals'?
- As we have seen, T-duality exchanges momentum (related to TE), with winding (related to T^*E).

  A natural geometric framework for T-duality is therefore a framework which treats TE and T^*E on equal footing.

$$\Downarrow$$

GENERALIZED GEOMETRY

Generalized geometry

Replace structures on TE by structures on $TE \oplus T^*E$

- Bilinear form on sections $(X, \Xi) \in \Gamma(TE \oplus T^*E)$

$$\langle (X_1, \Xi_1), (X_2, \Xi_2) \rangle = \tfrac{1}{2}(\imath_{X_1}\Xi_2 + \imath_{X_2}\Xi_1)$$

- (twisted) Courant bracket

$$[\![\, (X_1, \Xi_1), (X_2, \Xi_2) \,]\!]_H =$$
$$([X_1, X_2], \mathcal{L}_{X_1}\Xi_2 - \mathcal{L}_{X_2}\Xi_1 - \tfrac{1}{2} d\left(\imath_{X_1}\Xi_2 - \imath_{X_2}\Xi_1\right) + \imath_{X_1}\imath_{X_2} H)$$

where $H \in \Omega^3_{\mathrm{cl}}(E)$

Generalized Geometry (cont'd)

- Clifford algebra

$$\{\gamma_{(X_1,\Xi_1)}, \gamma_{(X_2,\Xi_2)}\} = 2\langle (X_1,\Xi_1),(X_2,\Xi_2)\rangle$$

- Clifford module $\Omega^\bullet(E)$

$$\gamma_{(X,\Xi)}\cdot\Omega = \imath_X\Omega + \Xi\wedge\Omega$$

- (twisted) Differential on $\Omega^\bullet(E)$

$$d_H\Omega = d\Omega + H\wedge\Omega$$

Properties of the Courant bracket

For $A, B, C \in \Gamma(TE \oplus T^*E)$, $f \in C^\infty(E)$,

(a)

$$[\![A, B]\!] = -[\![B, A]\!]$$

(b)

$$\mathrm{Jac}(A, B, C) = [\![[\![A, B]\!], C]\!] + \mathrm{cycl} = d\mathrm{Nij}(A, B, C)$$

with

$$\mathrm{Nij}(A, B, C) = \tfrac{1}{3}\left(\langle [\![A, B]\!], C\rangle + \mathrm{cycl}\right)$$

(c)

$$[\![A, fB]\!] = f[\![A, B]\!] + (\rho(A)f)B - \langle A, B\rangle df$$

where $\rho : TE \oplus T^*E \to TE$ is the projection.

[Note that isotropic, involutive subbundles $A \subset TE \oplus T^*E$ (Dirac structures) give rise to Lie algebroids.]

Properties of the Courant bracket (cont'd)

(d) Symmetries of $\langle \cdot, \cdot \rangle$ are given by orthogonal group $O(TM \oplus T^*M) \cong O(d,d)$.
A particular kind of orthogonal transformation is the so-called B-field transform. For $b \in \Omega^2(E)$

$$e^b(X, \Xi) = (X, \Xi + \imath_X b)$$

We have

$$e^b [\![A, B]\!]_H = [\![e^b A, e^b B]\!]_{H+db}$$

Courant bracket as a derived bracket

We have the following 'Cartan formulas'

$$
\begin{aligned}
\{\gamma_{(X_1,\Xi_1)}, \gamma_{(X_2,\Xi_2)}\} &= 2\langle (X_1, \Xi_1), (X_2, \Xi_2)\rangle \\
\{d_H, \gamma_{(X,\Xi)}\} &= \mathcal{L}_{(X,\Xi)} \\
[\mathcal{L}_{(X_1,\Xi_1)}, \gamma_{(X_2,\Xi_2)}] &= \gamma_{(X_1,\Xi_1)\circ(X_2,\Xi_2)} \\
[\mathcal{L}_{(X_1,\Xi_1)}, \mathcal{L}_{(X_2,\Xi_2)}] &= \mathcal{L}_{(X_1,\Xi_1)\circ(X_2,\Xi_2)} = \mathcal{L}_{[\![(X_1,\Xi_1),(X_2,\Xi_2)]\!]}
\end{aligned}
$$

