Proceedings

Ninth Annual Workshop
in Geometric Topology

The Colorado College
June 11-13, 1992

The Ninth Annual Workshop in Geometric Topology was held at The Colorado College in Colorado Springs, Colorado on June 11-13, 1992. The participants were:

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These proceedings contain the notes compiled by Fred Tinsley of two one-hour talks given by the principal speaker Mladen Bestvina, summaries of talks given by other participants, and a problem list compiled at the end of the workshop. The success of the workshop was due in large part to funding provided by the National Science Foundation (DMS-9101515) and The Colorado College. Both have supported these workshops in the past and we wish to express our gratitude for their continuing support.

Jim Henderson
Fred Tinsley

TABLE OF CONTENTS

Stable Actions on Real Trees

by

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Compiled from Notes
by
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Stable Actions on Real Trees

I. Introduction

Definition: T^{metric} is an $\mathbb{R}-tree$ if it is a countable union of finite metric trees.

General Question: What can be said about the structure of a group that acts by isometries on a given $\mathbb{R}-tree$ provided the vertex and edge stabilizers are understood?

Fact: In the case that T is simplicial, then the group is obtained from the vertex stabilizers by amalgamating and HNN-extending over the edge stabilizers.

Example 1: Theorem (Paulin): If Γ is negatively curved and $Out(\Gamma)\,(= Aut(\Gamma)/Inner(\Gamma))$ is infinite, then Γ admits a non-trivial action on an $\mathbb{R}-tree$ with all edge stabilizers virtually cyclic.

Example 2: If each of A, B has an infinite cyclic subgroup of infinite index, then $Out(A *_{\mathbb{Z}} B)$ is infinite.

Main Theorem (Bestvina-Feighn): Let G be a finitely presented group with a non-trivial, minimal stable action on an $\mathbb{R}-tree\ T$. Then either G splits over an extension E-by-cyclic for some edge stabilizer E, or else T is a line.

Corollary: Γ of Example 1 splits over a virtually cyclic group.

Rips developed the techniques necessary for the proof in:

Theorem (Rips): If a finitely presented group G acts freely on an $\mathbb{R}-tree$, then G is the free product of free abelian groups and surface groups.

These notes indicate a proof of Rips' theorem.

Definition: An action of G on an $\mathbb{R}-tree\ T$ is stable if for any collection of edges $I_1 \supset I_2 \supset \ldots$ in T with $\bigcap I_i = pt$, then there is an integer N so that $Fix(I_i) = Fix(I_{i+1})$ for all $i \geq N$. An action is non-trivial if no point is fixed by G. An action is minimal if no proper subtree is $G-invariant$.

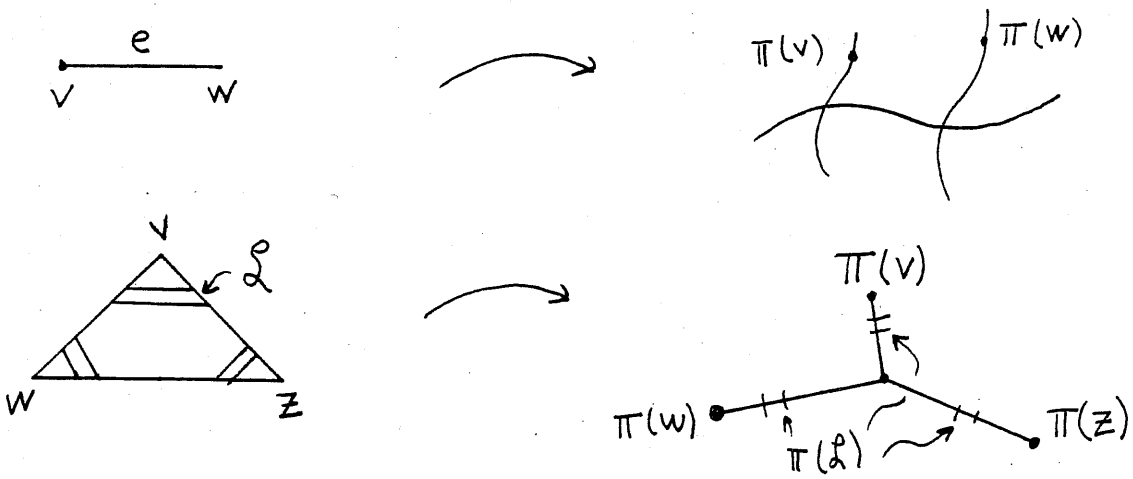
Example 3: Free actions are stable.

Example 4: A copy of $\mathbb{Q}$ cannot reside inside a negatively curved group. If G is negatively curved and the edge stabilizers are virtually cyclic, then the action is stable.

II: Laminations on 2-complexes and Resolutions

Suppose G is a finitely presented group which acts freely on an $\mathbb{R}-tree\ T$, K is a finite 2-complex with $\pi_1(K) = G$, and $\tilde{K}$ is the universal cover of K. Construct an equivariant map $\pi : \tilde{K} \to T$ as follows. Choose an equivariant set D, countably dense in T, that contains all vertices of T. First, define π on the vertices of $\tilde{K}$ equivariantly so $\pi(v) \in D$. Then, using a Cantor-type function, extend π to an edge e (with boundary v and w) so that $\pi^{-1}(d) = arc$ for all $d \in D \bigcap [\pi(v), \pi(w)]$. Here, $[\pi(v), \pi(w)]$ is the unique geodesic in T joining $\pi(v)$ and $\pi(w)$.

Finally, extend π naturally to each 2-simplex as indicated in the following diagram:



Locally, the leaves of the lamination, $\mathcal{L}$, are (Cantor set) $\times\, I$ (see previous diagram). $\mathcal{L}$ inherits a transverse measure from $\tilde{K}$ and is said to be a measured lamination. By construction, K inherits $\mathcal{L}$ from $\tilde{K}$.

The equivariant map π is called a resolution.

Key Fact: If the action is free, then

$$\pi_1(l) \to \pi_1(K) = G$$

is trivial for all leaves $l \in \mathcal{L}$ and also for any complimentary component. In particular, a loop in the complement of $\mathcal{L}$ represents a trivial element of G.

Example 5: Geodesic lamination on a hyperbolic surface. Here, $\pi_1(Surface)$ acts freely on an $\mathbb{R}$-tree.

Theorem (Morgan-Shalen): $\mathcal{L}$ can be represented as the union of finitely many clopen sublaminations. Each sublamination consists of either a parallel family of leaves, a twisted family of leaves, or a minimal piece (every leaf is dense).

The minimal case is the interesting case.

Definition: A *band* B is a set of the form $b \times I$ where b is an arc of the real line. The components $b \times \partial I$ are the *bases* of B.

Definition: A *band complex* is formed as follows:

(1) Start with a measured graph Γ

(2) Attach bands (laminated) to Γ along their bases in a measure preserving fashion.

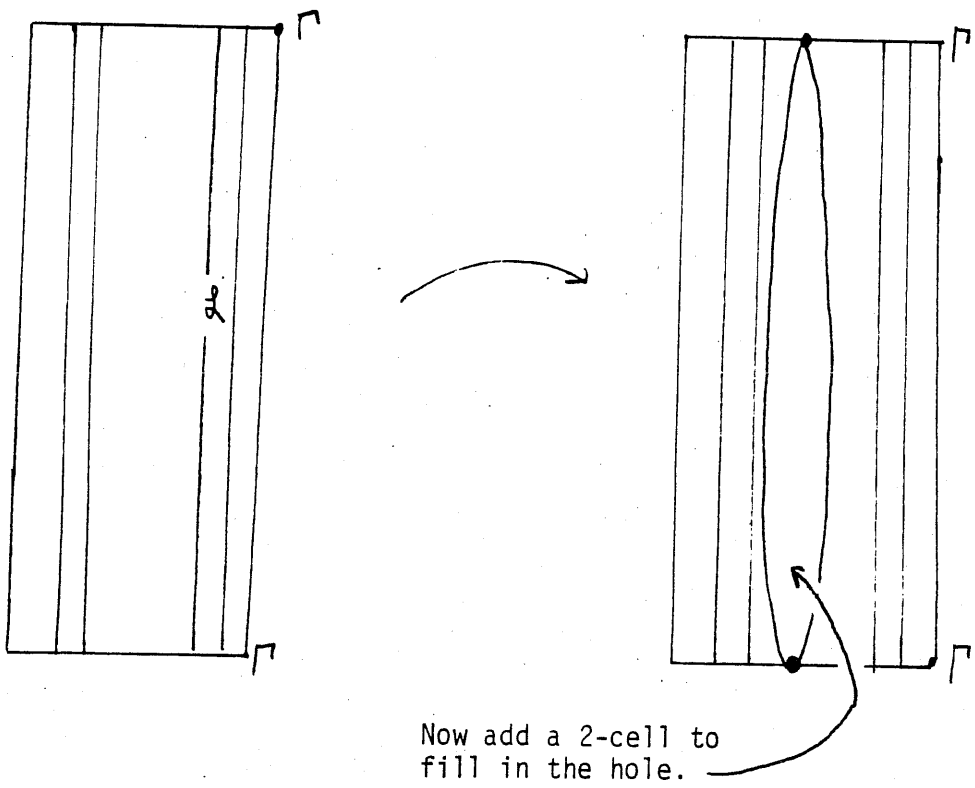
(3) Attach 1-cells and 2-cells with attaching regions disjoint from lamination.

Clearly, K, defined as above, may be given the structure of a band complex such that Γ is the disjoint union of its edges.

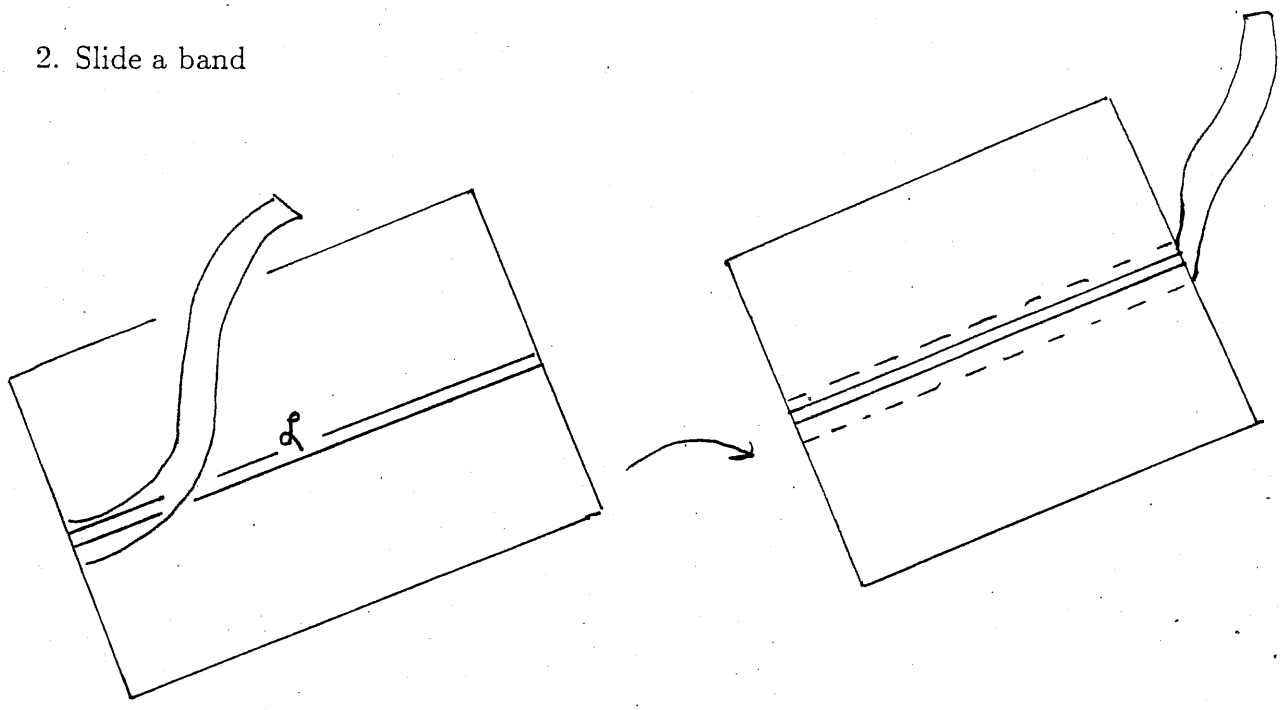
III. Moves

A band complex may be placed into a standard form via a machine consisting of two processes, each of which consists of a finite sequences of certain legal moves. The resulting complex has the same fundamental group as the original complex and its universal cover still admits a resolution to the original $\mathbb{R}$-tree.

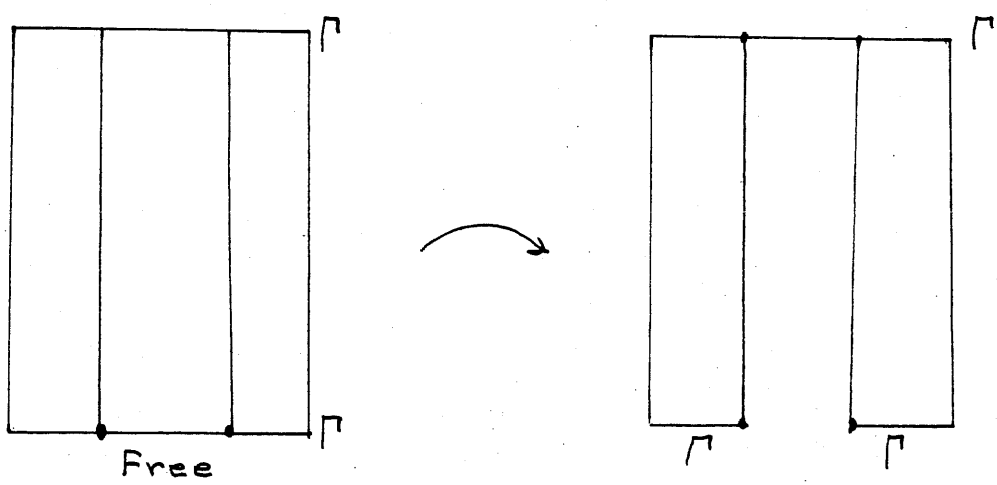
1. Split a band



2. Slide a band

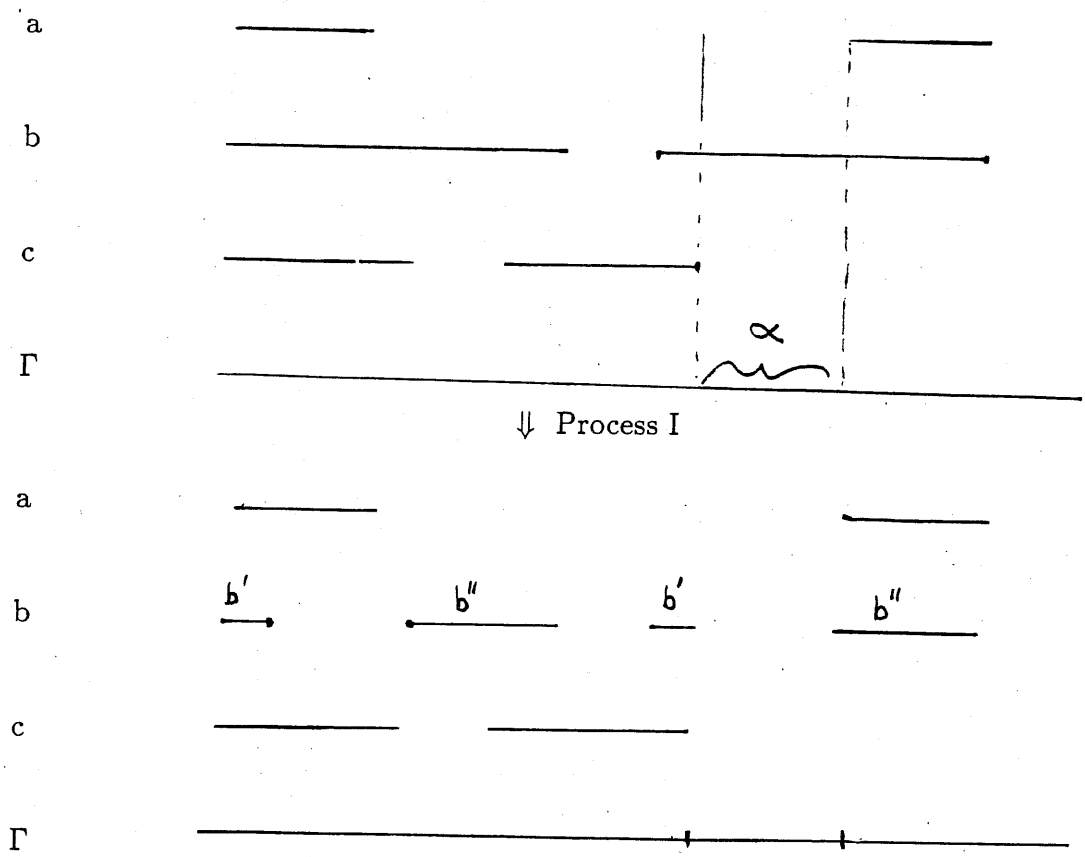


3. Collapse a band from a face free of other bands.



Process I: There is an $x \in \Gamma \bigcap \mathcal{L}$ that belongs only to one face. In this case collapse from a maximal free arc.

Example 6: In the following graphic three bands (a-c) are attached to Γ via vertical projection. Collapse through α to obtain the next stage:



Another free arc presents itself. In fact, for this example Process I would continue forever.

Process II: Each $x \in \Gamma \bigcap \mathcal{L}$ is covered by at least two faces. Orient Γ and order its components. Among all bases containing the initial point, let c be the longest one. Slide all other bases containing the initial point along c. Then collapse from the free arc c containing the initial point.

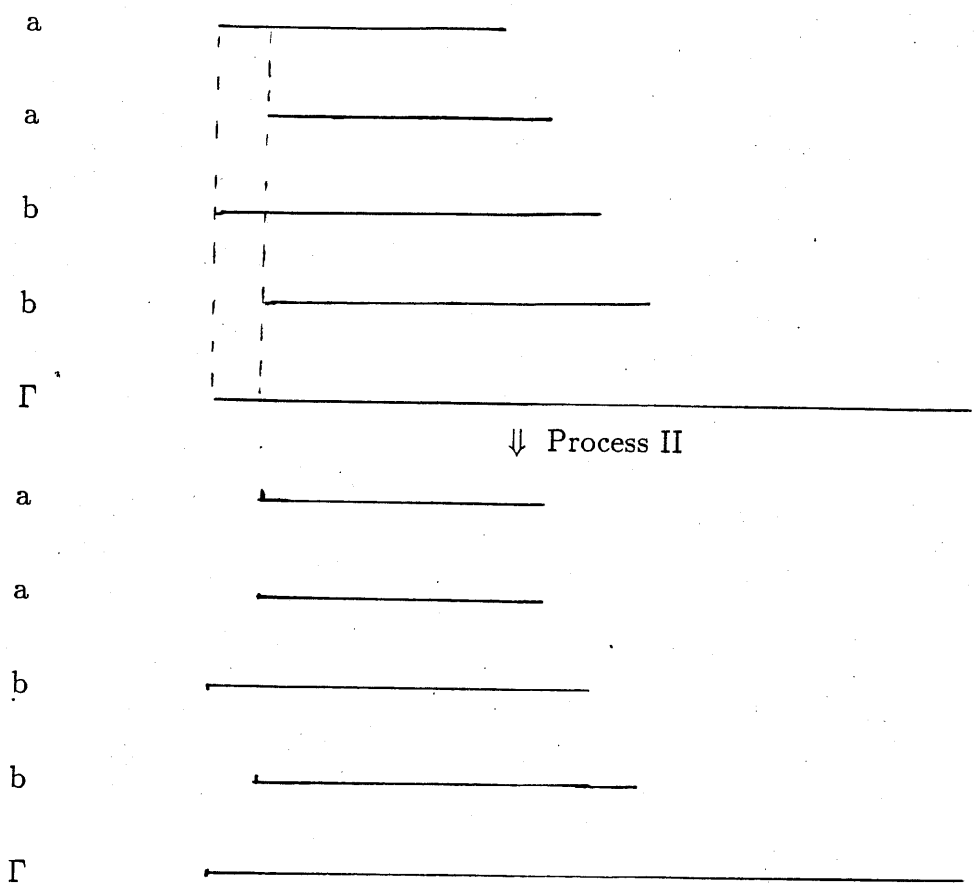
There are three cases:

Case 1: An attaching region slides onto its dual. Remove the band as it is a non-contributor to π_1.

