Proceedings of the Twenty-First Annual Workshop in Geometric Topology

University of Wisconsin–Milwaukee
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Preface

The Twenty-First Annual Workshop in Geometric Topology was held at the University of Wisconsin–Milwaukee on June 10-12, 2004. A list of the participants can be found later in these proceedings.

The principal speaker for the workshop was Professor Peter Teichner of the University of California, Berkeley. Professor Teichner presented a series of three one-hour lectures titled "New obstructions for embedding 2-spheres into 4-manifolds." Some details about these lectures are included later in these proceedings.

As always, the workshop included a number of shorter talks contributed by the participants, and concluded with a problem session. Summaries of several contributed talks are printed in these proceedings, as is a summary of the problem session.

Support. Financial support for the workshop was provided by the National Science Foundation (Grant DMS-0407583) and by the University of Wisconsin–Milwaukee.

Organizers. The Workshops in Geometric Topology are organized by:

- ◊ Fredric Ancel, University of Wisconsin–Milwaukee,
- ◊ Dennis Garity, Oregon State University,
- ◊ Craig Guilbault, University of Wisconsin–Milwaukee,
- ◊ Frederick Tinsley, Colorado College,
- ◊ Gerard Venema, Calvin College, and
- ◊ David Wright, Brigham Young University.

The organizers also serve as editors of these proceedings.

History of the Workshops in Geometric Topology

Year	Workshop Location	Principal Speaker
2005	Colorado College	Thomas Farrell
2004	University of Wisconsin–Milwaukee	Peter Teichner
2003	Park City, Utah (BYU)	Martin Bridson
2002	Calvin College	Alexander Dranishnikov
2001	Oregon State University	Abigail Thompson
2000	Colorado College	Robert Gompf
1999	University of Wisconsin–Milwaukee	Robert Edwards
1998	Park City, Utah (BYU)	Steve Ferry
1997	Oregon State University	James Cannon
1996	Colorado College	Michael Freedman
1995	University of Wisconsin–Milwaukee	Shmuel Weinberger
1994	Park City, Utah (BYU)	Michael Davis
1993	Oregon State Univ. and Newport, OR	John Bryant
1992	Colorado College	Mladen Bestvina
1991	University of Wisconsin–Milwaukee	Andrew Casson
1990	Oregon State University	Robert Daverman
1989	Brigham Young University	John Luecke
1988	Colorado College	John Hempel
1987	Oregon State University	Robert Edwards
1986	Colorado College	John Walsh
1985	Colorado College	Robert Daverman
1984	Brigham Young University	none

List of Participants (2004)

Ric Ancel	University of Wisconsin-Milwaukee
Anthony Bedenokovic	Bradley University
Nikolay Brodskiy	University of Tennessee
James Cannon	Brigham Young University
Stephen Chan	University of California, Los Angeles
James Conant	University of Tennessee
Greg Conner	Brigham Young University
Bob Daverman	University of Tennessee
Tadek Dobrowolski	Pittsburg State University
Robert Edwards	University of California, Los Angeles
Steve Ferry	Rutgers University
Hanspeter Fischer	Ball State University
Tom Fleming	University of California, San Diego
Paul Fonstad	University of Wisconsin-Milwaukee
Dennis Garity	Oregon State University
Kailash Ghimire	Oregon State University
Brent Gorbutt	Brigham Young University
Craig Guilbault	University of Wisconsin-Milwaukee
Yusuf Z. Gurtas	Suffolk County CC
Denise Halverson	Brigham Young University
Rena Hull	University of California, Santa Barbara
Margaret May	University of Wisconsin-Milwaukee
Mark Meilstrup	Brigham Young University
Atish Mitra	University of Tennessee
Christopher Mooney	University of Wisconsin-Milwaukee
Boris Okun	University of Wisconsin-Milwaukee
David Radcliffe	University of Minnesota
Konstantin Salikhov	University of Maryland
Carrie Schermetzler	University of Wisconsin-Milwaukee
Rob Schneiderman	Courant Institute

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List of Participants (continued)

Tim Schroeder	University of Wisconsin-Milwaukee
David Snyder	Texas State University, San Marcos
Peter Teichner	University of California, Berkeley
Tom Thickstun	Texas State University, San Marcos
Mat Timm	Bradley University
Fred Tinsley	Colorado College
Anthony Van Groningen	University of Wisconsin-Milwaukee
Violeta Vasilevska	University of Tennessee
Gerard Venema	Calvin College
Shmuel Weinberger	University of Chicago
Julia Wilson	SUNY Fredonia
Bobby Winters	Pittsburg State University
David Wright	Brigham Young University

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Principal Lectures

New Obstructions to Embedding 2–spheres in S^4

PETER TEICHNER

June 10-12, 2004

The principal speaker for the Twenty-First Annual Workshop in Geometric Topology was Professor Peter Teichner of the University of California, Berkeley. His three-lecture series on "New Obstructions for Embedding 2–spheres in S^4" was the centerpiece of the workshop. The official abstract for these lectures was the following:

ABSTRACT. *In joint work with Rob Schneiderman, we have developed a new obstruction theory for the embedding problem for 2-spheres in 4-manifolds. It is given in terms of the intersection theory of Whitney towers, immersed in the 4-manifold, and it is related to Milnor invariants and the Kontsevich integral in the easiest cases (where the 4-manifold is given by attaching 2-handles to a link in the 3-sphere). As a consequence, we give an intersection theoretic explanation of the Milnor invariants, and we relate them to the existence of embedded gropes in the 4-ball.*

In this sequence of talks, we shall give an outline of the theory, explain the main results, and discuss the remaining open problems. There are 3 papers, all joint with Rob Schneiderman (and one also joint with Jim Conant) available on my homepage.

At the time these lectures were given, all of the main results had already been written up and were made available to workshop participants—primarily in preprint form. For this reason, the traditional writeup of the main lectures is not included in these proceedings. Instead, we provide full bibliographic information and electronic links for the corresponding papers. In addition, we have posted on the workshop website, scanned copies of the over-head slides used in each of the three lectures.

Papers

Rob Schneiderman and Peter Teichner, *Higher order intersection numbers of 2-spheres in 4-manifolds*, Algebraic & Geometric Topology, 1 (2001), 1-29. (www.maths.warwick.ac.uk/agt/AGTVol1/agt-1-1.abs.html).

Rob Schneiderman and Peter Teichner, *Whitney towers and the Kontsevich integral*, Proceedings of a Conference in Honor of Andrew Casson, UT Austin 2003., Geo. & Top. Monogr. 7 (2004), 101-134.
(www.maths.warwick.ac.uk/gt/GTMon7/paper4.abs.html).

James Conant, Rob Schneiderman and Peter Teichner, *Jacobi identities in low-dimensional topology*, to appear in Compositio Mathematica.
(xxx.lanl.gov/abs/math.GT/0401427).