where

$$\mathcal{L}_{(X,\Xi)} \cdot \Omega = \mathcal{L}_X \Omega + (d\Xi + \imath_X H) \wedge \Omega$$

and the Dorfmann bracket is defined by

$$(X_1, \Xi_1) \circ (X_2, \Xi_2) = ([X_1, X_2], \mathcal{L}_{X_1}\Xi_2 - \imath_{X_2} d\Xi_1 + \imath_{X_1}\imath_{X_2} H)$$

T-duality for principal circle bundles

Given a principal circle bundle E with H-flux H

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \big\downarrow \pi \\ & & M \end{array} \qquad H = H_{(3)} + A \wedge H_{(2)}\,, \;\; F = dA$$

there exists a T-dual principal circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & \widehat{E} \\ & & \big\downarrow \widehat{\pi} \\ & & M \end{array} \qquad \widehat{H} = H_{(3)} + \widehat{A} \wedge F\,, \;\; \widehat{F} = H_{(2)} = d\widehat{A}$$

Theorem [Cavalcanti-Gualtieri]

(a) We have an isomorphism of differential complexes $\tau : (\Omega^\bullet(E)_{S^1}, d_H) \to (\Omega^\bullet(\widehat{E})_{S^1}, d_{\widehat{H}})$

$$\tau(\Omega_{(k)} + A \wedge \Omega_{(k-1)}) = -\Omega_{(k-1)} + \widehat{A} \wedge \Omega_{(k)}$$
$$\tau \circ d_H = -d_{\widehat{H}} \circ \tau$$

Hence, τ induces an isomorphism on twisted cohomology

(b) We can identify $(X, \Xi) \in \Gamma(TE \oplus T^*E)_{S^1}$ with a quadruple $(x, f; \xi, g)$

$$X = x + f\partial_A\,, \qquad \Xi = \xi + gA$$

and define a map $\phi : \Gamma(TE \oplus T^*E)_{S^1} \to \Gamma(T\widehat{E} \oplus T^*\widehat{E})_{S^1}$

$$\phi(x + f\partial_A + \xi + gA) = x + g\partial_{\widehat{A}} + \xi + f\widehat{A}$$

The map ϕ is orthogonal wrt pairing on $TE \oplus T^*E$, hence τ induces an isomorphism of Clifford algebras

Theorem (cont'd)

(c) For $(X, \Xi) \in \Gamma((TE \oplus T^*E)_{S^1})$ we have

$$\tau(\gamma_{(X,\Xi)} \cdot \Omega) = \gamma_{\phi(X,\Xi)} \cdot \tau(\Omega)$$

Hence τ induces an isomorphism of Clifford modules

(d) For $(X_i, \Xi_i) \in \Gamma((TE \oplus T^*E)_{S^1})$ we have

$$\phi([\![(X_1, \Xi_1), (X_2, \Xi_2)]\!]_H) = [\![\phi(X_1, \Xi_1), \phi(X_2, \Xi_2)]\!]_{\widehat{H}}$$

Hence ϕ gives a homomorphism of twisted Courant brackets

Theorem (cont'd)

(e) Generalized metric on $TE \oplus T^*E$

$$\mathcal{G} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}$$

Note that $\mathcal{G}^2 = 1$. We have

$$\begin{aligned} C_+ &= \mathrm{Ker}(\mathcal{G} - 1) = \{(X, (g+b)(X)),\ X \in \Gamma(TE)\} \\ &= \mathrm{graph}(g + b : TE \to T^*E)\,, \end{aligned}$$

The transformed generalized metric $\widehat{\mathcal{G}}$ is given by

$$\widehat{C}_+ = \mathrm{graph}(\widehat{g} + \widehat{b} : T\widehat{E} \to T^*\widehat{E})$$

where $(\widehat{g}, \widehat{b})$ are given by the Buscher rules.