Case 2: Process II may be applied again and some point of $\Gamma \bigcap \mathcal{L}$ belongs to 3 bases.

Case 3: Every point in $\Gamma \bigcap \mathcal{L}$ belongs to exactly 2 bases. This is the surface group case.

Example 7: In this example there are two bands attached to Γ. One base of band (a) slides along band (b) to line up exactly with the other base of band (a).



As a result, the sliding band, (a), becomes an annulus. The complex is simpler since this annulus does not contribute to π_1. Formal analysis requires a notion of complexity.

Complexity of a bandcomplex: A *block* is a component of the union of bases. Then

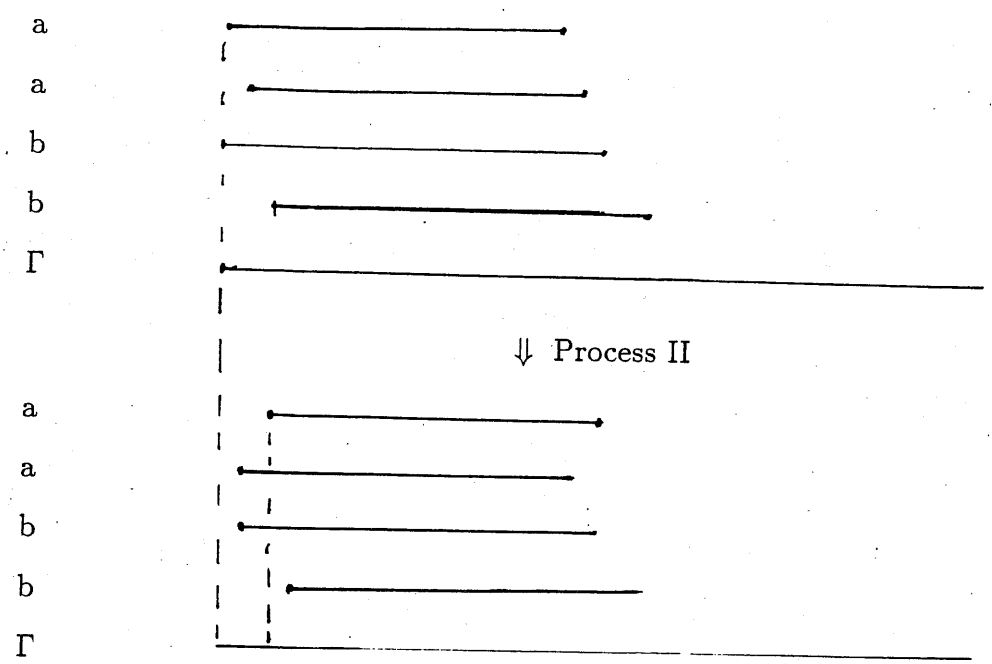
$$comp(\text{block}) = max\,(0, (\text{number of bases in the block-2})$$

and

$$comp(\text{band complex}) = \sum comp(\text{block})$$

In example 6 observe that the complexity (= 4) does not change. However, in example 7, the complexity decreases (from 4 to 2) when the annulus is removed.

Example 8: Again there are two bands attached to Γ. One base of band (a) again slides along base (b) but not precisely onto the other base. However, the overlap between the bases is large compared to the translation length. Here, Process II continues forever.



The Machine:

Apply Processes I and II as follows: If a free arc is present, apply process I; if not, apply Process II until complexity decreases or stabilizes (If complexity decreases, a free arc may present itself). Analysis of complexity shows that three possibilities result:

Possibility 1 (thin case): Process I continues forever.

Possibility 2 (Surface case): Process II continues and every point is covered exactly twice.

Possibility 3 (axial case): Process II continues forever and at least one point is covered three or more times.

Surface case: $G \cong H * F$ where H is the free product of surface groups and F is a free group. ($\Gamma \bigcup$ bands) is a surface with boundary which contains $\mathcal{L}$ in its interior. Since the boundary components are contained in the complement of $\mathcal{L}$, they are trivial in the complex. Thus, π_1(complex) is a surface group. There may also be arcs attached. Those attached to the same component of the complement of $\mathcal{L}$ do not contribute to π_1; the others give the free part of G. In this case, each leaf has the homotopy type of $\mathbb{R}$ with circles of bounded size attached.

Axial case: $G \cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) * F$ where F is a free group. Just check that when a and b are paths in $\mathcal{L}$, $aba^{-1}b^{-1}$ is a loop which actually lives in the complement of $\mathcal{L}$ and, thus, is trivial. In this case, the leaves contain arbitrarily large circles and, thus, are intrinsically high dimensional.

Thin case: G is free. If two bands attach to Γ in exactly the same way and intersect no other bands, consider them as a continuation.

Definition: $B_1, B_2, \cdots, B_n$ forms a *long band* if B_i, B_{i+1} are as above for $i = 1, \cdots, n-1$, $B_i \bigcap B_j = \emptyset$ for $|i-j| > 1$, $B_1 \bigcup B_n$ contains the intersection with other bands of $\bigcup B_i$, and the sequence is maximal with respect to these properties.

During each application of Process I, treat a long band as a single band. The idea is to adjust the band complex so that a band is free of attached two disks. Then cut the band and apply induction to obtain that G is free.

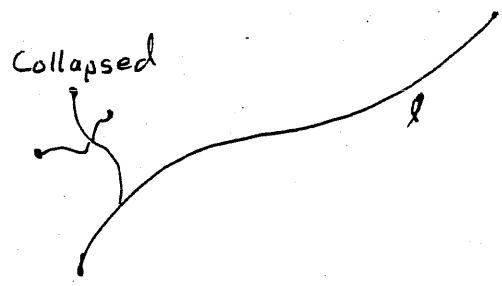
Outline of steps:

1. There is an N such that at each stage the band complex has less than or equal to N long bands.

2. For all n there exists a leaf segment of length n that is eventually collapsed.

3. There exists a leaf l that contains a ray every segment of which is eventually collapsed.

4. This leaf l is a one-ended tree and every finite subtree is eventually collapsed.



5. This l is dense; thus, all cell bases have diameters which go to zero and there are more and more of them.

6. Thus, eventually a band is naked, ie, free of attaching regions.

7. Cut along the naked band and induct on the number of generators, ie, $G \cong A * B$ where neither A_i is trivial.

Preservation of Absolute (Neighborhood) Retracts and of Soft Maps by Covariant Topological Functors

Taras Banakh

We say that a covariant functor $F: \mathcal{M}etr \longrightarrow \mathcal{T}op$ preserves

(i) *embeddings* provided for every embedding $e: X \longrightarrow Y$ with $X, Y \in \mathcal{OB}(\mathcal{C})$ the map $F(e): F(X) \longrightarrow F(Y)$ is an embedding as well;

(ii) *homotopies* iff for every homotopy $\{H_t: X \longrightarrow Y\}_{t\in[0,1]}$, where $X, Y \in \mathcal{OB}(\mathcal{C})$, the homotopy $\{F(H_t): F(X) \longrightarrow F(Y)\}_{t\in[0,1]}$ is continuous as a map $F(X)\times[0,1] \longrightarrow F(Y)$.

We denote by $Q=[-1,1]^{\omega}$ the Hilbert cube and by $l_2(A)=\{(x(a))_{a\in A}\in\mathbb{R}^A \mid \sum_{a\in A}|x(a)|^2<\infty\}$ the standard Hilbert space of density A.

By $A(N)R(\mathcal{M})$ and $A(N)E(\mathcal{M})$ we denote respectively the class of absolute (neighborhood) retracts and the class of absolute (neighborhood) extensors for the class of metric spaces.

Theorem 1. Let A be a cardinal and $F: \mathcal{M}etr \longrightarrow \mathcal{T}op$ be a functor that preserves embeddings and homotopies.

(i) If $F(l_2(A))\in AE(\mathcal{M})$ then $F(X)\in AE(\mathcal{M})$ for every absolute retract X with $dens(X)\le A$;

(ii) if $F(U)\in ANE(\mathcal{M})$ for every open set $U\subset l_2(A)$ then $F(X)\in ANE(\mathcal{M})$ for every absolute neighborhood retract X with $dens(X)\le A$.

Corollary 1. Let $F: \mathcal{M}etr \longrightarrow \mathcal{T}op$ be a functor that preserves embeddings and homotopies.

(i) If $F(Q)\in AE(\mathcal{M})$ then $F(X)\in AE(\mathcal{M})$ for every separable absolute retract X;

(ii) if $F(M)\in ANE(\mathcal{M})$ for every separable Q-manifold M then $F(X)\in ANE(\mathcal{M})$ for every separable absolute neighborhood retract X.

There is also a map version of the concept of an $A(N)E(\mathcal{M})$-space, namely, the concept of a (locally) soft map. A map $p: X \longrightarrow Y$ of topological spaces is defined to be (locally) soft, provided for every metric space Z, its closed subspace $Z_0\subset Z$ and maps $f: Z_0 \longrightarrow X$, $g: Z \longrightarrow Y$ with $p\circ f=g|Z_0$ there exists an extension $\bar{f}: U \longrightarrow X$ of f, where $U=Z$ ($U\supset Z_0$ is an open set in Z) such that $p\circ\bar{f}=g|U$.

Let a functor $F: \mathcal{M}etr \longrightarrow \mathcal{T}op$ preserves embeddings. We say that the functor F preserves *preimages* iff for every map $f: X \longrightarrow Y$ of metric spaces and every $A \subset Y$ we have $F(f)^{-1}(F(A)) = F(f^{-1}(A))$.

The following statements are map versions of the above results:

Theorem 2. Let A be a cardinal and $F: \mathcal{M}etr \longrightarrow \mathcal{T}op$ be a functor that preserves embeddings, preimages and homotopies. Let $pr_1: l_2(A) \times l_2(A) \longrightarrow l_2(A)$ be the natural projection onto the first factor.

(i) If $F(pr_1)$ is a soft map then the map $F(f)$ is soft for every soft map $f: X \longrightarrow Y$ of metric spaces with $dens(X) \leq A$;

(ii) if $F(pr_1|U)$ is a locally soft map for any open $U \subset l_2(A) \times l_2(A)$ then the map $F(f)$ is locally soft, provided $f: X \longrightarrow Y$ is a locally soft map of metric spaces with density $\leq A$.

Corollary 2. Let A be a cardinal and $F: \mathcal{M}etr \longrightarrow \mathcal{T}op$ be a functor that preserves embeddings, preimages and homotopies. Let $p_1: Q \times Q \longrightarrow Q$ be the natural projection onto the first factor.

(i) If $F(p_1)$ is a soft map then the map $F(f)$ is soft for every soft map $f: X \longrightarrow Y$ of separable metric spaces.

(ii) if $F(p_1|U)$ is a locally soft map for any open $U \subset Q \times Q$ then the map $F(f)$ is locally soft, provided $f: X \longrightarrow Y$ is a locally soft map of separable metric spaces.

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DETECTING HYPERHOPFIAN GROUPS

by
Robert J. Daverman

Definitions. A group G is <u>hopfian</u> if every epimorphism $\Psi:G \to G$ is an isomorphism; dually, G is <u>cohopfian</u> if every injection $\Phi:G \to G$ is an isomorphism. Expanding on this, we call a finitely presented group G <u>hyperhopfian</u> if every homomorphism $\psi:G \to G$ with $\psi(G)$ normal and $G/\psi(G)$ cyclic is an isomorphism.

Hyperhopficity is not an Abelian phenomenon; the property isn't even held by groups which split off a cyclic direct factor. Simple groups have it, as do the fundamental groups of all compact surfaces with negative Euler characteristic (a class which includes all finitely generated nonabelian free groups). Also, hyperhopfian groups are hopfian, by definition, but they are only partially cohopfian. The key hyperhopfian feature is: if $\mu:G \to \Gamma$ is an injection having normal image with $\Gamma/\mu(G)$ cyclic and if $\theta:\Gamma \to G$ is an epimorphism, then $\theta\mu:G \to G$ is an automorphism.

Call a closed manifold N <u>hopfian</u> if it is orientable and every degree one map $N \to N$ is a homotopy equivalence. Whether $\pi_1(N)$ a hopfian group necessarily makes N a hopfian manifold is part of a significant, old unsolved problem, due to Hopf and more recently reexamined by Hausmann. Nevertheless, this terminology aids in explaining the topological interest in hyperhopfian groups.

Theorem. Every closed, hopfian n-manifold N with hyperhopfian fundamental group is a codimension 2 fibrator (i.e., every closed map $p:M \to B$ from an orientable $(n+2)$-manifold M to a metric space B such that each $p^{-1}b$ is homeomorphic to N is an approximate fibration).

Last year's Workshop included a discussion of the following:

Contrasting Theorem. Every closed, hopfian n-manifold N with $\pi_1(N)$ hopfian and $\chi(N) \neq 0$ is a codimension 2 fibrator.

Below are listed some conditions implying a group is hyperhopfian. Proofs can be found in "Hyperhopfian groups and approximate fibrations", to appear in Compositio Mathematica.

Theorem. A finite group Γ isomorphic to the fundamental group of a closed 3-manifold is hyperhopfian if and only if Γ has no cyclic direct factor.

Theorem. Suppose the group G has a presentation consisting of s generators and t relations, $s>t+1$. Then G is hyperhopfian if and only if G is hopfian.

Corollary. Free groups on $s>1$ generators are hyperhopfian.

Corollary. Fundamental groups of closed surfaces S with $\chi(S)<0$ are hyperhopfian.

Theorem. If G_1, G_2 are nontrivial, finitely generated groups such that $G_1 * G_2$ is hopfian and $G_2 \neq Z_2$, then $G_1 * G_2$ is hyperhopfian.

Corollary. Suppose the hopfian group G is a nontrivial free product where $G \neq Z_2 * Z_2$, and suppose N^3 is a closed orientable 3-manifold with $\pi_1(N^3) \cong G$. Then N^3 is a codimension 2 fibrator.

ON THE HYPERSPACE OF ALL ANR'S IN THE PLANE

TADEUSZ DOBROWOLSKI, HELMA GLADDINES AND JAN VAN MILL

ABSTRACT. If X is a space then $\mathrm{ANR}(X)$ denotes the subspace of 2^X consisting of all ANR's. We prove that $\mathrm{ANR}(\mathbb{R}^2)$ is an absorber in $2^{[-1,1]^2}$ for the class of spaces that can be written as the difference of two absolute $F_{\sigma\delta}$'s. As a consequence, $\mathrm{ANR}(\mathbb{R}^2)$ is homeomorphic to $B^\infty \times (Q^\infty \setminus B^\infty)$, where Q denotes the Hilbert cube and B its pseudo-boundary.

Preliminary version presented by T. Dobrowolski

1. INTRODUCTION

For a space X, let 2^X and $C(X)$ denote the hyperspace of all nonempty subcompacta and nonempty subcontinua of X, respectively. It is known that, for a Peano continuum X, $2^X \approx Q$, where Q denotes the Hilbert cube $\prod_{n=1}^{\infty}[-1,1]_n$ (see [5]). By $\mathrm{ANR}(X)$, $\mathrm{ANR}_c(X)$ and $\mathrm{AR}(X)$ we denote the subspaces of 2^X and $C(X)$ consisting of all ANR's, all connected ANR's and AR's, respectively. It is natural to ask what is the topological structure of the spaces $\mathrm{ANR}(X)$, $\mathrm{ANR}_c(X)$ and $\mathrm{AR}(X)$, or more generally, of the pairs $(2^X, \mathrm{ANR}(X))$, $(C(X), \mathrm{ANR}_c(X))$ and $(C(X), \mathrm{AR}(X))$. We will be concerned only with the first pair and treat the case of $X = \mathbb{R}^2$.

Let B denote the pseudo-boundary of Q, i.e.,

$$B = \{x \in Q : (\exists i \in \mathbb{N})(|x_i| = 1)\}.$$

In [4], Cauty proved that the subspace $\{I \in C(\mathbb{R}^2) : I \text{ is an arc}\}$ is homeomorphic to B^∞ and in [7] Gladdines and van Mill proved that the subspaces $\{P \in C(\mathbb{R}^n) : P \text{ is a Peano continuum}\}$, $n \geq 3$, are also homeomorphic to B^∞.

Our main result is that the space $\mathrm{ANR}(\mathbb{R}^2)$ is homeomorphic to

$$\hat{B} = B^\infty \times (Q^\infty \setminus B^\infty).$$

We use the theory of absorbing sets in the Hilbert cube and some ideas from [7]; see also [6]. In fact, we prove that $\mathrm{ANR}([-1,1]^2)$ is an absorber for the class of all spaces X that can be written as $A \setminus B$, where A and B are absolute $F_{\sigma\delta}$'s. Our main result then follows easily.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

2. Absorbing sets apparatus

All spaces under discussion are separable and metrizable. Any space that is homeomorphic to Q is called a **Hilbert cube**.

Let A be a closed subset of a space X. We say that A is a **Z-set** provided that every map $f: Q \to X$ can be approximated arbitrarily closely by a map $g: Q \to X \setminus A$. A countable union of Z-sets is called a **σZ-set**. A **Z-embedding** is an embedding the range of which is a Z-set.

Let $\mathcal{M}$ be a class of spaces that is topological and closed hereditary. A subset $A \subseteq X$ is called **strongly $\mathcal{M}$-universal** in X if for every $M \in \mathcal{M}$ with $M \subseteq Q$, every embedding $f: Q \to X$ that restricts to a Z-embedding on some compact subset K of Q, can be approximated arbitrarily closely by a Z-embedding $g: Q \to X$ such that $g \mid K = f \mid K$ while moreover $g^{-1}[A] \setminus K = M \setminus K$.

Let X be a Hilbert cube. A subset $A \subset X$ is called an **$\mathcal{M}$-absorber** in X if:

(1) $A \in \mathcal{M}$;
(2) there is a σZ-set $S \subset X$ with $A \subseteq S$;
(3) A is strongly $\mathcal{M}$-universal in X.

We use the following version of the Uniqueness Theorem on absorbers proved in [6].

A. *Let X be a Hilbert cube and let A and B be $\mathcal{M}$-absorbers for X. Then there is a homeomorphism $h: X \to X$ with $h[A] = B$. Moreover, h can be chosen arbitrarily close to the identity.*

Absorbers for the class F_σ of all σ-compact spaces were first constructed by Anderson and Bessaga and Pełczyński. A basic example of such an absorber in Q is B. For details, see [1] and [9, Chapter 6]. The space B^∞ in Q^∞ is an absorber for the Borel class $F_{\sigma\delta}$. This was shown in [2]; see also [6].

Let Γ denote the class of all spaces X that are homeomorphic to $A \setminus B$, for certain $F_{\sigma\delta}$-subsets A and B of Q.

B. *The set $\hat{B} = B^\infty \times (Q^\infty \setminus B^\infty)$ is a Γ-absorber in $\hat{Q} = Q^\infty \times Q^\infty$.*

Applying the Uniqueness Theorem on absorbers we get the folowing corollary.

C. *Let X be a Hilbert cube and let A be an absorber in X for the class Γ. Then there is a homeomorphism of pairs*

$$(\hat{Q}, \hat{B}) \approx (X, A).$$

In particular, A is homeomorphic to $\hat{B}$.

3. Main result

First we will determine the Borel type of the space $\mathrm{ANR}(\mathbb{R}^2)$. The result that makes this possible is the following characterization of plane ANR's due to Borsuk [3]: a compact subset $X \subseteq \mathbb{R}^2$ is an ANR if and only if X is locally connected and $\mathbb{R}^2 \setminus X$ has finitely many components only.

Let D and E denote $[-2,2]^2$ and $(-2,2)^2$, respectively. Observe that 2^E is an open subspace of 2^D and that $2^D \setminus 2^E$ is a Z-set in 2^D.