Slides from the Lectures

Lecture 1. *Intersection Theory for Whitney Towers:*
www.uwm.edu/Dept/Math/conf/topology/Lecture1.pdf

Lecture 2. *Whitney Towers and Milnor invariants:*
www.uwm.edu/Dept/Math/conf/topology/Lecture2.pdf

Lecture 3. *Gropes in 3– and 4–space:*
www.uwm.edu/Dept/Math/conf/topology/Lecture3.pdf

Unique path lifting and the shape group

Hanspeter Fischer* and Andreas Zastrow†

June 2004

Abstract

If a paracompact Hausdorff space X admits a (classical) universal covering space, then the natural homomorphism $\varphi : \pi_1(X) \to \check{\pi}_1(X)$ from the fundamental group to its first shape homotopy group is an isomorphism.

We present a partial converse: a path connected topological space X admits a *generalized* universal covering space if $\varphi : \pi_1(X) \to \check{\pi}_1(X)$ is injective. This generalized notion of universal covering $p : \tilde{X} \to X$ at which we arrive, enjoys most of the usual properties with the possible exception of evenly covered neighborhoods.

General Assumption. *(X, x_0) will be a pointed path connected topological space.*

§1 Introduction. Recall that a continuous map $p : \bar{X} \to X$ is called a *covering* of X, and $\bar{X}$ is called a *covering space* of X, if for every $x \in X$ there is an open subset U of X with $x \in U$ and such that U is *evenly covered* by p, that is, $p^{-1}(U)$ is the disjoint union of open subsets of $\bar{X}$ each of which is mapped homeomorphically onto U by p. In the classical theory, one assumes that X is, in addition, locally path connected and wishes to classify all path connected covering spaces of X and to find among them a *universal covering space*, that is, a covering $p : \tilde{X} \to X$ with the property that for every covering $q : \bar{X} \to X$ by a path connected space $\bar{X}$ there is a covering $\bar{q} : \tilde{X} \to \bar{X}$ such that $q \circ \bar{q} = p$. If X is locally path connected, we have the following well-known result, which can be found, for example, in [13] and [14]:

Every simply connected covering space of X is a universal covering space. Moroever, X admits a simply connected covering space if and only if X is semilocally simply connected, in which case the coverings $p : (\bar{X}, \bar{x}) \to (X, x_0)$ with path connected $\bar{X}$ are in direct correspondence with the conjugacy classes of subgroups of $\pi_1(X, x_0)$, via the monomorphism $p_\# : \pi_1(\bar{X}, \bar{x}) \to \pi_1(X, x_0)$.

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Outside of semilocally simply connected spaces, the theory is not as pleasant. While it is still possible, based on Fox's concept of overlay, to classify specific types of covering spaces via the fundamental pro-group pro-$\pi_1(X, x_0)$ [8, 11], no universal covering space might be available.

For most applications, however, the particular usefulness of a universal covering space does not lie in the evenly covered neighborhoods, but rather in the following properties:

(U_1) The space $\tilde{X}$ is path connected, locally path connected and simply connected.

(U_2) The map $p : \tilde{X} \to X$ is a continuous surjection.

(U_3) For every path connected and locally path connected topological space Y, every continuous $f : (Y, y) \to (X, x)$ with $f_\#(\pi_1(Y, y)) = 1$, and every $\tilde{x}$ in $\tilde{X}$ with $p(\tilde{x}) = x$, there exists a *unique* continuous lift $g : (Y, y) \to (\tilde{X}, \tilde{x})$ with $f = p \circ g$.

(U_4) The group of covering transformations $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ is isomorphic to $\pi_1(X, x_0)$.

(U_5) The map $p : \tilde{X} \to X$ is open so that $\tilde{X}/G \approx X$, where $G = \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$.

Note that Properties U_1, U_2 and U_3 uniquely characterize $p : \tilde{X} \to X$ and that together they imply U_4. Note also that, in the presence of U_1 and U_2, we can expect U_5 to hold only if X is locally path connected.

In the absence of a universal covering, still certain Hurewicz fibrations [14] and certain Serre fibrations [1] $p : E \to X$ with simply connected E and additional helpful properties are sometimes available. However, these fibrations lack, in general, most of the properties on the above list, notably local path connectivity of E on which the other properties hinge—even if X itself is locally path connected.

Our approach is fundamentally different. Seeking the middle ground between restricting ourselves to overlays and considering very general fibrations, we examine the "standard" construction of the classical universal covering and ask the question: under what circumstances will it have properties U_1–U_5? This approach is in the spirit of [15] as well as [2]. We therefore make the following

Definition. If a map $p : \tilde{X} \to X$ satisfies Properties U_1–U_3, we call it the *generalized universal covering* of X and we call $\tilde{X}$ the *generalized universal covering space* of X.

Remark. Every generalized universal covering is a Serre fibration, since it has the homotopy lifting property with respect to $[0, 1]^n$ for all n. Consequently, the homomorphisms $p_\# : \pi_i(\tilde{X}) \to \pi_i(X)$ are isomorphisms for $i > 1$. However, a generalized universal covering need not be a covering or a Hurewicz fibration (see Example 3).

§2 The "standard" construction. Note that a generalized universal cover $\tilde{X}$ of X, if it exists, must be in one-to-one correspondence with the homotopy classes of paths in X which emanate from x_0. Accordingly, there is only one way to define the set $\tilde{X}$: let $\mathcal{P}(X)$ be the set of all continuous maps $\alpha : [0, 1] \to X$ such that $\alpha(0) = x_0$. On

$\mathcal{P}(X)$ consider the equivalence relation given by $\alpha \sim \beta$ if and only if $\alpha(1) = \beta(1)$ and α is homotopic to β within X, relative to their common endpoints. Let $[\alpha]$ denote the equivalence class of α and let $\tilde{X}$ denote the set of all such equivalence classes. *We will denote the equivalence class containing the constant path at x_0 by $\tilde{x}_0$.*

If now $\hat{p} : \hat{X} \to X$ is a generalized universal covering and if $\hat{p}(\hat{x}_0) = x_0$, then the function which assigns to a point $\hat{x}$ of $\hat{X}$ the homotopy class $\hat{p}_\#([\alpha])$, where α is any path in $\hat{X}$ from $\hat{x}_0$ to $\hat{x}$, is a bijection from $\hat{X}$ onto $\tilde{X}$. Consequently, there is also no ambiguity as to what the projection function $p : \tilde{X} \to X$ ought to be: we define $p([\alpha]) = \alpha(1)$.