Proof (sketch)

We have

$$d_H = \bar{d} + H_{(3)} + F\partial_A + A \wedge H_{(2)}$$

which proves

$$\tau \circ d_H = -d_{\widehat{H}} \circ \tau$$

The isomorphism of Clifford algebra and modules follows just as easily, and the statement on the Courant bracket follows from the Cartan formulas. □

Dimensionally reduced Courant bracket

$$
\begin{aligned}
&[\![(x_1, f_1; \xi_1, g_1), (x_2, f_2; \xi_2, g_2)]\!]_{F,H} = \\
&\qquad ([x_1, x_2], x_1(f_2) - x_2(f_1) + \imath_{x_1}\imath_{x_2}F; \\
&\qquad (\mathcal{L}_{x_1}\xi_2 - \mathcal{L}_{x_2}\xi_1) - \tfrac{1}{2}d(\imath_{x_1}\xi_2 - \imath_{x_2}\xi_1) + \imath_{x_1}\imath_{x_2}H_{(3)} \\
&\qquad + \tfrac{1}{2}(df_1 g_2 + f_2 dg_1 - f_1 dg_2 - df_2 g_1) \\
&\qquad + (g_2\imath_{x_1}F - g_1\imath_{x_2}F) + (f_2\imath_{x_1}H_{(2)} - f_1\imath_{x_2}H_{(2)}), \\
&\qquad\quad x_1(g_2) - x_2(g_1) + \imath_{x_1}\imath_{x_2}H_{(2)})
\end{aligned}
$$

Generalization to principal torus bundles

We have

$$H = H_{(3)} + A_i \wedge H^i_{(2)} + \tfrac{1}{2} A_i \wedge A_j \wedge H^{ij}_{(1)} + \tfrac{1}{6} A_i \wedge A_j \wedge A_k \wedge H^{ijk}_{(0)}$$

such that

$$\begin{aligned} d_H = & \bar{d} + H_{(3)} + F_{(2)i} \partial_{A_i} + \tfrac{1}{2} F_{(1)ij} \partial_{A_i} \wedge \partial_{A_j} + \tfrac{1}{6} F_{(0)ijk} \partial_{A_i} \wedge \partial_{A_j} \wedge \partial_{A_k} \\ & + A_i \wedge H^i_{(2)} + \tfrac{1}{2} A_i \wedge A_j \wedge H^{ij}_{(1)} + \tfrac{1}{6} A_i \wedge A_j \wedge A_k \wedge H^{ijk}_{(0)} \end{aligned}$$

The $F_{(1)ij}$ and $F_{(0)ijk}$ are known as nongeometric fluxes

Nongeometric fluxes

Let $\{e_a\}$ be a basis of $\Gamma(TE)$, such that $[e_a, e_b] = f_{ab}{}^c e_c$, and $\{e^a\}$ be a dual basis of $\Gamma(T^*E)$, then the Courant bracket can be expressed as

$$
\begin{aligned}
[\![e_a, e_b]\!] &= f_{ab}{}^c e_c + h_{abc} e^c \\
[\![e_a, e^b]\!] &= q^{bc}{}_a e_c - f_{ac}{}^b e^c \\
[\![e^a, e^b]\!] &= 0 r^{abc} e_c + q^{ab}{}_c e^c
\end{aligned}
$$

where $H = \frac{1}{6} h_{abc} e^a \wedge e^b \wedge e^c$.
Together with certain conditions on the structure constants this defines a Courant algebroid.

Theorem [Bouwknegt-Garretson-Kao]: T-duality provides an isomorphism of (certain) Courant algebroids.

THANK YOU FOR LISTENING !!

THANKS BOB AND GREG !!

THANKS JONATHAN !!