For each $n \in \mathbb{N}$, put

$$\mathcal{E}_n = \{A \in 2^E : E \setminus A \text{ has at most } n \text{ components}\}.$$

In addition, let $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$.

Lemma. *$\mathcal{E}$ is a $G_{\delta\sigma}$-subset of 2^E.*

Here is our key result which allows us verify the strong Γ universality.

Proposition 1. *If $A \subseteq Q$ is an $F_{\sigma\delta}$-subset then for every nonempty open rectangle $R \subset E$ there is an embedding $f: Q \to 2^R \subseteq 2^E$ such that*

(1) *If $x \notin A$ then $f(x)$ is an* ANR.
(2) $f^{-1}[\mathcal{E}] = Q \setminus A$.

Let $\mathcal{L}$ denote the subspace of 2^E consisting of all nonempty locally connected subcompacta of E. It is easy to see that $\mathcal{L}$ is an $F_{\sigma\delta}$-subset of 2^X ([8]; see also [7]). By the above cited result of Borsuk, $\mathrm{ANR}(E) = \mathcal{E} \cap \mathcal{L}$; this implies that $\mathrm{ANR}(E) \in \Gamma$. We can do better.

Proposition 2. *Let $A \subseteq Q$ be in Γ. Then there exists an embedding $f: Q \to 2^E$ such that $f^{-1}[\mathrm{ANR}(E)] = A$.*

Now, with the use of Proposition 2 we are able to repeat an argument of [7, Theorem 5.1] to show the strong Γ–universality of $\mathrm{ANR}(E)$ in 2^D. The fact that $\mathrm{ANR}(E)$ is contained in a σZ-set in 2^D can be easily shown. Since $\mathrm{ANR}(E) \in \Gamma$, our main result follows.

Theorem. $\mathrm{ANR}(E)$ *is a Γ-absorber in 2^D.*

Corollary. $\mathrm{ANR}(\mathbb{R}^2)$ *is homeomorphic to $\hat{B}$.*

References

[1] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite–Dimensional Topology*, PWN, Warszawa, 1975.

[2] M. Bestvina and J. Mogilski, *Characterizing certain incomplete infinite-dimensional absolute retracts*, Michigan Math. J. **33** (1986), 291–313.

[3] K. Borsuk, *Theory of Retracts*, PWN, Warszawa, 1967.

[4] R. Cauty, *L'espace des arcs d'une surface*, Trans. Amer. Math. Soc. **332** (1992), 193–209.

[5] D. W. Curtis and R. M. Schori, *Hyperspaces of Peano continua are Hilbert cubes*, Fund. Math. **101** (1978), 19–38.

[6] J. J. Dijkstra, J. van Mill, and J. Mogilski, *The space of infinite-dimensional compact spaces and other topological copies of $(\ell_f^2)^\omega$*, Pac. J. Math. **152** (1992), 255–273.

[7] H. Gladdines and J. van Mill., *Hyperspaces of Peano continua of euclidean spaces*, submitted, Fund. Math..

[8] K. Kuratowski, *Evalution de la classe borélienne ou projective d'un ensemble de points à l'aide des symbols logiques.*, Fund. Math. **17** (1931), 249–272.

[9] J. van Mill, *Infinite-Dimensional Topology: prerequisites and introduction*, North-Holland Publishing Company, Amsterdam, 1989.

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This is the summary of the talk I gave at the Topology workshop at Colorado College in June, 1992.

Enhanced cohomology and obstruction theory

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This talk was about my PhD thesis. I would like to thank my advisor Professor L.S.Husch for his help.

Suppose (K,A) is a simplicial pair, $f:A\to Y$ - a map, and the question is: can one extend f to $\bar{f}:K\to Y$? The classical obstruction theory attempts to answer this inductively on the skeleta of K. An extension $f^{m-1}:K^{[m-1]}\cup A\to Y$ yields a class $[c^m]\in H^m(K,A;\pi_{m-1}(Y))$, which is 0 iff $f^{m-1}\big|_{K^{[m-2]}\cup A}$ extends to a map $f^m:K^{[m]}\cup A\to Y$.

The next step entails looking at $[c^{m+1}]\in H^{m+1}(K,A;\pi_m(Y))$; the trouble is that if this is not 0, we have to redo f^m and try for $[c^{m+1}]=0$ again.

In a simple case $Y=S^{n-1}$, the above process can be restated geometrically as follows.

$f:A\to S^{n-1}$ extends to $f^{n-1}:K^{[n-1]}\cup A\to S^{n-1}=\mathrm{bd}D^n$. Now extend f^{n-1} to $g^n:K^{[n]}\cup A\to D^n$ in general position with respect to the centre 0 of D^n. The preimage of 0 is then finite and we can look at small n-1-spheres around these points and see them mapped to $D^n\setminus 0\simeq S^{n-1}$ with degree ± 1. Adding those degrees for the points within an n-simplex gives the value of c^n on that simplex. That $[c^n]=0$ means we can homotope g^n rel. $K^{[n-2]}\cup A$ to cancel the preimage points. $H^n(K,A)$ is now translated geometrically in a way that involves sets of isolated points in $K^{[n]}\setminus K^{[n-1]}$, each equipped

with a number 1 or -1. Coboundaries are reflected as certain operations on such sets of points; they show what can happen when g^n is homotoped rel. $K^{[n-2]} \cup A$.

What we just described is primary obstruction at work; for reasons mentioned before secondary cohomology obstruction is not well-defined. In order to capture the first 2 non-trivial extension steps, even for S^{n-1} as the range, in a single well-defined obstruction, we need a different set of functors, called enhanced cohomology.

When we extend f^{n-1} to $g^{n+1}:K^{[n+1]} \cup A \to D^n$, we see a 1-dimensional preimage of 0. In each $n+r$-cell of K, $r=0,1$, it is a PL proper r-submanifold. We can then take a nice regular neighborhood of the preimage in $K^{[n+1]}$ and its boundary is mapped to S^{n-1} with degree (in a suitable sense) ± 1. The preimage together with this map constitutes an "enhanced cochain". Some of these are called enhanced cocycles, and what can happen to the preimage and the map during a homotopy of g^n is reflected in enhanced coboundaries. Cochains can be added in the set-theoretic manner up to cohomology class; if the 2 terms we want to add intersect, we first isotope them apart.

In general, for any $n \geq 4$, and any finite regular CW-complex K, we obtain an abelian group, $EH^n(K)$, called the n-th enhanced cohomology group of K. These are contravariant functors (using continuous maps between CW-complexes). In fact, if $f \simeq g$, then $f^* = g^*$, so if K and L are homotopy equivalent, then $EH^*(K) = EH^*(L)$. We have an exact sequence

$$H^{n-1}(K,\mathbb{Z}) \xrightarrow{Sq^2} H^{n+1}(K,\mathbb{Z}_2) \xrightarrow{I} EH^n(K) \xrightarrow{F} H^n(K) \to 0$$

(the role of Sq^2 was suggested to me by J.Stasheff and M.Bestvina). In particular,

$$EH^{n-1}(\text{closed, 2-connected PL } n\text{-manifold, } n\geq 5)=\mathbb{Z}_2,$$

so $EH^* \neq H^*$ (any coefficients) for finite CW-complexes.

The enhanced cohomology groups can be computed by a finite algorithm.

They provide a well-defined, necessary and sufficient obstruction to extend a map from a subpolyhedron A of K to $K^{[n+1]} \cup A$ to some n-2-connected spaces as the range. They provide a well-defined, necessary and sufficient obstruction to embed an n-dimensional polyhedron in $\mathbb{R}^{2n-1}$, $n\geq 5$ (and also $\mathbb{R}^{2n}$, where our obstruction reduces to the Shapiro-Wu obstruction).

ON MULTIPLE HOMOGENEITY OF PRODUCTS OF MENGER SPACES

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Section 1. Introduction and Background.

This paper is based on a talk given at the Workshop in Geometric Topology held at Colorado College in June 1992. The results outlined in Section Three will be submitted for publication elsewhere.

We begin with definitions of the Menger Spaces. These spaces were originally defined by Menger in 1932 [Mg]. An inductive definition is as follows. Let M_n^0 be $I^{2n+1} \subset \mathbb{R}^{2n+1}$. Inductively assume that M_n^k is a union of $(2n+1)$–dimensional cells with sides of length $(1/3)^k$. Subdivide each cell in M_n^k into 3^{2n+1} smaller cells by subdividing each side in thirds. Then M_n^{k+1} is the union of all of these smaller cells that intersect the k–skeleton of M_n^k. The space μ_n is then defined as $\bigcap_{i=0}^{\infty} M_n^i$. Figure 1 below shows the spaces M_1^1 and M_1^2.

Figure 1. The Spaces M_1^1 and M_1^2

Key words and phrases. Menger Space, Universal Curve, Homogeneity.

The zero-dimensional Menger space, μ_0, is the Cantor Set. The one-dimensional Menger space, μ_1 is the universal curve characterized by R.D. Anderson in [An1] and [An2]. In 1984, M. Bestvina characterized all the remaining Menger Spaces [Be]. We take this characterization by Bestvina as our formal definition of the Menger Spaces.

Definition 1.1 [Be]. The Menger universal n-dimensional space μ_n is the unique space satisfying the following conditions:

(1) μ_n is a compact n-dimensional metric space.

(2) μ_n is locally $(n-1)$–connected (LC^{n-1}).

(3) μ_n is $(n-1)$–connected (C^{n-1}).

(4) μ_n satisfies the Disjoint n-cells Property (DD^nP).

Definition 1.2. A space X is *k–connected* if every map of S^k into X extends to a map of B^{k+1} into X. A space X is *locally k–connected* if for each point $p \in X$ and for each neighborhood U of p, there exists a neighborhood V of p so that each map of S^k into V extends to a map of B^{k+1} into U. A space X satisfies the *Disjoint k-Cells Property* if for each $\epsilon > 0$ and for each pair of maps f_1 and f_2 from I^k into X, there are maps g_1 and g_2 from I^k into X with $g_1(I^k) \cap g_2(I^k) = \emptyset$ and $d(g_i, f_i) < \epsilon$.

These conditions in Definition 1.1 yield the result that μ_n is a universal n–dimensional separable metric space, i.e. μ_n is n-dimensional and contains a copy of every separable metric n–dimensional space. The details are given in [Be]. We are interested in the homogeneity properties of Menger Spaces and of products of Menger Spaces. The relevant definitions are provided next.

Definition 1.3. A space X is *homogeneous* if and only if for each pair of points p and q in X, there is a homeomorphism $h : X \to X$ with the property that $h(p) = q$. A space X is *n–homogeneous* if for each pair of n–point subsets of X, A and B, there is a homeomorphism $h : X \to X$ with the property that $h(A) = B$. A space X is *countable dense homogeneous* if and only if for each pair of countable dense subsets A and B of X, there is a homeomorphism $h : X \to X$ with the property that $h(A) = B$.

It is well known that the Cantor set (μ_0), the Hilbert Cube, and all manifolds satisfy these types of homogeneity. R. D. Anderson established that μ_1 also satisfies these types

of homogeneity [An1, An2]. M. Bestvina established the analogous results for the higher dimensional Menger spaces.

Theorem 1.4 [Be, pg. 73]. *Each Menger Space μ_n is k–homogeneous for each k, and is countable dense homogeneous.*

. In 1980, K. Kuperberg, W. Kuperberg and W. R. R. Transue showed that $\mu_1 \times \mu_1$ was not 2-homogeneous [KKT]. (They also showed that $\mu_1 \times S^1$ was not 2-homogeneous.) Their results depended on certain one–dimensional facts that do not generalize to higher dimensions. We outline the results in [KKT] is Section 2. In section 3, we sketch how replacing the one–dimensional arguments in [KKT] with higher dimensional Čech Homology arguments allows us to generalize the results in [KKT]. In particular, we are able to show that $\mu_m \times \mu_n$ is not 2–homogeneous for all values of n and m where $\max\{m, n\} \geq 1$.

Section 2. Non 2–homogeneity of $\mu_1 \times \mu_1$.

This section contains an outline of some of the results in [KKT].

Theorem 2.1 [KKT]. *$\mu_1 \times \mu_1$ is not 2–homogeneous.*

This result depends on the following result of M. L. Curtis and M. K. Fort.

Lemma 2.2 [CK]. *If X is a 1–dimensional space and if $f : S^1 \to X$ is an inessential loop in X, then f is inessential in $f(S^1)$.*

This lemma becomes false if S^1 is replaced by S^2 and one–dimensional is replaced by 2–dimensional. For example, consider the CW pair $(X, \mathbb{P}^2)$ where X is obtained from $\mathbb{P}^2$ by attaching a 2–cell via a map that takes S^1 to a generator of $\pi_1(\mathbb{P}^2)$. Here $\mathbb{P}^2$ is real projective 2–space. The natural quotient map from S^2 onto $\mathbb{P}^2$ is essential in $\mathbb{P}^2$, but is inessential in X.

This lemma is used to show the following result:

Lemma 2.3 [KKT]. *If X is a one–dimensional continuum and if f_1 and f_2 are two essential loops in X with disjoint images, then f_1 and f_2 are not homotopic.*

Sketch of Proof.

Lemma 2.3 is established by assuming that f_1 and f_2 are homotopic and considering the quotient space $Y \equiv X/f_2(S^1)$. The space Y is one–dimensional. If the map $p \circ f_1$ is

inessential in Y, Lemma 2.2 implies that it is inessential in its image. This contradicts the fact that f_1 is essential in X. ■

The non 2–homogeneity of $\mu_1 \times \mu_1$ occurs as a result of the product structure of any self homeomorphism $\mu_1 \times \mu_1$. Specifically, the non 2–homogeneity follows from:

Theorem 2.4 [KKT]. *If h is any self homeomorphism of $X = \mu_1 \times \mu_1$, then one of the following holds:*

(1) *there are homeomorphisms h_1 and h_2 of μ_1 such that for every point $x = (x_1, x_2)$ of X , $h(x) = (h_1(x_1), h_2(x_2))$.*

(2) *there are homeomorphisms h_1 and h_2 of μ_1 such that for every point $x = (x_1, x_2)$ of X , $h(x) = (h_1(x_2), h_2(x_1))$.*

To see how Theorem 2.1 follows from this, choose points $x_1 = (a, b)$, $x_2 = (a, c)$, and $x_3 = (d, e)$ in X, where a, b, c, d and e are distinct. Then the above theorem immediately implies that there is no self homeomorphism h so that $h(\{x_1, x_2\}) = \{x_1, x_3\}$.

For completeness, and to prepare for the outline in section 3, we provide a sketch of the proof of Theorem 2.4 above.

Sketch of Proof of Theorem 2.4 .

Let h be any homeomorphism as in the statement of the theorem. We first show that for any $a \in \mu_n$, $h(\{\{a\} \times \mu_n\})$ is contained in either a horizontal or vertical slice of X. If not, there are points (a, b) and (a, c) in X with $h(a, b) = (\alpha, \beta)$ and $h(a, c) = (\gamma, \delta)$ where $\alpha \neq \gamma$ and $\beta \neq \delta$. Use uniform continuity to choose an embedding e of S^1 into μ_1 so that

$$p_1 \circ h \circ e_1(S^1) \cap p_1 \circ h \circ e_2(S^1) = \emptyset \text{ and } p_2 \circ h \circ e_1(S^1) \cap p_2 \circ h \circ e_2(S^1) = \emptyset.$$

Here, e_1 is the map into X with first coordinate map e and second coordinate map the constant map to $\{b\}$, e_2 is the map into X with first coordinate map e and second coordinate map the constant map to $\{c\}$, p_1 is projection onto the first coordinate, and p_2 is projection onto the second coordinate.

Both e_1 and e_2 are essential, since e is. The maps e_1 and e_2 are homotopic since μ_1 is path connected. So $h \circ e_1$ and $h \circ e_2$ are homotopic and essential. So at least one of $p_1 \circ h \circ e_1$ and $p_2 \circ h \circ e_1$ must be essential. Assume $p_1 \circ h \circ e_1$ is essential. Then the maps $p_1 \circ h \circ e_1$ and $p_1 \circ h \circ e_2$ contradict Lemma 2.3 since they are homotopic.

Thus, for any $a \in \mu_n$, $h(\{\{a\} \times \mu_n\})$ is contained in either a horizontal or vertical slice of X. Without loss of generality, assume $h(\{\{a\} \times \mu_n\})$ is contained in a vertical slice. A similar argument then shows that for each $p \in \mu_n$, $h(\{\mu_n \times \{p\}\})$ is contained in a horizontal slice. This completes the sketch of the proof. ■

Section 3. Non 2–homogeneity of $\mu_m \times \mu_n$.

To generalize the results in Section 2, the following Theorem is needed. The proof of this Theorem can be found in [Ga].

Theorem 3.1. *For each μ_n, the following results hold:*

(1) *Any embedding of S^n in μ_n is essential both with respect to homotopy and with respect to n-thČech homology.*

(2) *If f_1 and f_2 are any maps from S^n into μ_n that are essential with respect to n-th Čech homology, and if f_1 and f_2 have disjoint images, then f_1 and f_2 are not homotopic.*

(3) *For each $\epsilon > 0$ and for each point $p \in \mu_n$, there is an embedding $f : S^n \to \mu_n$ with image contained in the ϵ neighborhood of p.*

(4) *If $f : S^n \to \mu_n \times \mu_m$, with $m > n$, is a map that is essential with respect to n-th Čech homology, then $p_1 \circ f$ is essential with respect to n-th Čech homology, where p_1 is projection onto the first coordinate.*

(5) *If $f : S^m \to \mu_n \times \mu_m$, with $m > n$, is a map that is essential with respect to m-th Čech homology, then $p_2 \circ f$ is essential with respect to n-th Čech homology, where p_2 is projection onto the second coordinate.*

The main Theorem in [Ga] is the following. The exceptional case of $\mu_0 \times \mu_0$ fails because μ_0 is not path–connected.

Theorem 3.2. *$\mu_m \times \mu_n$ for $\max\{m, n\} \geq 1$ is not 2–homogeneous.*

The case where $m > n$ follows from Theorem 3.3 below just as Theorem 2.1 followed from Theorem 2.4. The general case needs a separate analysis when $n = m$. The complete details on the proof of the following theorem can be found in [Ga]. We sketch a partial proof of this Theorem.

Theorem 3.3. *If h is any self homeomorphism of* $X = \mu_n \times \mu_m$ *, where* $m > n$*, then there are homeomorphisms* h_1 *and* h_2 *of* μ_n *and* μ_m *such that for every point* $x = (x_1, x_2)$ *of* X *,* $h(x) = (h_1(x_1), h_2(x_2))$.

Sketch of Proof. Let h be any homeomorphism of X as in the statement of the Theorem, and let $\{p\}$ be any point of μ_n. Assume that there are points q and r of μ_m so that $h(p,q) = (\alpha, \beta)$ and $h(p,r) = (\gamma, \delta)$ where $\alpha \neq \gamma$. Use uniform continuity to choose an embedding e of S^n into μ_n so that

$$p_1 \circ h \circ e_1(S^n) \cap p_1 \circ h \circ e_2(S^n) = \emptyset.$$

Here, e_1 is the map into X with first coordinate map e and second coordinate map the constant map to $\{q\}$, e_2 is the map into X with first coordinate map e and second coordinate map the constant map to $\{r\}$, and p_1 is projection onto the first coordinate.

Theorem 3.1 implies that e is essential with respect to n-th Čech homology, so both e_1 and e_2 must also be essential with respect to n-th Čech homology. Also, since $\max\{m, n\} \geq 1$, The maps e_1 and e_2 are homotopic. Now consider the maps $h \circ e_1$ and $h \circ e_2$. Both are essential with respect to n-th Čech homology. It follows from Theorem 3.1 that both $p_1 \circ h \circ e_1$ and $p_1 \circ h \circ e_2$ are essential with respect to n-th Čech homology. Since these maps are homotopic, this contradicts condition 2 in Theorem 3.1. So h takes each vertical slice of X into a vertical slice.

A similar argument shows that h takes each horizontal slice into a horizontal slice. This completes the sketch of the proof. ∎

References

[An1] R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. **68** (1958), 313–324.

[An2] ———, *1-dimensional continuous curves and a homogeneity theorem*, Ann. of Math. **68** (1958), 1–16.