Next, we need to decide on the correct topology for $\tilde{X}$. For each $[\alpha] \in \tilde{X}$ and each open subset U of X containing $\alpha(1)$, let $B([\alpha], U)$ denote the set of all $[\beta] \in \tilde{X}$ for which there exists a continuous map $\gamma : [0, 1] \to U$ such that $\gamma(0) = \alpha(1)$, $\gamma(1) = \beta(1)$ and $[\beta] = [\alpha \cdot \gamma]$; where $\alpha \cdot \gamma$ denotes the usual concatenation of the paths α and γ. Notice that $B([\alpha], X) = \tilde{X}$ for all $[\alpha] \in \tilde{X}$ and that if $[\beta] \in B([\alpha], U)$, then $B([\beta], U) = B([\alpha], U)$. Moreover, if $U \subseteq V$, then $B([\alpha], U) \subseteq B([\alpha], V)$. It follows that the collection of all such sets $B([\alpha], U)$ forms a basis for a topology on $\tilde{X}$. From here on forward, we will endow $\tilde{X}$ with this topology.

In the event that X is locally path connected and semilocally simply connected, $\tilde{X}$ is the *classical* universal covering space of X and $p : \tilde{X} \to X$ is the *classical* universal covering of X as defined in [13] and [14]. However, the topology, which we just defined on $\tilde{X}$ is, in general, finer than the quotient topology inherited from the compact-open topology on $\mathcal{P}(X)$. While the two topologies agree when X is locally path connected and semilocally simply connected, the compact-open topology does not, in general, render $\tilde{X}$ locally path connected, as can be observed in Example 1.

Example 1. If $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - 1/n)^2 = (1/n)^2 \text{ for some } n \in \mathbb{N}\}$ is the Hawaiian Earring with base point $x_0 = (0, 0)$ and if $\tilde{X}$ is given the quotient topology inherited from the compact-open topology on $\mathcal{P}(X)$, then $\tilde{X}$ is not locally path connected. Indeed, if l_n denotes the simple closed loop of X of radius $1/n$ based at x_0, then the sequence $\tilde{x}_n = [l_1][l_n][l_1]$ converges to $\tilde{x} = [l_1][l_1]$ in this topology, although there are no small paths connecting $\tilde{x}_n$ and $\tilde{x}$.

In the following lemmas, we list some basic properties of $p : \tilde{X} \to X$. Their fairly straightforward proofs, which can be found in [15, 2], are omitted here.

Lemma 1. *The projection $p : \tilde{X} \to X$ is a continuous surjection; it is open if and only if X is locally path connected.* □

Lemma 2. *Suppose that Y is path connected and locally path connected, that $f : (Y, y) \to (X, x)$ is continuous with $f_\#(\pi_1(Y, y)) = 1$ and that $\tilde{x} \in \tilde{X}$ with $p(\tilde{x}) = x$. Then there is a continuous function $g : (Y, y) \to (\tilde{X}, \tilde{x})$ with $p \circ g = f$.*

Specifically, denoting $\tilde{x} = [\alpha]$, we define $g : Y \to \tilde{X}$ as follows: for $w \in Y$, choose any path $\tau : [0, 1] \to Y$ from $\tau(0) = y$ to $\tau(1) = w$, and put $g(w) = [\alpha \cdot (f \circ \tau)]$. □

Lemma 3. *The space $\tilde{X}$ is path connected and locally path connected; even if X is not locally path connected. But $\tilde{X}$ might not be simply connected (see Example 2).* $\square$

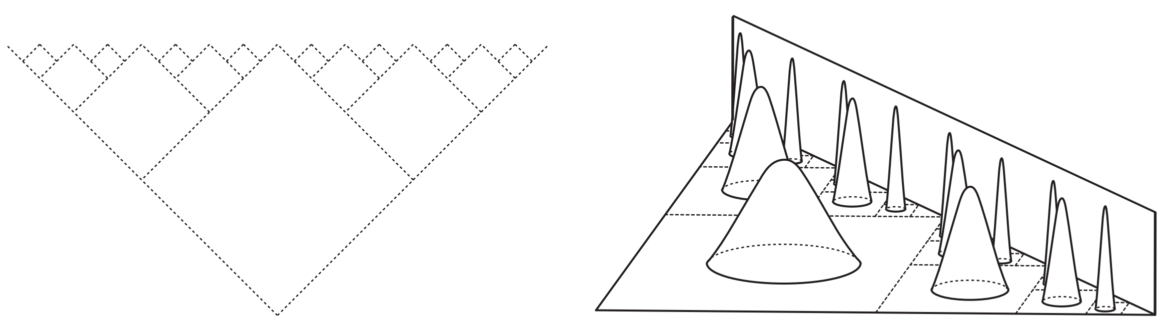
Lemma 4. *Let $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$ with $p(\tilde{x}_1) = p(\tilde{x}_2)$. Then there is a homeomorphism $g : (\tilde{X}, \tilde{x}_1) \to (\tilde{X}, \tilde{x}_2)$ with $p \circ g = p$. Indeed, for $\tilde{x} \in \tilde{X}$ we may define $g(\tilde{x}) = [\beta \cdot \bar{\alpha} \cdot \gamma]$, where $\tilde{x}_1 = [\alpha]$, $\tilde{x}_2 = [\beta]$, $\tilde{x} = [\gamma]$ and $\bar{\alpha}(t) = \alpha(1-t)$.*

Taking $\tilde{x}_1 = \tilde{x}_0$, we see that the action of $\pi_1(X, x_0)$ on $\tilde{X}$ given by $[\beta].[\gamma] = [\beta \cdot \gamma]$, yields an isomorphism of $\pi_1(X, x_0)$ onto a subgroup of $Aut(\tilde{X} \xrightarrow{p} X)$. $\square$

Definition. We say that a map $\hat{p} : \hat{X} \to X$ has the *unique path lifting property*, if whenever we are given two continuous maps $g_1, g_2 : [0,1] \to \hat{X}$ such that $\hat{p} \circ g_1 = \hat{p} \circ g_2$ and $g_1(0) = g_2(0)$, we can conclude that $g_1 = g_2$.

Remark. Suppose a map $\hat{p} : \hat{X} \to X$ has the unique path lifting property. Let Y be any path connected space, $f : (Y, y) \to (X, x)$ any continuous map, and $\hat{x} \in \hat{X}$ with $\hat{p}(\hat{x}) = x$. If there exists a continuous map $g : (Y, y) \to (\hat{X}, \hat{x})$ such that $\hat{p} \circ g = f$, then it is unique.

Example 2. This example is adapted from [15]. Let X be the compact subspace of $\mathbb{R}^3$ obtained as follows: subdivide the interior of an isosceles right triangle into infinitely many squares, accumulating along the hypotenuse as shown below (left), change its embedding into $\mathbb{R}^3$ by elevating the centers of all squares to unit level as indicated, and take the closure in $\mathbb{R}^3$. (For a locally connected non-compact version of this example, instead of taking the closure, only add the boundary of the triangle.)



We claim that $p : \tilde{X} \to X$ does not have the unique path lifting property. Indeed, let $f : [0,1] \to X$ be the path, which runs along the hypotenuse of our triangle with unit speed from left to right. There is, of course, the standard lift $g_1 : [0,1] \to \tilde{X}$ given by $g_1(s) = [f_s]$, where $f_s(t) = f(st)$. However, another continuous lift $g_2 : [0,1] \to \tilde{X}$ is given by $g_2(t) = [\alpha_t]$, where $[\alpha_t]$ is the unique homotopy class of a path α_t which begins at the upper left corner of our triangle, goes straight to the bottom vertex, and then increases back up to $f(t)$, only using the dotted boundary lines of our squares. While $p \circ g_1 = p \circ g_2 = f$ and $g_1(0) = g_2(0)$, we have $g_1(t) \neq g_2(t)$ for all $t > 0$. Hence, $\tilde{X}$ is not simply connected by the following lemma.

Lemma 5. *$p : \tilde{X} \to X$ has the unique path lifting property if and only if $\tilde{X}$ is simply connected.* $\square$

Definition. We call X *homotopically Hausdorff at the point $x \in X$*, if for every $g \in \pi_1(X, x) \setminus \{1\}$ there is an open set $U \subseteq X$ with $x \in U$ such that there is no loop $\alpha : (S^1, *) \to (U, x)$ with $[\alpha] = g$. If X is homotopically Hausdorff at every one of its points, then X is said to be *homotopically Hausdorff.*

Lemma 6. *Suppose X is Hausdorff/metrizable. Then $\tilde{X}$ is Hausdorff/metrizable if and only if X is homotopically Hausdorff.* $\square$

Lemma 7. *If $p : \tilde{X} \to X$ has the unique path lifting property, then X is homotopically Hausdorff.* $\square$

We summarize all of these observations in the following

Proposition 1. *Suppose $p : \tilde{X} \to X$ has the unique path lifting property. Then*

(a) *The map $p : \tilde{X} \to X$ satisfies properties U_1, U_2, U_3 and U_4.*

(b) *If X is locally path connected, then $p : \tilde{X} \to X$ also satisfies U_5.*

(c) *If X is Hausdorff or metrizable, then so is $\tilde{X}$.*

§3 The first shape homotopy group. We briefly recall the definition of the *first shape homotopy group* $\check{\pi}_1(X, x_0)$ of X at x_0 from [12]. For every open cover $\mathcal{U}$ of X, designate one element $* \in \mathcal{U}$ with $x_0 \in *$. Let $\mathcal{C}$ be the collection of all pointed normal covers $(\mathcal{U}, *)$ of X. (Recall that a *normal cover* $\mathcal{U}$ of X is an open cover of X, which admits a partition of unity subordinated to $\mathcal{U}$. This partition of unity can always be chosen to be locally finite.) Then $\mathcal{C}$ is naturally directed by refinement. Denote by $(N(\mathcal{U}), *)$ a geometric realization of the pointed nerve of $\mathcal{U}$, that is, a geometric realization of the abstract simplicial complex $\{\Delta \mid \emptyset \neq \Delta \subseteq \mathcal{U}, \bigcap_{U \in \Delta} U \neq \emptyset\}$ with distinguished vertex $*$. For every $(\mathcal{U}, *), (\mathcal{V}, *) \in \mathcal{C}$ such that $(\mathcal{V}, *)$ refines $(\mathcal{U}, *)$, choose a pointed simplicial map $p_{\mathcal{U}\mathcal{V}} : (N(\mathcal{V}), *) \to (N(\mathcal{U}), *)$ with the property that the vertex corresponding to an element $V \in \mathcal{V}$ gets mapped to a vertex corresponding to an element $U \in \mathcal{U}$ with $V \subseteq U$. (Any assignment on the vertices which is induced by the refinement property will extend linearly.) Then $p_{\mathcal{U}\mathcal{V}}$ is unique up to pointed homotopy and we denote its pointed homotopy class by $[p_{\mathcal{U}\mathcal{V}}]$. For each $(\mathcal{U}, *) \in \mathcal{C}$ choose a pointed map $p_{\mathcal{U}} : (X, x_0) \to (N(\mathcal{U}), *)$ such that $p_{\mathcal{U}}^{-1}(St(U, N(\mathcal{U}))) \subseteq U$ for all $U \in \mathcal{U}$, where $St(U, N(\mathcal{U}))$ denotes the open star of the vertex of $N(\mathcal{U})$ which corresponds to U. (For example, define $p_{\mathcal{U}}$ based on a locally finite partition of unity subordinated to $\mathcal{U}$.) Again, such a map $p_{\mathcal{U}}$ is unique up to pointed homotopy and we denote its pointed homotopy class by $[p_{\mathcal{U}}]$. Then $[p_{\mathcal{U}\mathcal{V}} \circ p_{\mathcal{V}}] = [p_{\mathcal{U}}]$. The so-called (pointed) *Čech expansion*

$$(X, x_0) \xrightarrow{([p_{\mathcal{U}}])} \Big((N(\mathcal{U}), *), [p_{\mathcal{U}\mathcal{V}}], \mathcal{C} \Big)$$

is an HPol_*-expansion, so that we can define the *first shape homotopy group* of X, based at x_0, by $\check{\pi}_1(X, x_0) = \lim\limits_{\longleftarrow} \Big(\pi_1(N(\mathcal{U}), *), p_{\mathcal{U}\mathcal{V}\#}, \mathcal{C} \Big)$.

Since the maps $p_{\mathcal{U}}$ induce homomorphisms $p_{\mathcal{U}\#} : \pi_1(X, x_0) \to \pi_1(N(\mathcal{U}), *)$ such that $p_{\mathcal{U}\#} = p_{\mathcal{U}\mathcal{V}\#} \circ p_{\mathcal{V}\#}$, whenever $(\mathcal{V}, *)$ refines $(\mathcal{U}, *)$, we obtain an induced homomorphism

$$\varphi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$$

given by $\varphi([\alpha]) = ([\alpha_{\mathcal{U}}])$ where $\alpha_{\mathcal{U}} = p_{\mathcal{U}} \circ \alpha$.

§4 An existence theorem.

Theorem 1. *Suppose X is paracompact Hausdorff. If X is locally path connected and semilocally simply connected, then $\varphi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is an isomorphism.*

Remark. For a proof of Theorem 1 in the compact metric case or in the locally simply connected case see [10] and [9], respectively.

PROOF. Since X is assumed to be paracompact Hausdorff, every open cover of X is normal. It therefore suffices to show that every open cover $\mathcal{U}$ of X is refined by an open cover $\mathcal{V}$ of X such that $p_{\mathcal{V}\#} : \pi_1(X, x_0) \to \pi_1(N(\mathcal{V}), *)$ is an isomorphism.