[Be] M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Memoirs Amer. Math. Soc. **71** (1988), no. 380.

[CF] M. L. Curtis and M. K. Fort, *The fundamental group of one-dimensional spaces*, Proc. Amer. Math. Soc. **10** (1959), 140–148.

[Ga] D. J. Garity, *Products of Menger spaces are not 2-homogeneous*, in preparation.

[KKT] K. Kuperberg, W. Kuperberg, and W. R. R. Transue, *On the 2-homogeneity of Cartesian products*, Fund. Math. **CX** (1980), 131–134.

[Mg] K. Menger, *Kurventheorie*, Teubner, Berlin–Leipzig, 1932.

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Complements of Globally 1-alg 2-spheres in 4-manifolds

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Let X^n be an n-manifold and $A \subset X^n$. A is *globally 1-alg* in X if for any neighborhood U of A there is a neighborhood V of A, $V \subset U$, so that loops which are null-homologous in $V-A$ are null-homotopic in $U-A$. This condition has proven to be valuable for studying complements of certain embeddings. For example, if $\Sigma^k \subset S^n$ is an embedded k-sphere (or shape k-sphere) with $k \leq n-3$, then $S^n-\Sigma^k \approx S^n-S^k$ iff Σ^k is globally 1-alg (see [Du] and [Ve]). Analogous results, but with knotting taken into consideration, are known when $k = n-2$. One example, due to Liem and Venema, is the following:

THEOREM ([L-V_1]) *Let $\Sigma^2 \subset S^4$ be an embedded shape 2-sphere. Then $S^4-\Sigma^2 \approx S^4-K^2$ for some locally flat 2-sphere K^2, or equivalently, Σ^2 has a neighborhood $N \approx S^2 \times D^2$ with $N-\Sigma^2 \approx (S^2 \times S^1) \times [0,1)$ iff Σ^2 is globally 1-alg in S^4.*

In [L-V_2] the following question is raised: If $\Sigma^2 \subset X^4$ is a globally 1-alg shape 2-sphere in a 4-manifold, does there exist a locally flat 2-sphere $K^2 \subset X^4$ with $X^4-\Sigma^2 \approx X^4-K^2$? Equivalently, one may ask whether every globally 1-alg shape 2-sphere Σ^2 in a 4-manifold X^4 has a neighborhood N homeomorphic to a disk bundle D over S^2 with $N-\Sigma^2$ homeomorphic to $D-S^2_0$ where S^2_0 is the 0-section of D. A weaker version simply asks whether the end of $X^4-\Sigma^2$ must be collarable.

If $\Sigma^2 \subset X^4$ is a globally *1-alg* 2-sphere and X^4 is compact, it follows from duality arguments that $X^4-\Sigma^2$ has a single end with (stable) cyclic fundamental group. In case this group is infinite cyclic, Liem and Venema have answered the above questions affirmatively, hence, we are led to the finite cyclic case. Moreover, [L-V_2] shows that under these circumstances the end of

$X^4-\Sigma^2$ satisfies all hypotheses of Siebenmann's (see [Si]) high dimensional collaring theorem. Remarkably, Kwasik and Schultz [K-S] have shown the existence of counterexamples to a 4-dimensional version of Siebenmann's theorem. Furthermore, many of their examples have ends with finite cyclic fundamental group. Hence, we are led naturally to the following:

<u>Question</u>. *Can any of the Kwasik-Schultz counterexamples to a 4-dimensional version of Siebenmann's collaring theorm be realized as a shape 2-sphere complement in compact 4-manifold?*

An affirmative answer to this question is provided by combining the following result with those of [K-S]..

MAIN THEOREM. *Let E be a connected 4-dimensional weak collar with $\pi_1(E) \cong \mathbb{Z}_n$. Then there is a closed 4-manifold Y, a shape 2-sphere $\Sigma^2 \subset Y$, and a neighborhood N of Σ^2 with $N-\Sigma^2 \approx E$ iff ∂E is $\mathbb{Z}$-homology equivalent to $L(n,1)$. Moreover, we may specify Y to be $S^2 \times S^2$ when n is even and $\mathbb{C}P^2\#(-\mathbb{C}P^2)$ when n is odd.*

A *weak collar* (see [F-Q]) is a manifold N with compact boundary and one end for which there is a proper map $f: N\times[0,1) \to N$ which is the identity on $N\times\{0\}$. A 3-manifold is $\mathbb{Z}$-homology equivalent to $L(n,1)$ provided it admits a degree 1 map onto $L(n,1)$ inducing $\mathbb{Z}$-homology isomorphisms in all dimensions. The significance of this condition is that these are precisely the 3-manifolds which bound a 4-manifold homotopy equivalent to S^2 (see [Gu_1]). Analysis of the construction by Kwasik-Schultz shows that their examples contain weak collars. Further analysis reveals that many have fundamental group $\mathbb{Z}_n$ and boundaries $\mathbb{Z}$-homology equivalent to $L(n,1)$.

A more complete discussion of this topic may be found in [Gu_2].

REFERENCES

[Du] P.F. Duvall, *Weakly flat spheres*, Michigan Math. J. 16(1969), 117-124.

[F-Q] M.H. Freedman and F. Quinn, *Topology of 4-Manifolds*, Princeton University Press, Princeton, New Jersey 1990.

[Gu_1] C.R. Guilbault, *Boundaries of 4-manifolds homotopy equivalent to S^2*, preprint.

[Gu_2] C.R. Guilbault, *Non-collarable ends of 4-manifolds: Some realization theorems*, to appear in Michigan Math. J..

[K-S] S. Kwasik and R. Schultz, *Desuspension of group actions and the ribbon theorem*, Topology 27(1988), 443-457.

[$L-V_1$] V.T. Liem and G.A. Venema, *Characterization of knot complements in the 4-sphere*, Top. and Appl. 41(1991), 231-245.

[$L-V_2$] V.T. Liem and G.A. Venema, *Complements of 2-spheres in 4-manifolds*, to appear in Proceedings of Topology Hawaii.

[Si] L.C. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Ph.D. dissertation, Princeton Univ., Princeton, N.J., 1965.

[Ve] G.A. Venema, *Neighborhoods of compacta in 4-manifolds*, Top. and Appl. 31(1989), 83-97.

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A Chaotic Embedding of the Whitehead Continuum

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This article summarizes a talk given at the 1992 Summer Topology Workshop held at Colorado College, June 11-13. This article gives an outline of my PhD dissertation, proofs and details will be submitted for publication elsewhere. I wish to acknowldege the help of my major professors: Dennis Garity and Richard Schori. I would also like to thank Professor Marcy Barge for his help.

1. Abstract

In this work we study the following problem: ***Which subsets of R^3 arise as chaotic local attractors for special self homeomorphisms h of R^3?*** R. F. Williams(1967), M. Misiureuicz(1985), W. Szczechla(1989), and M. Barge & J. Martin(1990) gave partial answers for this problem.

Barge and Martin showed that for any given continuous map $f : I \to I$, where I is a compact interval, there is an embedding of $\varprojlim(I, f)$ in R^2 and a homeomorphism $h : R^2 \to R^2$ such that $h(\varprojlim(I, f)) = \varprojlim(I, f)$, the restriction of h to $\varprojlim(I, f)$ is equal to $\hat{f}$, and $\varprojlim(I, f)$ is a global attractor for h. Here $\varprojlim(I, f)$ is the inverse limit of the sequence with bonding maps f and $\hat{f}$ is the induced homeomorphism on the inverse limit. Hence $\hat{f}$ on $\varprojlim(I, f)$ can be realized as the restriction of a homeomorphism h of the plane to its attractor.

In this work we extend these results to certain other compact subsets X of R^3. We show that X can be realized as local attractors for certain self homeomorphisms h of R^3 such that the restrictions of h to X are chaotic. These subsets X are cell-like sets arising as nested intersections of tori in a certain way. A typical example of these subsets is the Whitehead continuum, which is a non cellular embedding of the Knaster continuum in R^3. Technical difficulties arose in recognizing the self linking of certain subsets of R^3. This necessitated our working with inverse limits of pairs, and carefully analyzing a sequence of near homeomorphisms.

2. History of the Problem - Partial Answers

R.F. Williams [W] proved the following: Given a differentiable endomorphism of a branched one-dimensional manifold K, the inverse limit $\varprojlim(K, f)$ can be embedded in S^4 and the shift map $\hat{f}$ extended to a diffeomorphism of S^4 possessing $\varprojlim(K, f)$ as an attractor.

M. Misiureuicz [M] proved the following: If $\tau : I \to I$ is the tent map ($x \to 1 - |2x - 1|$), then:

A. For every manifold M where $dim(M) \geq 3$, there exists a C^∞ diffeomorphism $h : M \to M$ such that h restricted to its attractor Λ is topologically conjugate to $\hat{\tau}$ (which is chaotic).

B. For every manifold M where $dim(M) \geq 2$, there exists a homeomorphism $h : M \to M$ such that h restricted to its attractor Λ is topologically conjugate to $\hat{\tau}$.

The results A and B hold for all maps conjugate to $\hat{\tau}$, for example the quadratic map $x \to 4x(1 - x)$.

W. Szczechla [Sz], in a paper entitled "*Inverse Limits of Certain Maps as Attractors in 2 Dimensions*" extended Miziureuicz's results.

Barge and Martin [BM4] proved that if $f : I \to I$ is a map of a closed interval. Then $\varprojlim(I, f)$ can be realized as a global attractor for a homeomorphism of R^2.

In this work we extend some of the results of Barge and Martin to certain other compact subsets X of R^3. These subsets are cell-like sets arising as nested intersections of tori in a certain way. A typical example of these subsets is the Whitehead continuum.

3. The Whitehead Continuum

Let T_0 be a solid torus in R^3. Let T_1 be a solid torus in $Int(T_0)$ as shown in *Figure* 1. Let T_2 be a solid torus embedded in $Int(T_1)$ as T_1 is embedded in $Int(T_0)$. Continue this construction. This results in a sequence $T_0, T_1, T_2 \ldots$ of solid tori in R^3 such that for all nonnegative integers n, $T_{n+1} \subset Int(Tn)$. Assume the tori $T_0, T_1, T_2, \ldots$ are constructed efficiently to force 1-*dimensionality* of their intersection. For example, each T_i can be required to retract to its core curve under a retraction r_i with $diam(r_i^{-1}(p)) < \frac{1}{i}$ for each p. Then $W = \bigcap_{i=0}^{\infty} T_i$ is called the Whitehead continuum.

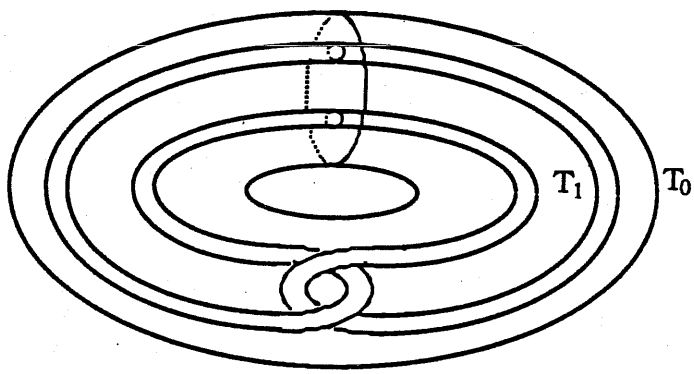

Figure 1

4. Properties of the Whitehead Continuum

For completeness, we list some of the properties of the Whitehead continuum. For more details, see [Da].

(i) The Whitehead continuum W is a noncellular subset of R^3, this is proved in *Section* 1.6 of the dissertation.

(ii) The continuum W is a cell-like subset of R^3. This follows from the fact that if U is a neighborhood of W then for some integer $k \geq 0$, $T_k \subset U$. Hence

$W \subset T_{k+1} \subset T_k \subset U$. Since T_{k+1} contracts to a point in T_k, W contracts to a point in U.

(iii) The continuum W is a UV^∞ continuum in R^3. This follows from the fact that W is cell-like and R^3 is an ANR (absolute neighborhood retract) [Da, Prop.1, p.123].

(iv) The continuum W is cellular in R^4. This follows from the fact that W is UV^∞ in R^3 [Mc3].

5. Objectives and Tools

For $i = 1, 2$ and 3, let B_i be a $3-cell$ satisfying $B_1 \subset Int(B_2)$ and $B_2 \subset Int(B_3)$. For $i = 1, 2$ and 3, let T_i be a solid torus satisfying $T_1 \subset Int(T_2)$ and $T_2 \subset Int(T_3)$. Choose T_i and B_i such that $T_i \subset Int(B_i)$ for $i = 1, 2, 3$.

Consider the solid torus $T_1 = S^1 \times D_1$ where D_1 is a 2-*cell*. Throughout this article, S^1 is taken to be the quotient space of $[0, 1]$ generated by identifying the endpoints $\{0\}$ and $\{1\}$ Let $G : B_3 \to B_3$ be a homeomorphism satisfying:

(1) The set $G(T_1)$ is a solid torus contained in $N\big(G(S^1), \frac{1}{n}\big) \subset Int(T_1)$.

(2) The torus $G(T_1)$ is embedded in T_1 just as T_1 is embedded in T_0 in *Figure* 1.

(3) The set $G(T_2) \subset Int(T_2)$.

(4) $G_{|B_3 - B_2} = id$

We will refer to such a homeomorphism as a Whitehead map.

Our objective is to construct a *near homeomorphism* $H : B_3 \to B_3$ satisfying:

(1) There is a sequence of homeomorphisms $H_{t_i} : B_3 \to B_3$ converging uniformly to H such that each H_{t_i} is a Whitehead map.

(2) There exists a homeomorphism $F : \varprojlim(B_3, H) \to \varprojlim(B_3, H_{t_i})$ such that $F(\varprojlim(T_1, H)) = \varprojlim(T_1, H_{t_i})$.

(3) The restriction of H to S^1 is the the function $\tau : S^1 \to S^1$ defined by

$$\tau(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ 2 - 2x, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

which is chaotic.

(4) The inverse $\varprojlim(T_1, H)$ is a local attractor for $\hat{H} : \varprojlim(B_3, H) \to \varprojlim(B_3, H)$.

Note that (2) implies that $\varprojlim(T_1, H)$ is embedded in $\varprojlim(B_3, H)$ just as the standard Whitehead continuum is embedded in B_3. Note alse that (3) implies $\hat{H}$ restricted to $\varprojlim(T_1, H)$ is chaotic.

While [Br, Theorem 3], stated below, supplies us with a homeomorphism $F : \varprojlim(B_3, H) \to \varprojlim(B_3, H_{t_i})$, it does not guarantee that $F(\varprojlim(T_1, H)) = \varprojlim(T_1, H_{t_i})$. This is rectified by proving a generalization of [Br, Theorem 3] for inverse sequences of pairs.

[Br, Theorem 3]. *Let $X^f_\infty = \varprojlim(X_i, f_i)$ where the X_i arecompact metric spaces. For $2 \leq i$, let G_i be a nonempty collection of maps from X_i into X_{i-1}. Suppose that for each $i \geq 2$ and $\epsilon > 0$ there exists a $g \in G_i$ such that $\|f_i - g\| < \epsilon$. Then there is a sequence (g_i) where $g_i \in G_i$ and X^f_∞ is homeomorphic to $\varprojlim(X_i, g_i) = X^g_\infty$.*

The homeomorphism in [Br, Theorem 3] is defined in [Br, Theorem 1] and [Br, Theorem 2]. For completeness we will state these theorems.

[Br, Theorem 1]. *Let $X^f_\infty = \varprojlim(X_i, f_i)$ and $X^g_\infty = \varprojlim(X_i, g_i)$ where the X_i are compact metric spaces. Suppose $\|f_{i+1} - g_{i+1}\| < a_i$, $i = 1, 2, \ldots$, where (a_i) is a Lebesgue sequence for (X_i, g_i). Then the function $F_N : X^f_\infty \to X_N$ defined by $F_N =$*

$\lim_{n\to\infty} g_{Nn}\pi_n$ *is well-defined and continuous. Moreover the function* $F : X^f_\infty \to X^g_\infty$ *defined by* $F(\underline{x}) = \big(F_1(\underline{x}), F_2(\underline{x}), \dots\big)$ *is well-defined, continuous, and onto.*

[Br, Theorem 2]. *Let* $X^f_\infty = \varprojlim(X_i, f_i)$ *and* $X^g_\infty = \varprojlim(X_i, g_i)$ *where the* X_i *are compact metric spaces. Suppose* $\|f_i - g_i\| < \min\big[c_{i-1}; \min_{k<i-1} L(c_{i-1}, g_{k,i-1})\big]$ *where* (c_i) *is a measure for* (X_i, f_i)*. Then the map* $F : X^f_\infty \to X^g_\infty$ *described in [Br, Theorem 1] is a homeomorphism .*

We now develop some notation. By the pair (X_i, Y_i) we mean a metric space X_i, equipped with a metric d_i, and a closed subset $Y_i \subseteq X_i$. By a map $f_i : (X_i, Y_i) \to (X_{i-1}, Y_{i-1})$ we mean a map $f_i : X_i \to X_{i-1}$ satisfying $f_i(Y_i) \subseteq Y_{i-1}$.

Let $\big((X_i, Y_i), f_i\big)$ denote the inverse sequence

$$(X_1, Y_1) \xleftarrow{f_1} (X_2, Y_2) \xleftarrow{f_2} (X_3, Y_3) \xleftarrow{f_3} \dots.$$

Let (X^f_∞, Y^f_∞) denote the inverse limit of the sequence $\big((X_i, Y_i), f_i\big)$. That is, let X^f_∞ and Y^f_∞ be the inverse limits of the sequences (X_i, f_i) and $(Y_i, f_{i|_{Y_i}})$ respectively.

By *Lemma* 1 and *Lemma* 2 of [Br], If the X_i are compact metric spaces then (X_i, g_i) has a Lebesgue sequence (a_i) and a measure (c_i). Definitions of these terms can be found in [Br].

The following theorem is a generalization of [Br, Theorem 1] .

<u>*Theorem 1*</u>. *Let* $(X^f_\infty, Y^f_\infty) = \varprojlim\big((X_i, Y_i), f_i\big)$ *and* $(X^g_\infty, Y^g_\infty) = \varprojlim\big((X_i, Y_i), g_i\big)$ *where the* X_i *are compact metric spaces and for all* i*,* Y_i *is a closed subset of* X_i*. Suppose* $\|f_{i+1} - g_{i+1}\| < a_i$*,* $i = 1, 2, 3, \dots$*; where* a_i *is a Lebesgue sequence for* (X_i, g_i)*. Then the function* $F_N : (X^f_\infty, Y^f_\infty) \to X_N$ *defined by* $F_N = \lim_{n\to\infty} g_{Nn}\pi_n$ *is well-defined and continuous. Moreover the function* $F : (X^f_\infty, Y^f_\infty) \to (X^g_\infty, Y^g_\infty)$ *defined by* $F(\underline{x}) = \big(F_1(\underline{x}), F_2(\underline{x}), \dots\big)$ *is well-defined, continuous, onto and* $F(Y^f_\infty) = Y^g_\infty$.

The following three theorems are generalizations of [Br, Theorem 2], [Br, Theorem 3] and [Br, Theorem 1] respectively. The proofs are identical to those found in [Br], hence they are omitted.

Theorem 2. *Let* $(X^f_\infty, Y^f_\infty) = \varprojlim((X_i, Y_i), f_i)$ *and* $(X^g_\infty, Y^g_\infty) = \varprojlim((X_i, Y_i), g_i)$ *where the* X_i *are compact metric spaces and for all* i, Y_i *is a closed subset of* X_i. *Suppose* $\|f_i - g_i\| < \min\left[c_{i-1}; \min_{k<i-1} L(c_{i-1}, g_{k,i-1})\right]$ *where* (c_i) *is a measure for* (X_i, f_i). *Then the map* $F : (X^f_\infty, Y^f_\infty) \to (X^g_\infty, Y^g_\infty)$ *described in Theorem* 6.1 *is a homeomorphism satisfying* $F(Y^f_\infty) = Y^g_\infty$.