Let $\mathcal{U}$ be any open cover of X. Since X is semilocally simply connected, there is an open cover $\mathcal{W}$ of X, which refines $\mathcal{U}$, such that for every $W \in \mathcal{W}$, $\pi_1(W) \to \pi_1(X)$ is trivial. Since $\mathcal{W}$ is a normal cover of X, there is an open cover $\mathcal{W}'$ of X which star-refines $\mathcal{W}$. That is, for every $W_1' \in \mathcal{W}'$ there is a $W \in \mathcal{W}$ such that for every $W_2' \in \mathcal{W}'$ with $W_1' \cap W_2' \neq \emptyset$ we have $W_2' \subseteq W$. Since X is locally path connected, there is an open cover $\mathcal{V}$ of X, which refines $\mathcal{W}'$ and all whose elements are path connected. We conclude that $\mathcal{V}$ is a cover of X by open path connected sets such that every loop which lies in the union of any two elements of $\mathcal{V}$ is contractible in X. Therefore, $p_{\mathcal{V}\#} : \pi_1(X, x_0) \to \pi_1(N(\mathcal{V}), *)$ is an isomorphism [3, pp. 269–271]. Since $\mathcal{V}$ refines $\mathcal{U}$, the theorem follows. $\square$

Theorem 2. *If $\varphi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is inective, then $p : \tilde{X} \to X$ has the unique path lifting property.*

PROOF. We will show that for any two continuous maps $g_1 : [0,1] \to \tilde{X}$ and $g_2 : [0,1] \to \tilde{X}$ such that $p \circ g_1 = p \circ g_2$, the set $\{t \in [0,1] \mid g_1(t) = g_2(t)\}$ is either empty or all of $[0,1]$. For every $t \in [0,1]$ choose continuous maps $\alpha_t, \beta_t : [0,1] \to X$ with $g_1(t) = [\alpha_t]$ and $g_2(t) = [\beta_t]$. Suppose, to the contrary, that there are $r, s \in [0,1]$ with $[\alpha_r] \neq [\beta_r]$ and $[\alpha_s] = [\beta_s]$. Let us also assume, without loss of generality, that $r < s$. Since $[\alpha_r \cdot \bar{\beta}_r] \neq 1$, then by assumption, there is a normal cover $\mathcal{U}$ of X such that $p_{\mathcal{U}\#}([\alpha_r \cdot \bar{\beta}_r]) \neq 1 \in \pi_1(N(\mathcal{U}), *)$. Let v be the greatest lower bound of the set $A = \{t \in [r,s] \mid p_{\mathcal{U}\#}([\alpha_t \cdot \bar{\beta}_t])] = 1 \in \pi_1(N(\mathcal{U}), *)\}$. Let $x = \alpha_v(1) = \beta_v(1)$. Since the collection $\{p_{\mathcal{U}}^{-1}(\mathrm{St}(U, N(\mathcal{U}))) \mid U \in \mathcal{U}\}$ is an open cover of X, we may choose $U \in \mathcal{U}$ so that $x \in V = p_{\mathcal{U}}^{-1}(\mathrm{St}(U, N(\mathcal{U})))$. By continuity of g_1 and g_2, we may choose $\delta > 0$ such that $[\alpha_t] \in B([\alpha_v], V)$ and $[\beta_t] \in B([\beta_v], V)$ for all $t \in [0,1]$ with $|t - v| < \delta$.

(i) Suppose $p_{\mathcal{U}\#}([\alpha_v \cdot \bar{\beta}_v]) = 1 \in \pi_1(N(\mathcal{U}), *)$. Then $r < v \leqslant s$. Choose $t \in (r, v)$ such that $|t - v| < \delta$. By definition of $B([\alpha_v], V)$ and $B([\beta_v], V)$, there are paths $\tau_1, \tau_2 : [0,1] \to V$ such $[\alpha_t] = [\alpha_v \cdot \tau_1]$ and $[\beta_t] = [\beta_v \cdot \tau_2]$. Since $\tau_1 \cdot \bar{\tau}_2$ is a loop in V, the loop $(p_{\mathcal{U}} \circ \tau_1) \cdot (\overline{p_{\mathcal{U}} \circ \tau_2})$ lies in the open star of the vertex corresponding to U in $N(\mathcal{U})$, where it can be homotoped to that vertex. Consequently, $p_{\mathcal{U}\#}([\alpha_t \cdot \bar{\beta}_t]) = p_{\mathcal{U}\#}([\alpha_v \cdot \tau_1 \cdot \bar{\tau}_2 \cdot \bar{\beta}_v]) = p_{\mathcal{U}\#}([\alpha_v \cdot \bar{\beta}_v]) = 1 \in \pi_1(N(\mathcal{U}), *)$. However $t < v$, so that v is not a lower bound for the set A. This is a Contradiction.

(ii) Now suppose $p_{\mathcal{U}\#}([\alpha_v \cdot \bar{\beta}_v]) \neq 1 \in \pi_1(N(\mathcal{U}), *)$. Then $r \leqslant v < s$. Using an argument similar to Part (i), we see that $p_{\mathcal{U}\#}([\alpha_t \cdot \bar{\beta}_t]) \neq 1 \in \pi_1(N(\mathcal{U}), *)$ for all $t \in [v, s)$ with $|t - v| < \delta$. Choose $u \in (v, s)$ with $|u - v| < \delta$. Then u is a lower bound for the set A, which is greater than v; another contradiction. $\square$

Combining Theorem 2 with Proposition 1, we obtain our main result:

Theorem 3. *Suppose $\varphi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is injective. Then $p : \tilde{X} \to X$ is a generalized universal covering of X satisfying U_1–U_4. If X is locally path connected, then $p : \tilde{X} \to X$ satisfies U_5. If X is Hausdorff or metrizable, then so is $\tilde{X}$.*

Applications. Spaces X for which $\varphi : \pi_1(X) \to \check{\pi}_1(X)$ is known to be injective include all subsets of closed surfaces [6], all 1-dimensional compacta [4], as well as certain inverse limits of consecutive connected sums of closed manifolds, which are trivialized in turn by the bonding maps (e.g. boundaries of certain Coxeter groups) [5].

Example 3. According to the above, Theorem 3 applies to the Hawaiian Earring X. Notice, however, that $p : \tilde{X} \to X$ is not a classical covering. Indeed, it is not even a Hurewicz fibration: if $Y = p^{-1}(\{x_0\}) \subseteq \tilde{X}$ is the fiber over the origin and α is a simple closed curve around any one of the loops of X, then the partial lifting $G : Y \times \{0\} \to \tilde{X}$, given by $G(y, 0) = y$, of $F : Y \times I \to X$, given by $F(y,t) = \alpha(t)$, cannot be extended to a full lift. (Otherwise, for every $t > 0$, the map $g : Y \to \tilde{X}$ given by $g(y) = G(y,t)$ would map the non-discrete fiber Y homeomorphically onto the discrete fiber $p^{-1}(\{\alpha(t)\})$.)

Full-length article. For a more comprehensive account of the above and related results and a discussion of generalized intermediate covering spaces, we refer the reader to our full-length article [7].

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A CONSTRUCTION OF RIGID CANTOR SETS IN R^3 WITH SIMPLY CONNECTED COMPLEMENT

DENNIS GARITY, DUŠAN REPOVŠ, AND MATJAŽ ŽELJKO

1. INTRODUCTION

This is a summary of a talk given by D. Garity on June 11, 2004 at the 21st annual Workshop in Geometric Topology held at the University of Wisconsin in Milwaukee. The results, with complete proofs, are being prepared for publication elsewhere.

A subset $A \subset \mathbb{R}^n$ is *rigid* if whenever $f\colon R^n \to R^n$ is a homeomorphism with $f(A) = A$ it follows that $f|_A = id_A$. There are known examples in R^3 of wild Cantor sets that are either rigid or have simply connected complement. However, until now, no examples were known having both properties.

The class of wild Cantor sets having of simply connected complement known as Bing-Whitehead Cantor sets seemed to suggest that no such example exists because every one-to-one mapping between two finite subsets of a Bing-Whitehead Cantor set $X \subset R^3$ is extendable to a homeomorphism of R^3 which takes X to X (see [Wr4] for details).

Two Cantor sets X and Y in R^3 are said to be *topologically distinct* or *inequivalent* if there is no homeomorphism of R^3 to itself taking X to Y. In this paper we show that in fact uncountably many inequivalent examples of rigid Cantor sets with simply connected complement exist. The key technique used is that of local genus, introduced in [Ze].

Sher proved in [Sh] that there exist uncountably many inequivalent Cantor sets in R^3. He showed that varying the number of components in the Antoine construction leads to these inequivalent Cantor sets. Shilepsky used this result and constructed a rigid Cantor set in R^3 (see [Sl]). Using slightly different approach Wright constructed a rigid Cantor set in R^3 as well (see [Wr2]) and using the Blankinship construction [Bl] Wright later extended

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this result to R^n, $n \geq 4$, (see [Wr3]). All these results rely heavily on the linking of the components of defining sequences for the Cantor sets. This linking yields non simply connected complements of the the Cantor sets, so these constructions cannot be modified to give examples of rigid Cantor sets with simply connected complement.

2. Local genus of points in a Cantor set

The following are some basic facts from [Ze] about the genus of a Cantor set and the local genus of points in a Cantor set.

Let $\mathcal{D}(X)$ be the set of all defining sequences for X. Let M be a handlebody. We denote the genus of M by $g(M)$. For a disjoint union of handlebodies $M = \bigsqcup_{\lambda\in\Lambda} M_\lambda$, we define $g(M) = \sup\{g(M_\lambda);\ \lambda \in \Lambda\}$.

Let $(M_i) \in \mathcal{D}(X)$ be a defining sequence for a Cantor set $X \subset R^3$. For any subset $A \subset X$ we denote by M_i^A the union of those components of M_i which intersect A. Define

$$\begin{aligned} g_A(X;(M_i)) &= \sup\{g(M_i^A);\ i \geq 0\} \quad \text{and} \\ g_A(X) &= \inf\{g_A(X;(M_i));\ (M_i) \in \mathcal{D}(X)\}. \end{aligned}$$

The number $g_A(X)$ is called *the genus of the Cantor set X with respect to the subset A*. For $A = \{x\}$ we call the number $g_{\{x\}}(X)$ *the local genus of the Cantor set X at the point x* and denote it by $g_x(X)$. For $A = X$ we call the number $g_X(X)$ *the genus of the Cantor set X* and denote it by $g(X)$.

3. Main results

Lemma 3.1. *Let $X \subset R^3$ be a Cantor set and $A \subset X$ a countable dense subset such that*

(1) *$g_x(X) \leq 2$ for every $x \in X \setminus A$,*
(2) *$g_a(X) > 2$ for every $a \in A$ and*
(3) *$g_a(X) = g_b(X)$ for $a, b \in A$ if and only if $a = b$.*

Then X is a rigid Cantor set in R^3.

The main theorem, which we will prove after detailing the construction, is the following.

Theorem 3.1. *For each increasing sequence $S = (n_1, n_2, \ldots)$ of integers such that $n_1 > 2$, there exists a wild Cantor set in R^3, $X = C(S)$, and a countable dense set $A = \{a_1, a_2, \ldots\} \subset X$ such that the following assertions hold.*

(1) *$g_x(X) \leq 2$ for every $x \in X \setminus A$,*
(2) *$g_{a_i}(X) = n_i$ for every $a_i \in A$ and*
(3) *$R^3 \setminus X$ is simply connected.*

An immediate consequence of this theorem is the following.

Theorem 3.2. *There exist uncountably many inequivalent rigid wild Cantor sets in R^3 with simply connected complement .*

4. THE CONSTRUCTION

Let us fix an increasing sequence $S = (n_1, n_2, \dots)$ of integers with $n_1 > 2$. We will construct inductively a defining sequence M_1, M_2, ... for a Cantor set $X = C(S)$.

To begin the construction, let M_1 be a unknotted genus n_1 handlebody.

4.1. **Stage $n+1$ if n is odd.**

If n is odd then by inductive hypothesis every component of M_n is a handlebody of genus higher than 2. Let N be a genus r component of M_n.

The manifold N can be viewed as an union of r handlebodies of genus 1, $T_1 \cup \ldots \cup T_r$, identified along some 2-discs in their boundaries as shown in Figure 1.

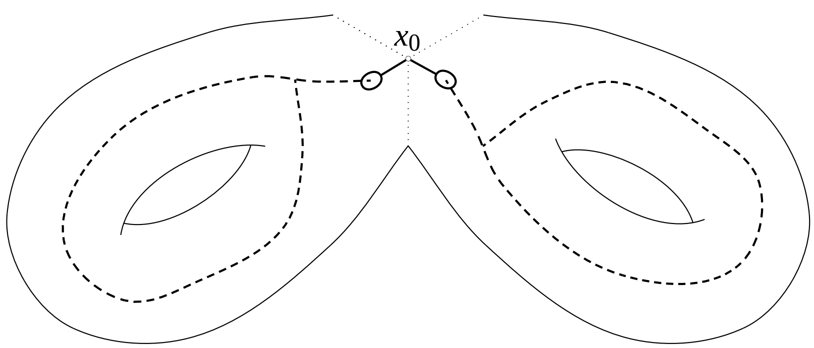

FIGURE 1. Manifold N

We replace the component N of genus r by a single smaller central genus r handlebody and a linked chains of genus 2 handlebodies. We use 6 genus 2 handlebodies for each handle of N. See Figure 2 for the linking pattern in one of the genus 1 handlebodies whose union is N.

Notice that the new components in N are actually unlinked if we regard them as handlebodies in R^3. Stage $n+1$ consists of all the new components constructed as above. The construction can be done so that each new component at stage $n+1$ has diameter less than half of the diameter of the component that contains it at stage n.