Theorem 3. *Let* $(X^f_\infty, Y^f_\infty) = \varprojlim((X_i, Y_i), f_i)$ *where the* X_i *are compact metric spaces and for all* i, Y_i *is a closed subset of* X_i. *For* $i \geq 2$, *let* G_i *be a nonempty collection of maps from* (X_i, Y_i) *into* (X_{i-1}, Y_{i-1}). *Suppose that for each* $i \geq 2$ *and* $\epsilon > 0$ *there exists a* $g \in G_i$ *such that* $\|f_i - g\| < \epsilon$. *Then there is a sequence* (g_i) *where* $g_i \in G_i$ *and a homeomorphism* $F : (X^f_\infty, Y^f_\infty) \to (X^g_\infty, Y^g_\infty)$ *satisfying* $F(Y^f_\infty) = Y^g_\infty$.

Theorem 4. Let $(X^f_\infty, Y^f_\infty) = \varprojlim((X_i, Y_i), f_i)$ where:

(1) For all i, there exists a homeomorphism $h_i : (X_i, Y_i) \to (X, Y)$, where X is a compact metric space and $Y \subset X$ is closed such that $h_i(Y_i) = Y$, and

(2) For all i, f_i is a near homeomorphism.

Then there is a homeomorphism $\phi : (X^f_\infty, Y^f_\infty) \to (X, Y)$ such that $\phi(Y^f_\infty) \subseteq Y$.

6. Construction and Conclusions

We now define three *pseudo-isotopies* P^1_t, P^2_t and P^4_t and an *isotopy* P^3_t of B_1 onto itself. The effects of these maps are represented graphically in *Figure* 2.

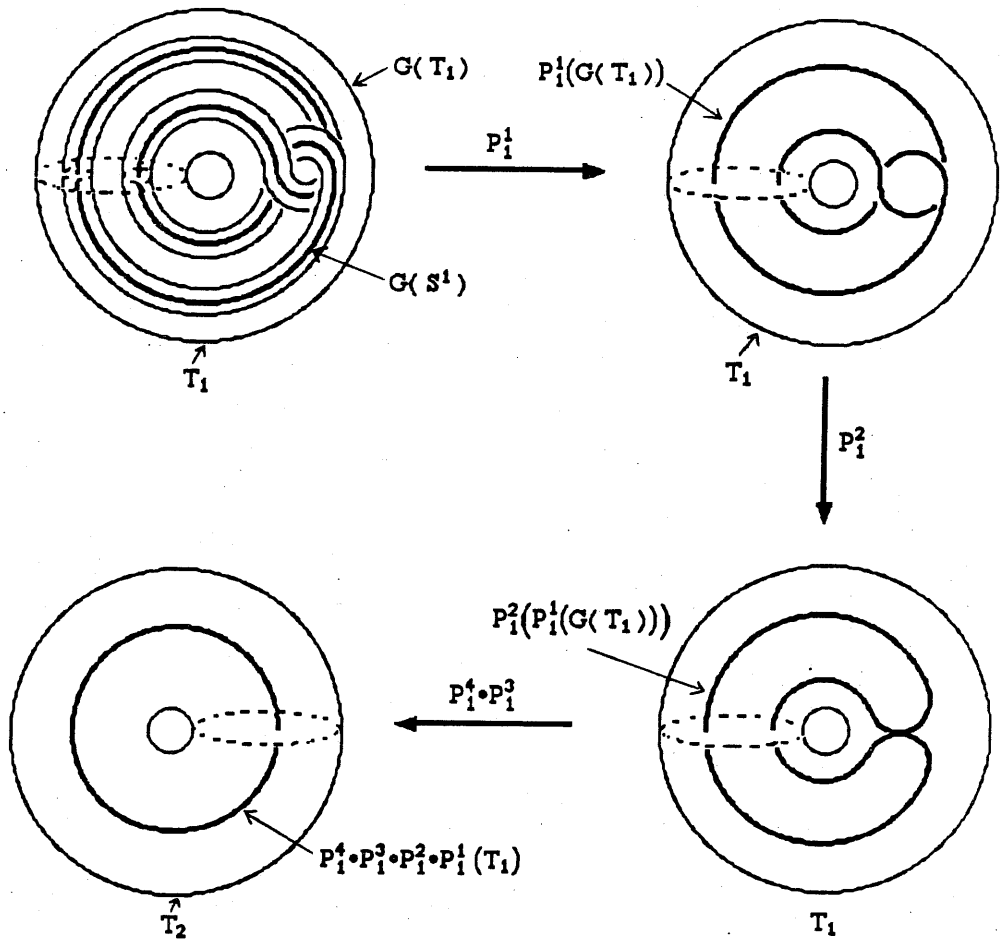

Figure 2

The map P_1^1 shrinks the solid torus $G(T_1)$ to $G(S^1)$ leaving $G(S^1)$ fixed. The map P_1^2 "eliminates" the self-linking of $G(S^1)$.

The map P_1^4 shrinks the torus T_1 to its core S^1. The map P_1^3 is defined such that for $0 \leq r \leq \frac{1}{2}$, $P_1^3 \circ P_1^2 \circ P_1^1 \circ G(r)$ is in $\{2r\} \times D_1$ and for $\frac{1}{2} \leq r \leq 1$, $P_1^3 \circ P_1^2 \circ P_1^1 \circ G(r)$ is in $\{2-2r\} \times D_1$.

Define $H : B_3 \to B_3$ by $H = P_1^4 \circ P_1^3 \circ P_1^2 \circ P_1^1 \circ G$. The maps $H_t = P_t^4 \circ P_t^3 \circ P_t^2 \circ P_t^1 \circ G$, for $0 \leq t \leq 1$ are required to satisfy:

(1) The maps H_t are homeomorphisms for $0 \leq t < 1$ and they converge uniformly to H as $t \to 1$. Hence H is a near homeomorphism.

(2) $H(T_2) = T_1$.

(3) $H_{|_{B_3 - Int(T_3)}} = id$.

(3) If $x \in S^1$ then $G(x \times D_1) \subset H^{-1}(x)$.

(4) For every point $x \in Int(T_3)$ there exists an integer $n \geq 0$ such that $H^n(x) \in S^1$.

Note that T_1 is a closed subset of B_3, $H(T_1) \subset T_1$ and $H_t(T_1) \subset T_1$ for all $t \in [0,1]$. It follows from *Theorem* 3 that there is a sequence H_{t_i}, $i = 1, 2, \ldots$, $H_{t_i} \in \{H_t : t = \frac{n}{n+1},\ and\ n \in \{1, 2, \ldots\}\}$ and a homeomorphism $F : \varprojlim ((B_3, T_1), H) \to \varprojlim ((B_3, T_1), H)$ such that $F(\varprojlim(T_1, H)) = \varprojlim(T_1, H_{t_i})$.

Let $K = \varprojlim(T_1, H)$ and $W = \varprojlim(T_1, H_{t_i})$. By *Theorem* 4, there is a homeomorphism $\Phi : \varprojlim ((B_3, T_1), H) \to (B_3, T_1)$. Now, consider the following diagram:

$$\begin{array}{ccccccccc}
T_1 & \xleftarrow{H_{t_1}} & T_1 & \xleftarrow{H_{t_2}} & T_1 & \xleftarrow{H_{t_3}} & \cdots & & W \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle H_{t_1}} & & \downarrow{\scriptstyle H_{t_1} H_{t_2}} & & & & \\
T_1 & \xleftarrow{i} & H_{t_1}(T_1) & \xleftarrow{i} & H_{t_1} H_{t_2}(T_1) & \xleftarrow{i} & \cdots & & \bigcap_{i=1}^{\infty} H_{t_1} H_{t_2} \ldots H_{t_i}(T_1)
\end{array}$$

This diagram defines a homeomorphism $h : W \to \bigcap_{i=1}^{\infty} H_{t_1} H_{t_2} \ldots H_{t_i}(T_1)$. Hence W is a standard Whitehead continuum (one with self-linking). Since $F : \varprojlim ((B_3, T_1), H) \to \varprojlim ((B_3, T_1), H_{t_i})$ takes $K = \varprojlim(T_1, H)$ onto $W = \varprojlim(T_1, H_{t_i})$, K is embedded in B_3 just as W is.

let h be the restriction of H to S^1 where S^1 is the core of T_1. Note that h is just the tent map on S^1. That is, considering S^1 as the quotient space of $[0,1]$ resulting from identifying the end points $\{0\}$ and $\{1\}$ then

$$h(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ 2 - 2x, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Now, consider the following diagram:

$$\begin{array}{ccccc}
\varprojlim(S^1,h) & \xrightarrow{i} & \varprojlim((B_3,T_1),H) & \xrightarrow{\Phi} & B_3 \\
\downarrow{\scriptstyle \hat{h}} & & \downarrow{\scriptstyle \hat{H}} & & \downarrow{\scriptstyle \Psi=\Phi\hat{H}\Phi^{-1}} \\
\varprojlim(S^1,h) & \xrightarrow{i} & \varprojlim((B_3,T_1),H) & \xrightarrow{\Phi} & B_3
\end{array}$$

Claim: $K = \varprojlim(T_1,H)$ is a local attractor for $\hat{H} : \varprojlim((B_3,T_1),H) \to \varprojlim((B_3,T_1),H)$.

To prove the claim, first note that since $H(T_1) = S^1$, it follows from the following diagram that $K = \varprojlim(T_1,H) = \varprojlim(S^1,h)$.

$$\begin{array}{ccccccccc}
T_1 & \xleftarrow{H} & T_1 & \xleftarrow{H} & T_1 & \xleftarrow{H} & \cdots & & K \\
\uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & & & \\
S^1 & \xleftarrow{h} & S^1 & \xleftarrow{h} & S^1 & \xleftarrow{h} & \cdots & & \varprojlim(S^1,H)
\end{array}$$

Since $H(S^1) = S^1$, then $\hat{H}(K) = K$.

Let $U = \{(x_1,x_2,\ldots) \in \varprojlim((B_3,T_1),H) : x_1 \in Int(T_2)\} = \pi^{-1}(Int(T_2))$. Clearly, U is open in $\varprojlim((B_3,T_1),H)$ and $K \subset U$. Now if $\underline{x} = (x_1,x_2,\ldots) \in U$, then $\hat{H}^n(\underline{x}) = (H^n(x_1),H^n(x_2),\ldots) \to K$ as $n \to \infty$.

Since $H(T_2) = T_1$, we have $\hat{H}(U) \subseteq \pi_1^{-1}(T_1)$ and hence $\overline{\hat{H}(U)} \subseteq \overline{\pi_1^{-1}(T_1)} = \pi_1^{-1}(T_1) \subset \pi_1^{-1}(T_2) = U$. Therefore $Cl(\hat{H}(U)) \subset U$.

It follows that $\hat{h} : \varprojlim(S^1,h) \to (S^1,h)$ is chaotic. Hence $K = \bigcap_{n\geq 0} \hat{H}^n(U)$ is a local chaotic attarctor for $\hat{H} : \varprojlim((B_3,T_1),H) \to \varprojlim((B_3,T_1),H)$.

Let $\Lambda = \Phi(K) = \bigcap_{n\geq 0} \Psi^n(\Phi(U))$. Since $\hat{H}_{|K}$ is *topologically conjugate* to $\Psi_{|\Phi(K)}$, then $\Phi(K)$ is a local chaotic attractor for Ψ.

7. Generalizations

Recall that we are studying the following problem: *Which subsets of R^3 arise as chaotic local attractors for self homeomorphisms of R^3?*

In the previous sections, we gave an outline of how the Whitehead continuum can be embedded in R^3 as a local chaotic attractor. In this section, we define two infinite classes of continua, $\mathcal{W} = \{W(n,m) : n \geq 1, m \geq 1\}$ and $\mathcal{K} = \{K_n : n \geq 2\}$ to which the previous construction generalizes.

Each of these continua is defined as the intersection of a nested sequence of solid tori. These continua have an important feature in common with the Whitehead continuum, namely the self-linking.

Defining $\mathcal{W}$.

Let T_0 be a solid torus in the interior of a 3-*cell* B_3. For all integers $n \geq 1$, $m \geq 1$, let $G_{nm} : B_3 \to B_3$ be a homeomorphism such that $T_1 = G_{nm}(T_0) \subset Int(T_0)$ is a solid torus which wraps around T_0 *n-times* in clockwise direction, then it self-links, and finally it wraps around T_0 *m-times* in counterclockwise direction as shown in *Figure* 3.

For all integers, $n \geq 1$ and $m \geq 1$, let $W(n,m) = \bigcap_{k \geq 0} G^k_{nm}(T_0)$. The continua $W(n,m)$ can be embedded in R^3 as local chaotic attractors.

Shown in *Figure* 3 are the first stages in the construction of $W(1,1), W(1,2), W(2,1), W(2,2)$, and $W(3,3)$. The solid torus T_1 is not shown in its entirty, only its core is shown.

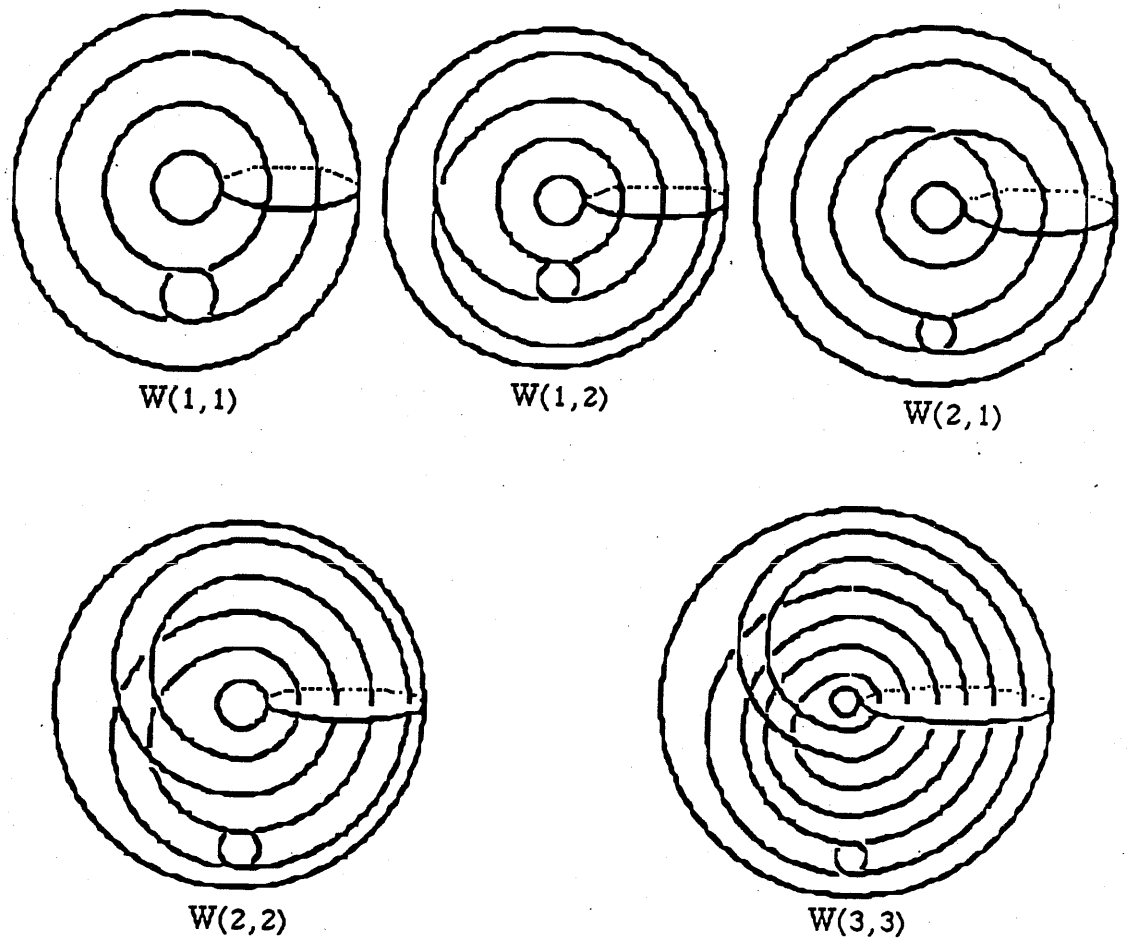

Figure 3

As we have seen earlier, after a few pseudo-isotopies (eliminating the self-linking), the homeomorphism G_{nm} is transformed into a near homeomorphism $H_{nm} : B_3 \to B_3$ such that the restriction of H_{nm} to S^1, the core of T_0, is the map $f_{nm} : S^1 \to S^1$ such that $W(n,m) = \varprojlim(S^1, f_{nm})$.

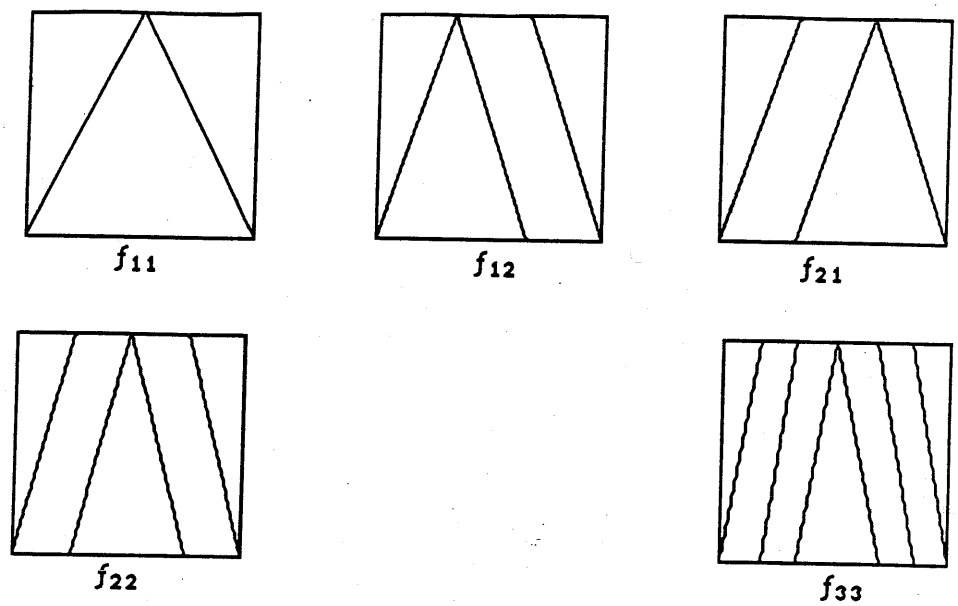

Figure 4

Shown in *Figure* 4 are the maps $f_{11}, f_{12}, f_{21}, f_{22}$, and f_{33}. Here S^1 is viewed as the quotient space of the interval $[0,1]$ resulting from identifying the end points $\{0\}$ and $\{1\}$.

For $n \geq 1$ and $m \geq 1$, the map $f_{nm} : S^1 \to S^1$ has the following property: If $J \subset S^1$ with nonempty interior, there exists an integer N such that $f_{nm}^k(J) = S^1$ for all integers $k \geq N$. Hence by [CM, Theorem C] f_{nm}^k is transitive for every $k > 0$. Clearly, f_{nm} has periodic points, hence [Si, Theorem 7.1] implies that f_{nm} is chaotic.

Defining $\mathcal{K}$.

For all integers $n \geq 2$, let $Q_n : B_3 \to B_3$ be a homeomorphism such that $T_1 = Q_n(T_0)$ is embedded in $Int(T_0)$ as shown in *Figure* 5.

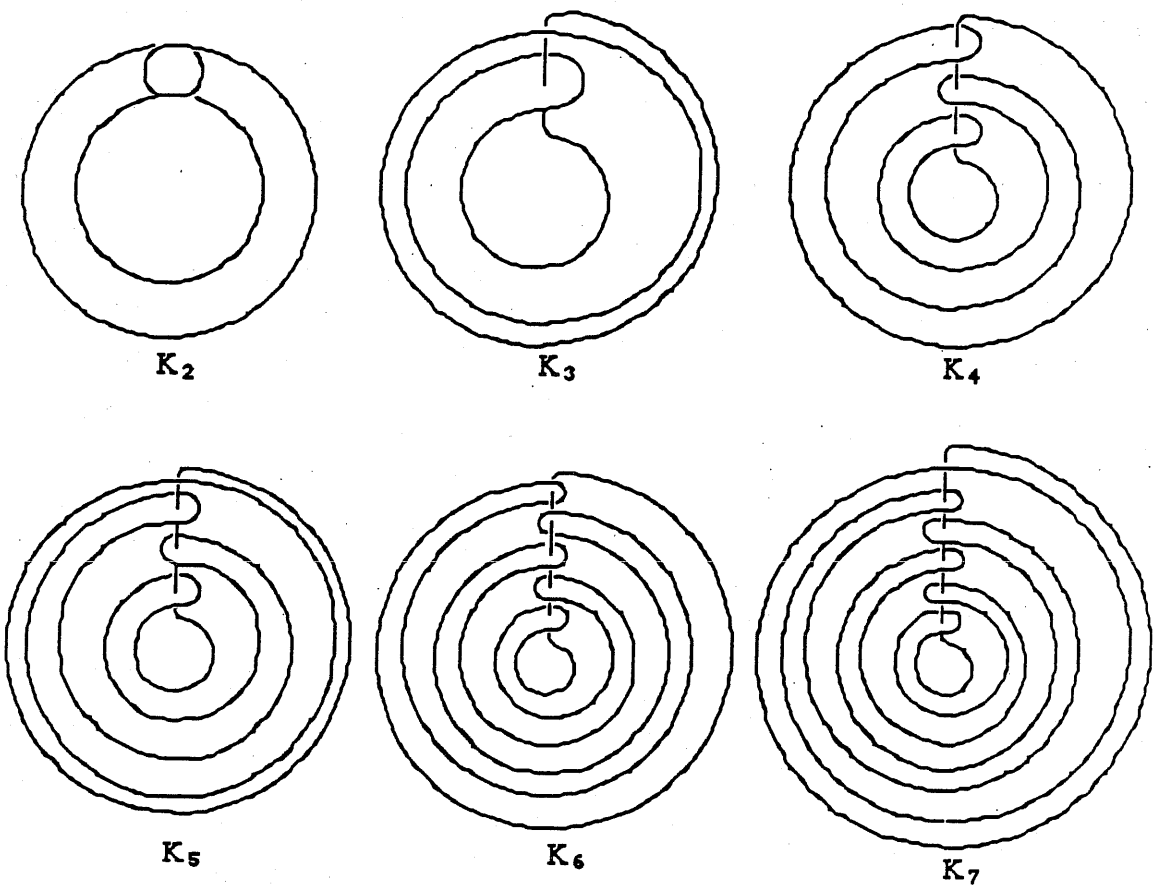

Figure 5

Shown in *Figure* 5 are the cores of $Q_i(T_0)$ for $i = 2, 3, \ldots, 7$. The images of T_0 under Q_n for $n > 7$ are not shown, but can be drawn by noticing the pattern developing in $Q_2(T_0), Q_3(T_0), \ldots, Q_7(T_0)$.

Let $K_n = \bigcap_{k \geq 0} Q_n^k(T_0)$ for $n \geq 2$.

The continua K_n can be embedded in R^3 as chaotic local attractors.

Again, as we have shown earlier, after a few pseudo-isotopies (eliminating the self-linking) the homeomorphism Q_n is transformed into a near homeomorphism $H_n : B_3 \to B_3$ such that the restriction of H_n to S^1, the core of T_0, is the map

$h_n : S^1 \to S^1$ such that $K_n = \varprojlim(S^1, h_n)$.

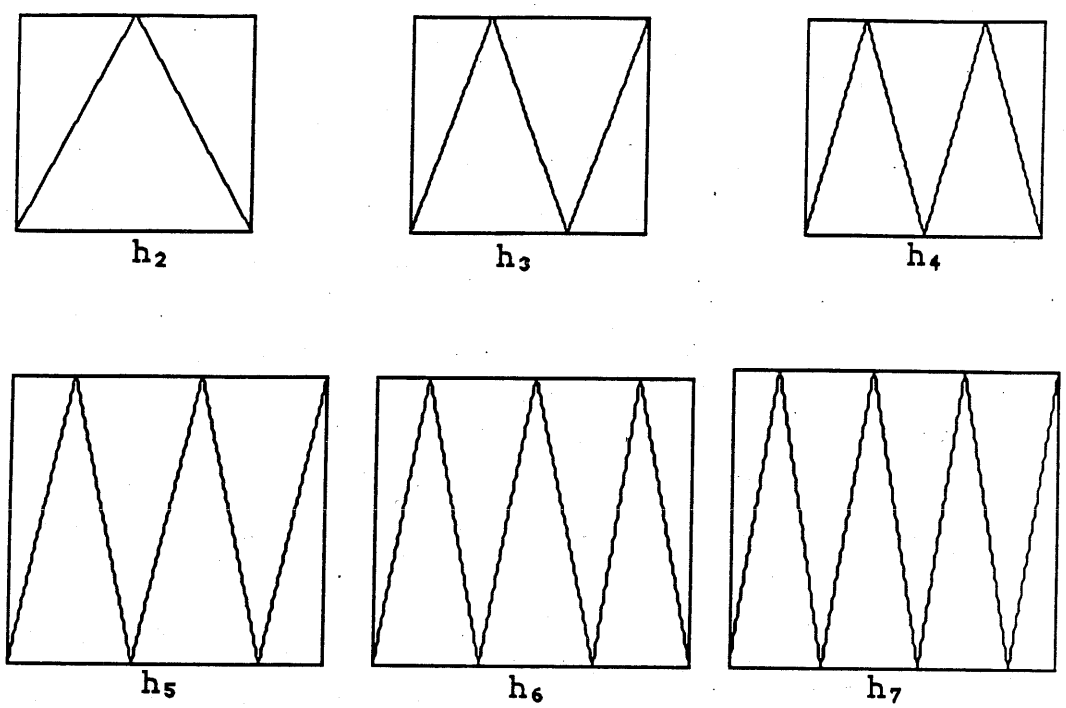

Figure 6

Shown in *Figure* 6 are the maps h_i for $i = 1, 2, \ldots, 7$. Recall that S^1 is viewed as the quotient space of the interval $[0, 1]$ resulting from identifying the end points $\{0\}$ and $\{1\}$. The maps $h_n : S^1 \to S^1$ are chaotic by [CM, Theorem C] and [Si, Theorem 7.1].

The continua $K_n \simeq \varprojlim(S^1, h_n) \simeq \varprojlim(I, h_n)$.

[W] that K_n is homeomorphic to K_m *if and only if* n and m have the same prime factors.

References

[Ba1] M. Barge, *A method for constructing attractors*, Ergodic Theory and Dynamical Systems **8** (1988), 331-349.

[Ba2] ——————, *Horseshoe maps and inverse limits*, Pacific J. Math. **121** (1986), 29-39.

[BM1] M. Barge and J. Martin, *Chaos, periodicity, and snake-like continua*, Trans. Amer. Math. Soc. **289** (1985), 355-365.

[BM2] ——————, *Dense periodicity on the interval*, Proc. Amer. Math. Soc. **94** (1985), 731-735.

[BM3] ——————, *Dense orbit on the interval*, Michigan Math. J. **34** (1987), 3-11.

[BM4] ——————, *The construction of global attractors*, Proc. Amer. Math. Soc. **110** (1990), 523-525.

[BR] M. Barge and R. Roe, *Circle maps and inverse limits*, Topology Appl. **36** (1990), 19-26.

[Be] Ralph Bennett, *On inverse limit sequences*, Master's Thesis, The University of Tennessee, 1962.

[Bl] Louis Block, *Periodic orbits of continuous mappings of the circle*, Trans. Amer. Math. Soc. **260** (1980), 553-562.

[Br] Morton Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc. **11** (1960), 478-483.

[CM] Ethan M. Coven and Irene Mulvey, *Transitivity and the Centre for maps of the circle*, Ergodic Theory and Dynamical Systems **6** (1986), 1-8.

[D] Robert J. Daverman, *Decompositions of Manifolds*, Accademic Press, Orlando, Florida, 1986.

[K] Keldyš L. V., *Topological Imbeddings in Euclidean Space*, Proc. of the Steklov Institute of Math. **81** (1968), 1-203.

[M] J. Martin, *Inverse limit spaces*, Xeroxed Notes, 1989.

[Mc1] D. R. McMillan, JR., *Cartesian products of contractible open manifolds*, Bull. Amer. Math. Soc. **67** (1961), 510-514.

[Mc2] ——————, *Some contractible 3-manifolds*, Trans. Amer. Math. Soc. **102** (373-382), .

[Mc3] ——————, *A criterion for celluarity in a manifold*, Ann. of Math. **79** (1964), 373-382.

[Mi] M. Misiurewicz, *Emdedding inverse limits of interval maps as attractors*, Fund. Math. **125** (1985), 23-40.

[Sc] Richard M. Schori, *Chaos: An introduction to some topological aspects*, in Continuum Theory and Dynamical Systems, Contemporary Mathematics, **117**,Morton Brown, editor, (1991), 149-161 American Mathematical Society, Providence, Rhode Island.

[Sz] Witold Szczechla , *Inverse limits of certain interval mappings as attractors in two dimensions*, Fund. Math. **133** (1989), 1-23.

[Wa] W. T. Watkins, *Homeomorphic classification of certain inverse limit spaces with open bonding maps*, Pacific J. Math. **103** (1982), 589-601.

[Wh] J. H. C. Whitehead, *A certain open manifold whose group is unity*, Quart. Jour. Math. Oxford **6** (1935), 268-279.

[W] R. F. Williams, *One-dimensional nonwandering sets*, Topology **6** (1967), 473-487.

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ANR QUOTIENTS OF MANIFOLDS WITH NON–NEARLY 1–MOVABLE FIBERS

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A subset X of an ANR Y is *nearly 1-movable* if for each open set U containing X there is an open subset V of U containing X such that for any neighborhood W of X and any map $f : \partial B^2 \to V$, f can be extended to a disk with holes whose other boundary components are mapped into W. Let M^n be a manifold and let $f : M^n \to Y$ be a map with Y an ANR. Let N_f denote the union of the non-degenerate point inverses of f. If $y \in Y$ is an isolated point of $f(N_f)$, then it follows that $f^{-1}(y)$ is nearly 1-movable. A question arises as to whether any non-degenerate point inverses of f can fail to be nearly 1-movable. Whenever $f(N_f)$ is compact and 0-dimensional, each $f^{-1}(y)$ for $y \in f(N_f)$ can be defined by nearly 1-movable sets in the following sense:

Proposition A. *[Daverman] If $f : M^n \to Y$ is a map from a manifold M to an ANR Y such that $f(N_f)$ is compact and 0-dimensional, then for each point $y \in f(N_f)$, $f^{-1}(y)$ can be written as the intersection of a countable collection of nearly 1-movable subsets of M. If the map f is acyclic, then the sets defining $f^{-1}(y)$ can be taken to be acyclic as well.*

Proof. Y contains an arc α threaded through the compact, 0-dimensional set $f(N_f)$. If π is the quotient map from Y to the decomposition space Y/α obtained by collapsing out α then π is acyclic, Y/α is an ANR, and the composition πf is acyclic when f is. $f^{-1}(\pi^{-1}(\pi(\alpha))) = f^{-1}(\alpha)$ is nearly 1-movable. More generally, if finitely many disjoint open arcs missing $f(N_f)$ are removed from α the quotient space of Y obtained by collapsing out the remaining subarcs of α is also an ANR and if A is one of these subarcs, then $f^{-1}(A)$ will be nearly 1-movable and acyclic when f is. The sequence defining a particular $f^{-1}(y)$ for $y \in f(N_f)$ can be obtained in this way using the 0-dimensionality of $f(N_f)$.

The purpose of this talk is to describe a construction which demonstrates a converse to the previous proposition. Specifically:

Proposition B. *If $X \subset M^n$ is not nearly 1-movable but can be written as the countable intersection of acyclic, nearly 1-movable sets, then there is a compact, 0-dimensional, upper semi-continuous decomposition G of $M^n \times E^1$ such that $(M^n \times E^1)/G$ is an ANR and each non-degenerate element of G fails to be nearly 1-movable.*

Proof. Let $A_0 \supset A_1 \supset A_2 \supset \dots$ denote a sequence of nearly 1-movable, acyclic sets in M^n and let $X = \cap A_k$. First consider a simpler decomposition, G_1, whose non-denegerate elements consist of the set $X \times \{0\}$ together with the sequence $\{A_k \times 1/2^k, k = 0, 1, \dots\}$. We can show that the quotient space is LC^1 at the image of $X \times \{0\}$ and this will be sufficient to establish that it is an ANR using the facts that the quotient must be finite dimensional, the local relative homology vanishes and a local version of the Hurewicz Theorem applies. G_1 is a decomposition producing an ANR quotient with a non-nearly 1-movable non-degenerate element. The sequence $\{A_k \times 1/2^k\}$ is used to "tame" the set $X \times \{0\}$.

We now describe how to augment this construction to produce a decomposition G in which every non-degenerate element fails to be nearly 1-movable. We will then outline a proof that the quotient space is an ANR.

Distinguish a point $x \in X$ and define the set

$$H_k(s,t) = (A_k \times \{s\}) \cup (\{x\} \times [s,t]) \cup (X \times \{t\}).$$

The non-degenerate elements of the decomposition G will be of the form $H_k(s,t)$ for some choice of k, s and t or of the form $X \times \{r\}$, the latter sets arising as limits of sequences of the former.

The construction of G begins by including the sets $H_k(3/2^{k+2}, 1/2^k)$, for $k = 0, 1, \dots$. Note that the sequence described converges (in the Hausdorff sense) to the set $X \times \{0\}$ which must be included to assure upper semi-continuity. If we stopped at this stage, we could show that the quotient is LC^1 at the image of $X \times \{0\}$ but it would be bad at other points since each $H_k(s,t)$ set is acyclic, but not nearly 1-movable.

For a given k, we include another sequence of sets converging to $H_k(.,.)$ from the right, using the gap in the E^1 factor between $H_k(.,.)$ and $H_{k-1}(.,.)$ or a small interval to the right of $H_0(3/4, 1)$. To preserve upper semi-continuity, a sequence to converge toward an $H_k(.,.)$ set will begin with index $k+1$. For example, one such sequence at the first iteration is $\{H_k(1/2 + 3/2^{k+3}, 1/2 + 1/2^{k+1}), k = 2, 3 \dots\}$. This process is iterated so that no non-degenerate element is isolated. A Cantor set worth of limiting copies of X will need to be included in order to get upper semi-continuity. Gaps can be left in the E^1 direction to force the non-degeneracy set to be 0-dimensional and since all of the non-degenerate elements have large size, the decomposition is closed (and compact).

Let π denote the quotient map from $M^n \times E^1$ to $Y = (M^n \times E^1)/G$. We will assume that $\pi(N_\pi)$ is $0 - LCC$ in Y. (Since codimension is not an issue we could cross with another interval factor if necessary.) Let $p \in \pi(N_\pi)$. We discuss the case where $p = \pi(H_k(s,t))$, for some triple (k,s,t); the case where $\pi^{-1}(p)$ is a copy of X is treated similarly.

Let U be a neighborhood of p. Determine a neighborhood P of A_k and an interval (a,b) contianing s so that $P \times [a,b] \subset \pi^{-1}(U)$. Similarly, find a neighborhood Q of X and an interval (c,d) containing t with $Q \times [c,d] \subset \pi^{-1}(U)$ and a neighborhood R of x with $R \times [a,d] \subset \pi^{-1}(U)$. Set $\widehat{V} = (P \times [a,b]) \cup (R \times [a,d]) \cup (Q \times [c,d])$ and let V be a neighborhood of p in Y with $\pi^{-1}(V) \subset \widehat{V}$.

Let $f : \partial B^2 \to Y$ be a loop in V. Using the facts that $\pi(N_\pi)$ is closed, 0-dimensional and $0 - LCC$ in Y we can find an infinite 1-complex Σ in B^2, having

mesh tending to 0 near ∂B^2, and an extension of f to $\partial B^2 \cup \Sigma$ so that each component of $B^2 - \Sigma$ is either a subset of ∂B^2 or an open 2-cell in the interior of B^2 whose boundary is mapped into a (small) subset of $V - \pi(N_\pi)$. If the necessary size conditions are imposed, the extension of f to all of B^2 can be accomplished by extending carefully over each of these 2-cells. This allows us to reduce the problem to a consideration of the case where the original loop f has image missing $\pi(N_\pi)$.

Accordingly, let f_0 denote the lift of f to $M^n \times E^1$. From the construction, we may assume that $Q \times (c, d)$ contains an element $g \in G$ of the form $H_l(\alpha, \beta)$. The group $\pi_1(\widehat{V})$ factors as $\pi_1(P) \times \pi_1(Q)$ so f_0 can be extended to a map f_1 of a disk-with-holes $H_1 \subset B^2$ whose two additional boundary components, J_1 and J_2, are mapped by f_1 into $P \times \{s\}$ and $Q \times \{\alpha\}$, respectively. Using the nearly 1-movability of A_k we can assume that P was chosen so that a loop in $P \times \{s\}$ can be extended to a map of a disk-with-holes with image in $\pi^{-1}(U)$ and whose other boundary components are mapped arbitrarily close to $A_k \times \{s\}$. In a similar but slightly more complicated way Q can have been chosen so that a loop in $Q \times \{\alpha\}$ can be extended to a map of a disk-with-holes with image in $\pi^{-1}(U)$ and whose other boundary components are mapped arbitrarily close to $A_l \times \{\alpha\}$. The desired extension of f to B^2 is obtained using a sequence of maps $\{\pi f_j\}$, where f_{j+1} extends f_j, f_j is defined on a disk-with-holes H_j in B^2 and the components of $\partial H_j - \partial B^2$ are mapped close to either p or $\pi(H_l(\alpha, \beta))$.

Remark. It is not known whether a non-nearly 1-movable set can have a defining sequence such as is discussed here. One possibility for such a construction is tied to deep set theoretic questions. See [**F. Tinsley**] in these proceedings.

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A TOPOLOGICAL ENTROPY INVARIANT FOR KNOTS IN S^3

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This paper summarizes a talk given by the author at the Ninth Annual Workshop in Geometric Topology held at The Colorado College, June 11-13, 1992. An expanded version will appear later.

1. Introduction. Let K be a knot in S^3. The symbol X denotes the *exterior* of K; i.e., the closure of S^3 minus a tubular neighborhood of K. The boundary ∂X will be regarded as $K \times S^1$. We recall that the knot K is *fibered* if there exists a smooth fibration $\phi : X \to S^1$ extending the projection $\partial X \to S^1$. If K is fibered, then X is diffeomorphic to a mapping torus $S \times [0,1]/\{(x,0) \sim (f(x),1)\}$, where S is a minimal-genus Seifert surface for K (called a *fiber*) and $f : S \to S$ is a diffeomorphism (called a *monodromy*). Well known examples of fibered knots include the figure-eight knot and all torus knots. (See [**10**] for further details.)

In [**6**] O. Kakimizu has shown that the exterior X of any knot K contains a codimension-0 submanifold X_0 such that

(1) $K \times S^1 \subset \partial X_0$;

(2) there exists a smooth fibration $\phi : X_0 \to S^1$ extending the projection $\partial X \to S^1$;

(3) X_0 is maximal and unique with respect to inclusion and isotopy;

(4) a fiber S_0 can be found inside any minimal-genus Seifert surface for K.

Following [**6**] we will call X_0 the *maximal fibered submanifold* of K.

Examples. (See [**6**].) 1. Any knot K is fibered if and only if $X_0 = X$.

2. If X is not fibered and X is atoroidal, then $X_0 =$ (collar of S) $\times S^1$. Whenever this conclusion holds, we will say that X_0 is *trivial.*

If $f : S_0 \to S_0$ is a monodromy of a maximal fibered submanifold for a knot K, then the pair of conjugacy classes $[f^{\pm 1}]$ in the mapping class group of S_0 depends only on the unoriented knot type of K (see [**3**, p. 34]). Consequently, any quantity that depends only on $[f^{\pm 1}]$ is a knot invariant defined for K.