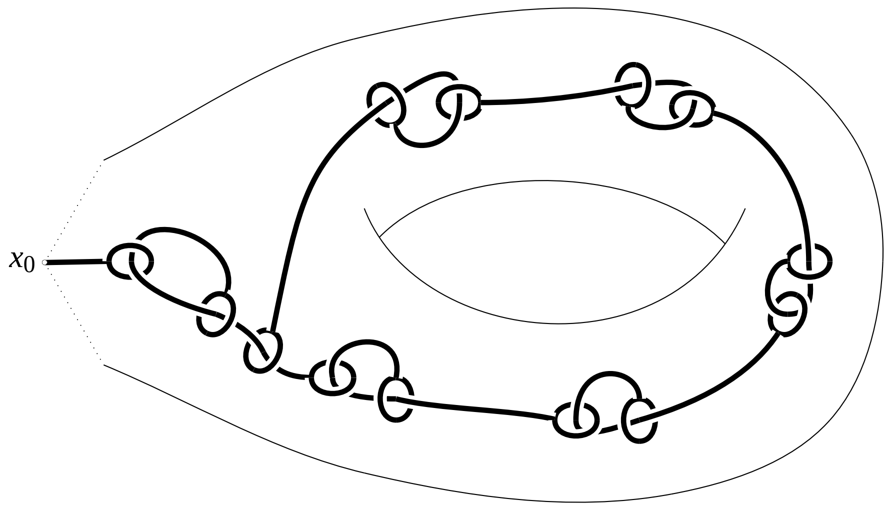

FIGURE 2. Linking along the spine of some handle of N

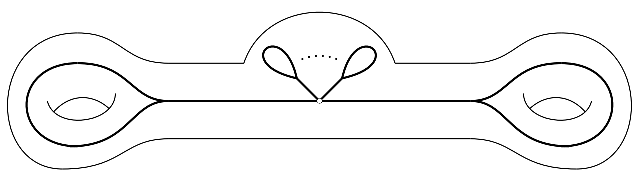

FIGURE 3. Modification in defining sequence

4.2. **Stage $n+1$ if n is even.** If n is even, we replace every genus r torus in M_n, $r > 2$, by a parallel interior copy of itself and every genus 2 torus by an embedded higher genus handlebody as shown in Figure 3.

More precisely, let us assume inductively that there exist handlebodies of genus $n_1, n_2, \ldots, n_N$ among the components of M_n. There are also K genus 2 components for some K and we replace one of these genus 2 handlebodies by a genus n_{N+1} handlebody, one by a genus n_{N+2} handlebody, $\ldots$ and one by a genus n_{N+K} handlebody. The components of M_{n+1} then consist of handlebodies of genus $n_1, \ldots n_{N+K}$.

This completes the inductive description of the defining sequence. Define the Cantor set associated with the sequence S, $X = C(S)$ to be

$$X = \bigcap_i M_i \ .$$

From the construction it is clear that X is a Cantor set.

4.3. **The countable dense subset** A**.** Each point p in X can be associated with a nondecreasing sequence of positive integers greater than 2 as follows. At stage $2n-1$, p is in a unique component. Let m_n be the genus of this component. The sequence we are looking for is $m_1, m_2, \ldots$. By construction, each m_{n+1} is either equal to m_n or is greater than m_n. It is greater than m_n precisely when the component of stage $2n$ containing p is a genus 2 torus. Let A be the set of points in X for which the associated sequence is bounded. Then A is countable and each point in A is associated with a sequence that is eventually constant. A is dense because each component of each M_i contains a point of$A$.

4.4. **Remaining Details.** The following results can be shown:

- The local genus at points of A is correct
- The local genus at points of $X \setminus A$ is correct,
- The complement of X is simply connected.

5. Questions

As stated in the introduction Bing-Whitehead Cantor sets have some strong homogeneity properties and therefore are not rigid.

- Does varying the numbers of consecutive Bing links and Whitehead links yield inequivalent Cantor sets? (This number cannot be arbitrary. See [Wr4] for details.)

The construction above gives a rigid Cantor set such that $g_x(X) \leq 2$ for $x \in X \setminus A$ and $g_{a_i}(X) = n_i$ for $a_i \in A$. Hence $g(X) = \infty$.

Let a positive integer r be given.

- Does there exist a rigid Cantor set X such that $g_x(X) = r$ for every $x \in X$? (For $r = 1$ the answer is affirmative. See [Sl], [Wr2].)
- Does there exist a rigid Cantor set X having simply connected complement such that $g_x(X) = r$ for every $x \in X$?

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A STABILIZATION THEOREM FOR OPEN MANIFOLDS

CRAIG R. GUILBAULT

Abstract. In this note we present a characterization of those one-ended open n-manifolds ($n \geq 5$), whose products with the real line are homeomorphic to interiors of compact $(n+1)$-manifolds with boundary.

1. Introduction

This work was motivated by a question asked to me recently by Igor Belegradek.

Question (Belegradek)**.** *Let M^n be an open manifold homotopy equivalent to an embedded compact submanifold, say a torus. Is $M^n \times \mathbb{R}$ homeomorphic to the interior of a compact manifold?*

For the purposes of this talk, we will focus on one-ended, high-dimensional manifolds; in particular, we assume that $n \geq 5$. (Although much of what we will do is valid in all dimensions; and all of what we do can be done without restriction on the number of ends.) We begin with a few standard definitions and examples.

- A manifold M^n is *open* if it is noncompact and has no boundary.
- A subset V of M^n is a *neighborhood of infinity* if $\overline{M^n - V}$ is compact.
- A neighborhood of infinity is *clean* if it is a codimension 0 submanifold and has bicollared boundary in M^n.
- M^n is *one-ended* if each neighborhood of infinity contains a connected neighborhood of infinity. (We assume this for convenience.)

Example 1. *$\mathbb{R}^n$ is an open n-manifold for all $n \geq 1$. If $n \geq 2$, then $\mathbb{R}^n$ is one-ended.*

Example 2. *Let P^n be a compact manifold with non-empty connected boundary. Then int (P^n) is a one-ended open manifold.*

Example 3. *(Disk with infinitely many handles) Let M^2 be the 2-manifold obtained by attaching a countably infinite discrete collection of handles to an open 2-disk.*

Example 4. *(The Whitehead manifold) In* [Wh]*, J.H.C. Whitehead constructed a, now-famous, example of a contractible (thus one-ended) open 3-manifold that is not homeomorphic to $\mathbb{R}^3$.*

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The following observations about the above examples help to motivate our work.