2. Some dynamical invariants for knots. Assume that K has nontrivial maximal fibered submanifold X_0 with monodromy $f : S_0 \to S_0$. The Nielsen-Thurston classification of hyperbolic surface automorphisms [**12**], [**4**] ensures that after an isotopy of f, there exists a finite f-invariant collection $\mathcal{C}$ (possibly empty) of disjoint, essential, simple closed curves that decompose S_0 into subsurfaces $T_1, \cdots, T_n$, and such that the following property holds:

For each $i = 1, \cdots, N$, let n_i be the smallest nonnegative integer such that $f^{n_i}(T_i) = T_i$. Then each mapping $g_i = f^{n_i}|_{T_i}$ is either pseudo-Anosov or periodic.

We will assume that $\mathcal{C}$ contains no proper subcollection with the property above. Consequently, $\mathcal{C}$ is unique up to isotopy [**1**]. For each $i = 1, \cdots, N$, let λ_i denote the n_i-th root of the stretching factor for g_i, if the mapping is pseudo-Anosov; if g_i is periodic, then let $\lambda_i = 1$.

Definition. (Compare with [**9**].) The *spectrum* of K, denoted by spec(K), is the set $\{\lambda_1, \cdots, \lambda_N\}$.

Remarks. 1. Each λ_i arises as the rate at which the length of some simple closed geodesic curve in S_0 grows under iteration of f (see [**7**, section V]).

2. The set of λ_i's associated to any hyperbolic surface automorphism was defined by W. Thurston [**13**], while the germ of the concept can be found in the papers of R.F. Williams [**14**], [**15**]. The invariant spec(K) is further motivated by an invariant for "braided links" recently defined by J. Los [**9**].

Definition. The *entropy* of K, denoted by γ_K, is the maximum of $\log \lambda_1, \cdots, \log \lambda_N$.

Remarks. 1. By [**4**] γ_K is the infimum of the topological entropies of all homeomorphisms isotopic to f. In fact, using the main result of [**5**] one can show that γ_K is realized as the topological entropy of a monodromy for some smooth fibration of X_0.

2. The invariant γ_K was introduced and studied for fibered knots in [**12**].

Examples. 1. The spectrum of any torus knot is $\{1\}$. More generally, if K is any knot obtained from the trivial knot by repeated cabling and connect-summing, then spec$(K) = \{1, \cdots, 1\}$.

2. The spectrum of any hyperbolic fibered knot K is $\{\lambda\}$, where $\lambda > 0$ is the stretching factor of any pseudo-Anosov map isotopic to a monodromy of K.

3. In [**9**] J. Los has exhibited a pair of mutant fibered knots that have different entropies. These knots have the same 2-variable Jones polynomial, Q-polynomial and Gromov invariant by [**8**], [**2**], and [**11**].

References

[**1**] J. Birman, A. Lubotzky, J. McCarthy, *Abelian and solvable subgroups of the mapping class group*, Duke J. Math. **50** (1983), 1107-1120.

[**2**] R.D. Brandt, W.B.R. Lickorish, K.C. Millett, *A polynomial invariant for unoriented knots and links*, Invent. Math. **84** (1986), 563-573.

[3] D. Eisenbud, W. Neumann, Three-dimensional Link Theory and Invariants of Plane Curve Singularities, Annals of Math. Studies, No. 110, Princeton Univ. Press, Princeton, NJ, 1985.

[4] A. Fathi, M. Shub, *Travaux de Thurston sur les surfaces*, Astérisque **66-67** (1979), 181-207.

[5] M. Gerber, A. Katok, *Smooth models of Thurston's pseudo-Anosov maps*, Ann. Sci. Ecole Norm. Sup. **15** (1982), 173-204.

[6] O. Kakimizu, *On maximal fibred submanifolds of a knot exterior*, Math. Ann. **284** (1989), 515-528.

[7] F. Laudenbach, *La théorie de Thurston sur les surfaces a bord*, Astérisque **66-67** (1979), 209-224.

[8] W.B.R. Lickorish, K.C. Millett, *A polynomial invariant of oriented links*, Topology **26** (1987), 107-141.

[9] J. Los, *Facteur de dilatation des entrelacs tréssés*, preprint.

[10] D. Rolfsen, Knots and Links, Publish or Perish, Inc., Berkeley, CA, 1976.

[11] D. Ruberman, *Mutations and volume of knots in S^3*, Invent. Math. **90** (1987), 189-215.

[12] D.S. Silver, *Growth rates and n-knots*, Top. Appl. **42** (1991), 217-230.

[13] W.P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. **19** (1988), 417-431.

[14] R.F. Williams, *One-dimensional non-wandering sets*, Topology **6**, (1967), 473-487.

[15] R.F. Williams, *Classification of one-dimensional attractors*, in Proc. of Symp. in Pure Math., Vol. 14, ed. by S. Chern and S. Smale, Amer. Math. Soc., Providence, RI, 1970, 341-361.

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Strange Acyclic Maps of ANR's

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Let $f:M \mapsto N$ be an acyclic map of closed n-manifolds. Then f induces $\bar{f} = f \times id:M \times \mathbb{R}^1 \mapsto N \times \mathbb{R}^1$, another acyclic map. Let $X = f^{-1}(y)$ for some $y \in N$. We say X is ***nearly 1-movable*** if for every neighborhood, U, of X, there is a neighborhood, V, of X, such that for every loop, s, in V, and neighborhood, W, of X, the loop s bounds a singular disk with holes so that each other boundary component lies in W. In other words, the loop, s, can be 'moved' arbitrarily close to X. The adjective, 'nearly', refers to the fact that the 'movement' does not fix a basepoint. This property of 1-movability is a property of X and not of a particular embedding.

For any set A closed in N and any set B in M with $f^{-1}(A) \subset int(B)$ denote by $G(A,B)$ the decomposition of B into points and possible nondegenerate elements $\{f^{-1}(y):y \in A\}$. We denote the decomposition map by $\alpha:B \mapsto B/G(A,B)$. Let $y_0 \in N$. We say $f^{-1}(y_0)$ is ***nasty in M*** if for every open neighborhood, U, of $f^{-1}(y_0)$ in M, there is a compact, 0-dimensional set $\mathbf{C}$ in $int(f(U))$ containing y_0 such that the decomposition space $U/G(\mathbf{C},U)$ is an ANR. Then the map f is nasty if each pointpreimage is nasty in M. Finally, we say f has ***locally, normally, finitely generated*** π_1 ***(lnfg)*** if for every $y \in N$ and neighborhood U of y in N, there exist a neighborhood V of y in N and a compact PL n-manifold L with $f^{-1}(V) \subset int(L)$ such that $Ncl\left(ker\left(\left(f|f^{-1}(V)\right)_{\#}\right),\pi_1(L)\right)$ is finitely generated as a normal subgroup of $\pi_1(L)$.

Since f is acyclic, $ker\left(\left(f|f^{-1}(V)\right)_{\#}\right)$ is perfect [reference], and thus, $Ncl\left(ker\left(\left(f|f^{-1}(V)\right)_{\#}\right),\pi_1(L)\right)$ is also perfect. Currently, there are no known examples of finitely presented groups with perfect, normal subgroups which are not finitely generated as normal subgroups. So, the condition in the definition of lnfg may be superflous. However, this appears to be a very difficult question.

The following relates nastiness to nearly 1-movability.

Theorem: Let M, N, and f be as above. If $f^{-1}(y)$ is nasty in M, then $f^{-1}(y)$ is the nested intersection in M of nearly 1-movable acyclic compacta. If $f^{-1}(y)$ is the nested intersection in M of nearly 1-movable acyclic compacta, then $\bar{f}^{-1}(y,t)$ is nasty in $M \times \mathbb{R}^1$.

Proof: See Lay's summary in these proceedings.

The following equivalence is our main result.

Main Theorem: Let M, N, f, and $\bar{f}$ be as above. Then $\bar{f}$ is nasty if and only if $\bar{f}$ is lnfg .

We would like to prove this theorem for f itself but currently the stable version is the best we can do.

Proof of Theorem: Suppose each $\bar{f}^{-1}(y_0,t)$ is nasty in $M \times R^1$. Let $(y_0,0) \in U \times (-\epsilon,\epsilon) \subset N \times \mathbb{R}^1$. We exhibit V and L for the definition of lnfg. This implication does not use stabilization so we prove it for M, N, and f. First, choose a compact, connected manifold $L \subset U$ with $f^{-1}(y_0) \subset int(L)$. Identify an f-saturated open set, W, with $f^{-1}(y_0) \subset W \subset int(L)$. By hypothesis there is a compact 0-dimensional set, $\mathbf{C}$, in $f(W)$ with $y_0 \in \mathbf{C}$ and the decomposition space, $W/G(\mathbf{C},W)$, an ANR. Then $Z = L/G(\mathbf{C},L)$ is also an ANR. Let $\alpha: L \mapsto L/G(\mathbf{C},L)$ be the decomposition map.

Since Z is locally contractible, cover $\alpha \circ f^{-1}(\mathbf{C})$ in Z by a finite collection of mutually disjoint open sets, E_i, so that each E_i contracts in $\alpha(W)$. Cover $\mathbf{C}$ in N by a finite collection of mutually disjoint open sets, B_j, so that $f^{-1}(B_j) \subset \alpha^{-1}(E_i)$, each j, some i. Now, let $V = \bigcup B_j$.

A straightforward geometric argument yields that $Ncl\left(ker\left(\left(f|f^{-1}(V)\right)_{\#}\right), \pi_1(L)\right) = ker(\alpha_{\#})$. Then it follows that since Y is an ANR of finite type, $ker(\alpha_{\#})$ must be the normal closure in $\pi_1(L)$ of a finite set of elements and, thus, the condition of lnfg holds.

For the converse, suppose the conditions of lnfg hold for $\bar{f}: M \times \mathbb{R}^1 \mapsto N \times \mathbb{R}^1$. Let $Map(\bar{f}) \cong M \times \mathbb{R}^1 \times [0,1) \bigcup_{\bar{f}} N \times \mathbb{R}^1 \times \{1\}$ denote the mapping cylinder of $\bar{f}$. Let s be a loop representing any non-trivial element of $ker(\bar{f}_{\#})$. We may assume s is embedded in $M \times \mathbb{R}^1 \times \{0\}$. Because of stabilization, we may also assume s bounds an embedded disk, D, in $Map(\bar{f})$ with $\mathbf{C} = D \bigcap (N \times \mathbb{R}^1 \times \{1\})$ 0-dimensional. Obtaining this 0-dimensional intersection is the only place where we need the extra $\mathbb{R}^1$ factor. We would like to eliminate this dependency, but it it is closely related to a longstanding, difficult question of Daverman about types of wildness.

The construction in the previous paragraph works just as well if M and $\bar{f}$ are replaced by any saturated open subset, $\bar{f}^{-1}(U)$, of M and $\bar{f}|\bar{f}^{-1}(U)$ respectively.

We apply this construction inductively to show that $\bar{f}$ is nasty. Let $(y_0, t) \in N \times \mathbb{R}^1$ and $\bar{f}^{-1}(y_0, t) \subset W^{open} \subset M \times \mathbb{R}^1$. To begin, choose U_1 with $(y_0, t) \in U_1^{open} \subset N \times \mathbb{R}^1$ and $\bar{f}^{-1}(clos(U_1)) \subset W$. By hypothesis (y_0, t) and U_1 yield L_1 and V_1 satisfying of lnfg. Apply the construction above to finitely many loops generating $Ncl\left(ker\left(\left(\bar{f}|\bar{f}^{-1}(V_1)\right)_{\#}\right), \pi_1(L_1)\right)$ to obtain a compact 0-dimensional set $\mathbf{C}_1 \subset V_1$. Add y_0 to $\mathbf{C}_1$ if necessary. By construction, $Ncl(ker\left(\left(\bar{f}|\bar{f}^{-1}(V_1)\right)_{\#}\right), \pi_1(L_1 \times [0,1) \bigcup \mathbf{C}_1)) \cong 1$. Now, suppose U_{k-1}, L_{k-1} V_{k-1}, and $\mathbf{C}_{k-1}$ have been constructed to satisfy all these conditions. Cover $\mathbf{C}_{k-1}$ with a finite, cover, $\mathcal{B}$, of open balls. Then find a finite refinement of $\mathcal{B}$ consisting of a mutually disjoint collection of open sets $\{U_{k,1}, \ldots, U_{k,m(k)}\}$ with $clos\,(U_{k,j}) \subset V_{k-1}$ and $dia(U_{k,j}) < \frac{1}{k}$. Let $U_k = \bigcup U_{k,j}$. For all $y \in \mathbf{C}_{k-1}$ apply the condition of lnfg to y and $U_{k,j(y)}$ where $y \in U_{k,j(y)}$. We obtain $L_{k,j(y)}$ and $V_{k,j(y)}$. Let $\left\{V_{k,j(y_1)}, \ldots V_{k,j(y_{u(k)})}\right\}$ be a finite subcover of $\mathbf{C}_{k-1}$. By construction, this cover is a refinement of the $U_{k,j}$'s. Apply the basic construction to a finite set of loops generating $Ncl\left(ker\left(\left(\bar{f}|\bar{f}^{-1}(V_{k,j(y_v)})\right)_{\#}\right), \pi_1(L_{k,j(y_v)})\right)$ to obtain a compact 0-dimensional set $\mathbf{C}_{k,j(y_v)} \subset V_{k,j(y_v)}$. Again,

$$Ncl\left(ker\left(\left(\bar{f}|\bar{f}^{-1}(V_{k,j(y_v)})\right)_{\#}\right),\pi_1(L_{k,j(y_v)}\times[0,1)\cup \mathbf{C}_{k,j(y_v)})\right)\cong 1$$

Let $\mathbf{C}_k = \left[\bigcup_{v=1}^{v=u(k)} \mathbf{C}_{k,j(y_v)}\right]\cup C_{k-1}$, $V = \bigcup_{v=1}^{v=u(k)} V_{k,j(y_v)}$, and $L = \left[L_{k,j(y_v)}\right]_{v=1}^{v=u(k)}$. Let

$$\mathbf{C} = \bigcap_{k=1}^{\infty} U_k$$

By construction, $\mathbf{C}$ is compact, 0-dimensional, and, for all k, $\mathbf{C}_k \subset \mathbf{C}$. We must show that $W/G(\mathbf{C},W)$ is an ANR.

It is sufficient to show that $W/G(\mathbf{C},W)$ is 1-LC at every point. This follows from the following two facts whose proofs we omit:

Fact 1: Given any point y^* and open neighborhood U of y^* in $W/G(\mathbf{C},W)$, there is an open set V with $y^* \in V \subset U$ such that any loop in $\bar{f}^{-1}(V)\times\{0\} \subset Map(\bar{f})$ bounds a disk in $\bar{f}^{-1}(U)\times[0,1)\cup \mathbf{C}\times\{1\} \subset Map(\bar{f})$.

Fact 2: Given any loop, s, in $W/G(\mathbf{C},W)$ and any $\epsilon > 0$, then s is ϵ-homotopic to a loop, s', in the complement of $\mathbf{C}$. In particular, s' lifts to W.

Planes in 3-manifolds of Genus One at Infinity

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In this paper, the symbol W will represent a noncompact 3-manifold and P will represent a plane (*i.e.* a space homeomorphic to $\mathbf{R}^2$) that is proper in W (*i.e.* $P \cap K$ is compact for every compact $K \subset W$).

If there is a compact $K_P \subset W$ such that P is not properly homotopic in W into $W - K_P$, then P is said to be *nontrivial in* W. Otherwise P is said to be *trivial* in W. The following is taken from [ST].

LEMMA 1. *If W is irreducible and P is trivial in W, then there is a proper embedding $f : \mathbf{R}^2 \times [0, \infty) \to W$ such that $f(\mathbf{R}^2 \times 0) = P$.*

Let $g \geq 0$ be an integer. We say that W is of *genus g at infinity* if

(1) for every compact $K \subset W$ there is a compact 3-manifold M_K such that $K \subset M_K - \partial M_K$ and ∂M_K is connected and of genus g.

(2) there is a compact $K \subset W$ such that there is no compact 3-manifold $M \subset W$ with connected boundary of genus g with $K \subset M - \partial M$.

LEMMA 2. *If W is irreducible and of genus zero at infinity, then W is homeomorphic to $\mathbf{R}^3$.*

The following are from [HM] and [K], respectively.

THEOREM 3. *If W is irreducible and of genus zero at infinity, then W contains no nontrivial plane.*

THEOREM 4. *If W is contractible, irreducible, and of genus one at infinity, then W contains no nontrivial plane.*

RHETORICAL QUESTION. *What can we say about W if it is irreducible, of genus one at infinity, and contains a nontrivial plane?*

Suppose that N is a noncompact 3-manifold such that ∂N has two components each of which is a plane and for every compact $K \subset W$ there is a closed 3-cell $C_K \subset W$ such that C_K contains K and meets each component of ∂N in a single disk. Then we say that N is a *nearnode with two faces.* A more detailed treatment can be found in [W].

Suppose that H is a noncompact 3-manifold and that the manifold N obtained from H by splitting along a proper plane Q is a nearnode with

two faces. Then we say that H is a *nearnode with one handle*. Note that $\pi_1(H) = \mathbf{Z}$ and H is irreducible in this case.

Let Q' and Q'' be the components of ∂N. Let $\eta : N \to H$ be the quotient map associated with the splitting of H along Q. Then $\eta(\partial N) = Q$. Therefore Q is nonseparating, and so by Lemma 1 it follows that Q is nontrivial in H. Let $K \subset H$ be compact. Let $K' = \eta^{-1}(K)$. It follows that K' is compact. Let C be a closed 3-cell in N such that C meets each component of ∂N in a disk and such that $K' \subset C - Fr(C)$. It is not difficult to argue that C can be chosen so that $\eta(Q' \cap C) \subset \eta(Q'' \cap C)$. It follows that $\eta(C)$ is a solid torus. Since $K \subset \eta(C)$ and since H is not $\mathbf{R}^3$, it follows that H is of genus one at infinity.

The following is proven in detail in §5 of [W].

THEOREM 5. *If $P \subset W$ is nontrivial and that W is irreducible and of genus one at infinity, then W split along P is a nearnode with two faces, i.e. W is a nearnode with one handle.*

OUTLINE OF PROOF: Let N be W split along P and let $\eta : N \to W$ be the quotient map. Suppose that $K' \subset N$ is compact and let $K^* = \eta(K')$.

Let K be a compact, connected subset of W such that P is not homotopic in W into $W - K$ and such that $K \cap P$ is connected. Note that since P is not homotopic into $W - K$, then K is not contained inside a ball in W.

Now choose a compact, connected 3-manifold $M \subset W$ such that

(1) ∂M is a torus,
(2) $K \subset M - \partial M$, and
(3) $\sharp(\partial M \cap P)$ is minimal with respect to (1) and (2).

STEP 1. *∂M is incompressible in $W - K$.*

PROOF: Compressing ∂M would put K in a ball in W.

STEP 2. *No component of $\partial M \cap P$ is contractible in ∂M.*

PROOF: It can be argued that if such a component existed, that a disk $D \subset P$ could be found such that $D \cap K = \emptyset$ and $\partial D = D \cap \partial M$. Step 1 yields a disk $E \subset \partial M$ with $\partial E = \partial D$. The irreducibility of $W - K$ (check this) yields a 3-cell $B \subset W - K$ bounded by $E \cup D$. Modifying M by digging B out of M or adding B to M would reduce $\sharp(\partial M \cap P)$.

STEP 3. *If J is a component of $\partial M \cap P$ that bounds a disk $E_J \subset P$, then $K \cap P \subset E_J - J$.*

PROOF: Step 1 and step 2 lead to a contradiction if it is assumed otherwise.

STEP 4. *M is a solid torus and each component of $\partial M \cap P$ is a meridian of M.*

PROOF: Step 3 implies that the components of $\partial M \cap P$ are concentric about $K \cap P$. The innermost disk compresses ∂M in W and is on the same side of the resulting 2-sphere as infinity.

STEP 5. *$\partial M \cap P$ is connected.*

PROOF: If $\partial M \cap P$ is not connected, it follows by steps 3 and 4 that there is an annulus component A of $P \cap cl(W - M)$ such that each component of ∂A is a meridian of M. Let D and E be disjoint meridian disks of M whose boundaries are the components of ∂A. Note that $A \cup D \cup E$ is a 2-sphere that must bound a 3-cell $B \subset W$. Note that $B \cup M$ is a solid torus. We now see that it is possible to reduce $\sharp(\partial M \cap P)$.

Now observe that $\eta(M)$ is a closed 3-cell that contains K' and meets each component of N in a disk.

REFERENCES

[HM] O. G. Harrold, Jr. and E. E. Moise, *Almost locally polyhedral spheres*, Annals of Mathematics **57** 3 (1953), 575–578.

[K] Shin'ichi Kinoshita, *On finite cyclic actions on contractible open 3-manifolds and strong irreducibility*, "Proceedings of the Second Conference on Compact Group Actions, Volume II. Lecture Notes in Mathematics 299," Springer-Verlag, 1972, pp. 323–327.

[ST] Peter Scott and Thomas Tucker, *Some examples of exotic non-compact 3-manifolds*, Quart. J. Math. Oxford **40** (1989), 481–499.

[W] Bobby Neal Winters, *Planes in 3-manifolds of finite genus at infinity*, in preparation.

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Tychonoff's Theorem

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A simple proof of Tychonoff's Theorem is given. The proof is, in spirit, much like Tychonoff's original proof which is also given.

Tychonoff's Theorem states that the arbitrary product of compact spaces is compact. There are two standard proofs of this theorem found in topology books. One proof uses Alexander's Lemma which states that a space is compact if every cover by a subbasis has a finite subcollection that covers [1, p. 139], [2, p.4]. An alternative proof by Bourbaki [1, p. 143], [3, pp. 232-3] uses a formulation of compactness in terms of closed sets instead of open sets; i.e., a space X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X satisfying the property that every finite subcollection has non empty intersection, then $\bigcap_{C \in \mathcal{C}} C$ is non empty. Of course, each of these proofs will show that the product of two compact spaces is compact. However, in each case the proof is far more complicated than any standard simple proof that the product of two compact spaces is compact. Indeed, Munkres [3, p. 229] seems to think that the Tychonoff Theorem is a "deep" theorem with no straightforward proof.

In this note we give two simple proofs that the product of two compact spaces is compact. Each of these proofs generalizes easily to Tychonoff's Theorem using only the fact that any set can be well-ordered. In the special case of the countably infinite product of compact spaces, this is just ordinary mathematical induction.

The first proof is Tychonoff's original proof [4]. Tychonoff's proof uses a non-trivial alternative formulation of compactness that seems almost forgotten among modern day topologists [2, p. 4]. The second proof uses an obvious formulation of compactness that is only slightly different from the usual covering definition. This proof is, in spirit, much like Tychonoff's. A slight variation of the second proof has been known and used by professors and students at the University of Wisconsin for many (over 30) years. But, to my knowledge, it is almost unknown to others.

Definitions.

For completeness, we provide definitions of products and the product topology.

Let $\{X_\alpha \mid \alpha \in J\}$ be a collection of sets. The product $\prod_{\alpha \in J} X_\alpha$ is defined to be the collection of all functions $f: J \to \bigcup_{\alpha \in J} X_\alpha$ such that $f(\alpha) \in X_\alpha$. We often write $f = (x_\alpha)_{\alpha \in J}$ or simply (x_α) where $f(\alpha) = x_\alpha$. For each α there is a *projection function* onto X_α, $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ defined by $P_\alpha(f) = f(\alpha)$. If each X_α is a topological space, then the *product topology* on $\prod_{\alpha \in J} X_\alpha$ is the *smallest* topology which makes the projection

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functions continuous. Let F be a finite subset of J and for each $\alpha \in F$, let U_α be an open subset of X_α. Sets of the form $U = \bigcap_{\alpha \in F} P_\alpha^{-1}(U_\alpha)$ form a basis for the product topology. These basic open sets can also be written as $U = \prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α and $U_\alpha = X_\alpha$ for all α except α in some finite set F. R. H. Bing called this topology the "finite gate topology." He thought of the basis elements as having a finite collection of gates. A point f of $\prod_{\alpha \in J} X_\alpha$ is in U if and only if f *goes through* the finitely many gates U_α, $\alpha \in F$; i.e., $f(\alpha) \in U_\alpha$ for each $\alpha \in F$. For a basic open set U, we say that U has a *gate* in X_α if $P_\alpha(U)$ is a proper subset of X_α. A nice illustration of the product topology is $\prod_{\alpha \in \mathbf{R}} \mathbf{R}_\alpha$ where **R** is the set of real numbers and each $\mathbf{R}_\alpha$ is a copy of the real numbers; i.e., all functions from the reals to the reals. A basic open set U is given by picking a finite set of real numbers F and a finite number of open intervals U_i, $i \in F$. A function f from the reals to the reals is in U if $f(i) \in U_i$ for each $i \in F$. The graph of f must pass through the gates $\{i\} \times U_i$.

Formulations of Compactness.

Definition. A topological space X is said to be compact if every open covering of X has a finite subcollection that covers. The following are equivalent formulations of compactness:

A. A topological space X is compact if and only if for each collection of open sets with the property that no finite subcollection covers, there is a point $x \in X$ so that x is not covered by the collection of open sets.

B. A topological space X is compact if and only if for each collection of closed subsets of X with the finite intersection property (the intersection of finitely many elements of the set is non empty) the intersection of all elements of the collection is non empty.

Definition. Let E be a subset of a topological space. We say that a limit point x of E is a *perfect limit point* of E if for every neighborhood U of x, the cardinality of $U \cap E$ is the same as the cardinality of E.

C. A topological space X is compact if and only if each infinite subset E of X has a perfect limit point.

The proofs of A and B are immediate. Alexander's Lemma uses A. The Bourbaki proof uses B. Our simple proof uses A. Tychonoff used C, and this fact requires some elementary cardinal arithmetic which probably explains why it is not well-known. We give a proof here for completeness.

Proof of C. Suppose X is compact and E is an infinite set with no perfect limit point. For each point x of X choose a neighborhood U_x so that the cardinality of $U_x \cap E$ is less than the cardinality of E. A finite subcollection $U_{x_1}, U_{x_2}, \cdots, U_{x_n}$, covers X. Then E is the

finite union of $U_{x_i} \cap E$. But the finite union of sets of cardinality less than E must also have cardinality less than E, a contradiction.

Suppose every infinite set has a perfect limit point. If X fails to be compact, there is an infinite collection $\{U_\alpha \mid \alpha \in J\}$, of open subsets of X which covers X and so that no finite subcollection covers. We may also assume that the set J has the minimum cardinality with this property. We further suppose that J is well ordered so that for each α, the cardinality of $\{\beta \in J \mid \beta < \alpha\}$ is less than the cardinality of J and $U_\alpha \not\subset \bigcup\{U_\beta \mid \beta < \alpha\}$. We define a set $E = \{x_\alpha \mid \alpha \in J\}$ so that $x_\alpha \in U_\alpha \setminus \bigcup\{U_\beta \mid \beta < \alpha\}$. The cardinality of E is the same as the cardinality of J. If x is a point of X, then x lies in some U_α, but the cardinality of $U_\alpha \cap E$ is less than the cardinality of E, contradicting the fact that every infinite set has a perfect limit point.

Tychonoff's Proof.

Theorem. Let X and Y be compact spaces then $X \times Y$ is compact.

Proof . Let E be an infinite subset of $X \times Y$. We first show that there is an $a \in X$ so that for each neighborhood U of a, the cardinality of $(U \times Y) \cap E$ is the same as the cardinality of E. If no such a exists, then for each $x \in X$, there exists an open set U_x containing x so that $(U_x \times Y) \cap E$ has cardinality less than E. By compactness a finite subcollection U_{x_1}, $U_{x_2}, \cdots, U_{x_m}$, covers X. Hence, $E = (X \times Y) \cap E = ((U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_m}) \times Y) \cap E$ $= \bigcup_{i=1}^{m} ((U_{x_i} \times Y) \cap E))$. This is a contradiction since the infinite set E cannot be written as the finite union of sets of cardinality less than E.

We now show that there is a $b \in Y$ so that for each basic open set of the form $U \times V$ containing (a,b), $(U \times V) \cap E$ has the same cardinality as E. If no such b exists, then for each $y \in Y$, there exists an open set $U_y \times V_y$ containing (a,y) so that the cardinality of $(U_y \times V_y) \cap E$ has cardinality less than E. By compactness a finite subcollection $V_{y_1}, V_{y_2}, \cdots, V_{y_n}$, covers Y.

Set $U = \bigcap_{i=1}^{n} U_{y_i}$. Then $(U \times Y) \cap E = (U \times (V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_n})) \cap E =$ $\bigcup_{i=1}^{n} ((U \times V_{y_i}) \cap E)) \subset \bigcup_{i=1}^{n} ((U_{y_i} \times V_{y_i}) \cap E))$. This is a contradiction since the infinite set $(U \times Y) \cap E$ cannot be contained in the finite union of sets of cardinality less than $(U \times Y) \cap E$. We now see that the point (a,b) is a perfect limit point of E, so $X \times Y$ is compact.

Tychonoff's Theorem. Let $\{X_\alpha \mid \alpha \in J\}$ be a collection of compact topological spaces. Then the product $\prod_{\alpha \in J} X_\alpha$ is compact.

Proof. We assume that J is well-ordered and that an infinite set E is given. Inductively define $a_\gamma \in X_\gamma$ so that if U is any basic open set containing $\prod_{\alpha \leq \gamma}\{a_\alpha\} \times \prod_{\alpha > \gamma} X_\alpha$, then the cardinality of $U \cap E$ is the same as the cardinality of E. Then the point (a_α) is a perfect limit point of E. Hence the product is compact.

The Simple Proof.

Theorem. Let X and Y be compact spaces then $X \times Y$ is compact.

Proof. Let $\mathcal{G}$ be a collection of open sets of $X \times Y$ so that no finite subcollection of $\mathcal{G}$ covers. We first show that there is an $a \in X$ so that for each neighborhood U of a no finite subcollection of $\mathcal{G}$ covers $U \times Y$. If no such a exists, then for each $x \in X$, there exists an open set U_x containing x so that a finite subcollection of $\mathcal{G}$ covers $U_x \times Y$. By compactness a finite subcollection $U_{x_1}, U_{x_2}, \cdots, U_{x_m}$, covers X. Hence, $X \times Y =$

$(U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_m}) \times Y = \bigcup_{i=1}^{m} (U_{x_i} \times Y)$. This is a contradiction because $X \times Y$ cannot be written as the finite union of sets each of which can be covered by a finite subcollection of $\mathcal{G}$.

We now show that there is a $b \in Y$ so that for each basic open set of the form $U \times V$ containing (a,b), no finite subcollection of $\mathcal{G}$ covers $U \times V$. If no such b exists, then for each $y \in Y$, there exists an open set $U_y \times V_y$ containing (a,y) so that $U_y \times V_y$ is covered by a finite subcollection of $\mathcal{G}$. By compactness a finite subcollection $V_{y_1}, V_{y_2}, \cdots, V_{y_n}$,

covers Y. Set $U = \bigcap_{i=1}^{n} U_{y_i}$. Then $U \times Y = U \times (V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_n}) =$

$\bigcup_{i=1}^{n}(U \times V_{y_i}) \subset \bigcup_{i=1}^{n}((U_{y_i} \times V_{y_i})$. This is a contradiction because $U \times Y$ cannot be contained in the finite union of sets each of which can be covered by a finite subcollection of $\mathcal{G}$. Thus the point (a,b) is not covered by any element of $\mathcal{G}$, and we see that $X \times Y$ is compact.

Tychonoff's Theorem. Let $\{X_\alpha \mid \alpha \in J\}$ be a collection of compact topological spaces. Then the product $\prod_{\alpha \in J} X_\alpha$ is compact.

Proof. We assume that J is well-ordered and that a covering $\mathcal{G}$ is given so that no finite subcollection of $\mathcal{G}$ covers. Inductively define $a_\gamma \in X_\gamma$ so that if U is any basic open set containing $\prod_{\alpha \leq \gamma}\{a_\alpha\} \times \prod_{\alpha > \gamma} X_\alpha$, then no finite subcollection of $\mathcal{G}$ covers U. Thus the point (a_α) is not covered by any element of G. Hence the product is compact.

References

[1] J. L. Kelley, *General Topology*, D. Van Nostrand Company, New York, (1955).

[2] K. Kuratowski, *Topology*, Volume II, Academic Press, New York, (1968).

[3] J. R. Munkres, *Topology a first course,* Prentice-Hall, New Jersey, (1975).

[4] A. Tychonoff, *Über die topologische Erweiterung von Räumen*, Math. Ann. **102**, (1929) 544-561.

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Problem Session

1. (Bestvina) First, some background: Defn: A group, G, with finite $KG_1 = K$ is a Z-group if the universal cover, $\tilde{K}$, can be embedded in an ER, E, so that $E - \tilde{K}$ is a Z-set in E and for all compact $C \subset \tilde{K}$, the set of translates of C is a null sequence in E.

Ex. 1: $G = Z$, $K = S^1$, $\tilde{K} \cong (0,1)$, and $E = [0,1]$.

Ex. 2: $G = F_2$, $K = S^1 \vee S^1$, $\tilde{K}$ embedded in the plane E^2, and $E = closure(\tilde{K}, E^2)$ with $E - \tilde{K}$ a Cantor set.

Ex. 3: G a negatively curved group (torsion free).

Ex. 4: CAT(O) groups.

Theorem (Ferry-Winberger): If $f : M^n \xrightarrow{h.e.} N^n$ (aspherical closed manifolds), $n \geq 6$, and $\pi_1(M) = \pi_1(N)$ is a Z-group, then f is homotopic to a homeomorphism.

Question: Is your favorite group a Z-group?

2. (Daverman) Suppose $p : M^3 \xrightarrow{CE} X$ with $D^2 \subset X$. For any $\epsilon > 0$ does there exist an embedding $\lambda : D^2 \to M^3$ so that $p\lambda$ is ϵ-close to inclusion: $D^2 \to X$?

3. (Daverman) Suppose Σ^n is a homology n-sphere. Does there exist a compact manifold N^{n+1} such that

$$\pi_1 \begin{pmatrix} \partial N^{n+1} \\ \| \\ \Sigma \end{pmatrix} \xrightarrow[incl]{} \pi_1\left(N^{n+1}\right) \text{ is 1:1} \tag{1}$$

$$H_*\left(N^{n+1}; \mathbf{Z}\right) = 0 \tag{2}$$

Is there a $\Sigma^3 \neq S^3$ for which the answer is yes? (Can get N^4 satisfying (1) and $H_1\left(N^4\right) = 0$)

4. (Daverman) Does there exist a closed manifold N^n such that if

$$p : M \to B^{finite\ dim'l\ metric}$$

is any closed map with $p^{-1}b \cong N$ for all $b \in B$ (M any manifold), then p is an approximate fibration?

5. (Daverman) Does there exist a finitely presented group which contains an infinite descending chain of perfect, normal subgroups?

6. (Garity) Which cell-like subsets of $\mathbb{R}^3$ can be realized as attractors of homeomorphisms?

7. (Garity) Does there exist an X which is LC^n for all n but which is not 2-homogeneous?

8. (Silver) Let K be an n-knot embedding of S^n in S^{n+2}. A Seifert manifold for K is a compact, connected, orientable $(n-1)$−manifold with $bdy(V) = K$. A Seifert manifold is minimal if the induced homorphism $\pi_1(int(V)) \mapsto \pi_1(S^{n+2} - K)$ is injective. Does every 2-knot have a minimal Seifert manifold? Answer is no for $n \geq 3$ (Hillman).

9. (Guilbault) Does there exist an embedded 2-sphere $\Sigma^2 \subset M^4$ with a neighborhood N such that $N - \Sigma^2$ is a counterexample to a 4-eimensional version of Siebenmann's thesis?

10. (Guilbault) There exist compact contractible n-manifolds C^n ($C^n \neq B^n$) which contain a pair of disjoint spines. Do all of these manifolds have disjoint spines?

11. (Dobrowolski) Let $L(\mathbf{R}^2)$ be the hyperspace consisting of locally connected continua in the space of all continua $C(\mathbf{R}^2)$. For $m = 1, 2, ...,$ let A_m be the set of all $X \in C(\mathbf{R}^2)$ such that X can be covered by at most m subcontinua each with diameter less than or equal one third the diameter of X. Can the identity on $C(\mathbf{R}^2)$ be (uniformly) approximated by maps $\psi : C(\mathbf{R}^2) \mapsto C(\mathbf{R}^2) \setminus A_m$?

12. (Dobrowolski) Consider two groups G_1 and G_2 with

$$G_1 = \left\{ \frac{n_k}{k} \in l^2 | n_k \in \mathbf{Z} \right\}$$

$$G_2 = \left\{ \frac{n_k}{k^2} \in l^2 | n_k \in \mathbf{Z} \right\}$$

G_1 and G_2 are 1-dimensional and totally disconnected; G_1 and G_2 are not isomorphic. Are G_1 and G_2 homeomorphic?

13. (Wright) Let M be the countably infinite sum of a fixed Whitehead manifold. Then M is the covering space of a non-compact 3-manifold. Can M be the covering space of a compact 3-manifold?

14. (Cannon) Suppose a closed 3-manifold admits a Riemannian metric with negative curvature. Does it admit a metric of constant negative curvature? Is π_1 residually finite? Does it have any non-trivial finite sheeted covers? Do negatively curved groups have torsion free subgroups of finite index?

Modification: Is this manifold homotopy equivalent to such a thing?

15. (Cannon) Thompson's group is the group of PL homeomorphisms of $[0, 1]$ fixing 0 and 1, having slopes powers of 2, and dyadic rational invariance as a set. This group is generated by two elements. Give a good algorithm for finding a word of minimal length in these generators. Is Thompson's group amenable?

16. (Banakh) Assume that a topological group $G \in ANE(Comp)$ can be represented as a direct limit of finite-dimensional compacta. Is G an $\mathbf{R}^\infty$-manifold? (Here, $\mathbf{R}^\infty$ is the standard direct limit of $\mathbf{R}^n$, $n = 1, 2, \cdots,$).

17. (Banakh) For a compactum K consider the free topological linear space over K, $L(K)$. Explicitly, ssume that K is embedded in a linear topological space X as a linearly independent set. Then $span(K) = \bigcup_{n=1}^{\infty} K_n$ where $K_n = \{\Sigma_{i=1}^n \lambda_i x_i : \Sigma_{i=1}^n |\lambda_i| \leq n, x_i \in K$

We define $L(K)$ to be the direct limit of the K_n's. Is for every compactum K the space $L(K)$ an absolute extensor for compacta?

18. (Banakh) Is every topological linear space an absolute extensor for compacta?

19. (Banakh) Let $(X_1, \ldots X_n)$ and $(X'_1, \ldots X'_n)$be n-tuples ($n < \infty$). Does $(X_i, X_j) \cong (X'_i, X'_j)$ for $1 \leq i, j \leq n$ imply $(X_1, \ldots X_n) \cong (X'_1, \ldots X'_n)$? (Here, $\cong$ means 'homeomorphic to')

20. (Banakh) Let $n > 1$. Does there exist a continuous, extension operator, $T : C[0,1] \mapsto C[-1,1]$ such that $T(f)[0,1] = f$, $f \in C[0,1]$, and $T(C^n[0,1]) \subset C^n[-1,1]$? Here, $C^n[a,b] \subset C[a,b]$ is the subspace of $C[a,b]$ consisting of all continuously differentiable functions.

21. (Banakh) Let $T : X \mapsto Y$ be a linear continuous operator from a separable Banach space X to a Banach space Y. Does $T(X)$ belong to the Borelian class

$$M_2(Y) = \left\{ A \subset Y : A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m} \right\}$$

where each $A_{n,m}$ is closed in Y?

22. (Banakh) Let $p : X \mapsto Y$ be a soft map of complete-metrizable spaces. Is the map $p \circ pr_X : X \times l_2 \mapsto Y$ the trivial l_2-bundle?

23. (Tinsley) Is there a compactum which is defined by nearly 1-movable acyclic compacta but is not nearly 1-movable itself? Is there a compactum which is defined by homology cells but is not nearly 1-movable itself?

24. (Tinsley) Can type 3 wildness arise via an acyclic extended mapping cylinder construction?