Facts.

a) Clearly Examples 1 and 2 are themselves interiors of compact manifolds; hence, so are their products with $\mathbb{R}$.

b) The manifold M^2 from Exercise 3 is not the interior of a compact 2-manifold; nor is $M^2 \times \mathbb{R}$ the interior of a compact 3-manifold. (*Exercise.* Why?)

c) The Whitehead manifold W^3 is not the interior of any compact 3-manifold; however, it is well-known that $W^3 \times \mathbb{R} \approx \mathbb{R}^4 \approx int(B^4)$. In fact, a result of Stallings [St] ensures that the product of *any* contractible n-manifold with a line is homeomorphic to $\mathbb{R}^{n+1}$

Reflection upon the above examples, together with past experience with noncompact manifolds, causes us to generalize our question to:

Generalized Belegradek Question (GBQ). If M^n is open and homotopy equivalent to a finite complex, is $M^n \times \mathbb{R}$ the interior of a compact $(n+1)$-manifold with boundary? (*As noted earlier, we restrict our attention to the case where M^n is one-ended and $n \geq 5$.*)

2. Results

In this section, we outline our solution to the GBQ in the one-ended case. As might be expected of any work on recognizing interiors of compact high-dimensional manifolds, we will employ the following celebrated result:

Theorem 2.1. *(Siebenmann, 1965) A one ended open n-manifold M^n $(n \geq 6)$ is the interior of a compact manifold with boundary iff:*

(1) *M^n is inward tame at infinity,*
(2) *π_1 is stable at infinity, and*
(3) *$\sigma_\infty(M^n) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\varepsilon(M^n))])$ is trivial.*

- Here *inward tame* means that for any neighborhood V of infinity, there exists a homotopy $H : V \times [0,1] \to V$ such that $H_0 = id$ and $\overline{H_1(V)}$ is compact. (Equivalently, we may require that all clean neighborhoods of infinity are are finitely dominated.)
- Combined, conditions 1) and 3) are equivalent to requiring that all clean neighborhoods of infinity have finite homotopy type. (For the purposes of this talk, we will refer to this property as *super-tame at infinity.*)

The following straightforward proposition begins our attack on the GBQ.

Proposition 2.2. *Let M^n be a connected open n-manifold.*

(1) *$M^n \times \mathbb{R}$ is inward tame at ∞ iff M^n is finitely dominated.*
(2) *$M^n \times \mathbb{R}$ is super-tame at ∞ iff M^n has finite homotopy type.*

Key Ingredient of Proof. $M^n \times \mathbb{R}$ has arbitrarily small neighborhoods of infinity of the form

$$U = (V \times \mathbb{R}) \cup (M \times [(-\infty, -r] \cup [r, \infty)])$$

where V is a clean neighborhood of infinity in M^n. $\square$

Equipped with Proposition 2.2 and Siebenmann's Theorem, it becomes clear that the answer to the GBQ depends only uponon the π_1-stability at infinity (or the lack thereof) in $M^n \times \mathbb{R}$. Siebenmann must have recognized this back in 1965 when he gave a positive answer to a weaker version of the GBQ—in particular, he allowed himself to cross with $\mathbb{R}^2$ instead of $\mathbb{R}$. The point there was that, by crossing with $\mathbb{R}^2$, π_1-stability at infinity becomes easy. (Verification of this fact is a good exercise.) Before proceeding, we review the meaning of π_1-stability at infinity.

A one-ended open manifold X of dimension at least 5, is π_1 *stable at infinity* if and only if there exists a sequence $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ of clean neighborhoods of infinity with, $\bigcap V_i = \emptyset$, such that each of the inclusion induced homomorphisms in the corresponding inverse sequence

$$\pi_1(V_0) \xleftarrow{\lambda_1} \pi_1(V_1) \xleftarrow{\lambda_2} \pi_1(V_2) \xleftarrow{\lambda_3} \cdots .$$

are isomorphisms. (Actually, the *definition* of π_1 stable at infinity simply requires that the above inverse sequence be 'pro-stable'. In dimensions ≥ 5, the desired isomorphisms can then be *arranged* using handle trading techniques developed by Siebenmann.)

A positive solution to the GBQ for $n \geq 5$ is obtained by proving the following:

Proposition 2.3. *If M^n is one-ended, open and finitely dominated, then $M^n \times \mathbb{R}$ is π_1-stable at ∞.*

Sketch of Proof. Let

$$U = (V \times \mathbb{R}) \cup (M \times [(-\infty, -r] \cup [r, \infty)])$$

where V is a connected neighborhood of ∞ in M^n.

If $G = \pi_1(M^n)$, then

$$\pi_1(U) \cong G *_H G$$

(a *free product with amalgamation*), where

$$H = image\,(\pi_1(V) \to \pi_1(M^n))$$

So π_1 'at infinity' looks like:

$$(G *_{H_1} G) \twoheadleftarrow (G *_{H_2} G) \twoheadleftarrow (G *_{H_3} G) \twoheadleftarrow \cdots$$

where $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$ is a sequence of neighborhoods of ∞ in M^n, and for each i

$$H_i = image\,(\pi_1(V_i) \to \pi_1(M^n)) .$$

To complete the proof, it suffices to show that $H_i = H_j$ for all i, j (when the V_i's are appropriately chosen). This is accomplished by proving:

Claim. *Let K be a compactum into which M^n deforms and let $V' \subseteq V \subseteq M^n - K$ be clean connected neighborhoods of ∞. Then any loop τ in V can be pushed into V' (with base point traveling along a given fixed base ray).*

To prove the claim, begin with an embedded 'base ray' r in M^n and assume τ is based on r. Choose a homotopy $H : M^n \times [0,1] \to M^n$ that pulls M^n into K and is 'nice' near r. (For example, points of r stay in r under H. See the discussion preceding Proposition 3.2 of [GuTi] for details.) Choose a third clean neighborhood $V'' \subseteq V'$ sufficiently small that $\tau \subseteq M^n - V''$. In addition, arrange that $\partial V''$ is connected and r pierces $\partial V''$ transversely in a single point p. Consider the restricted homotopy $H| : \partial V'' \times [0,1] \to M^n$. Adjust $H|$ so that it is transverse to τ. Then $H|^{-1}(\tau)$ will be a finite collection of circles in $\partial V'' \times [0,1]$. By the niceness of $H|$ near r (again see [GuTi, Prop.3.2]), one of these circles, call it τ', is taken in a degree 1 fashion onto τ by $H|$. Using the product structure, τ' can be pushed into $\partial V'' \times \{0\}$ within $\partial V'' \times [0,1]$. Composing this push with $H|$ pushes τ into $\partial V''$ in M^n, as desired. □

We conclude this note with a precise statement of our main result.

Theorem 2.4. *Let M^n be a one-ended open n-manifold ($n \geq 5$), then $M^n \times \mathbb{R}$ is homeomorphic to the interior of a compact $(n+1)$-manifold with boundary if an only if M^n is homotopy equivalent to a finite complex.*

Note. A complete write-up of this work—including the multi-ended case— is in preparation.

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S^4 admits no uscd into shape S^1's?

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Some history.

Ron Fintushel, building on work of Raymond and Orlik, Montgomery and Yang, classifies (1976-1978) locally smooth circle actions on: homotopy 4-spheres; simply connected 4-manifolds; and then 4-manifolds generally.

Pao (1978) follows with a classification of nonlinear actions.

Plotnick (1982) extends the results of both to homology 4-spheres, and then builds examples of such that admit no effective S^1 action and thus have fundamental groups that cannot belong to a 3-manifold.

Theorem. [Fintushel] Let M^* be the orbit space of a locally smooth S^1 action on the simply connected 4-manifold M, with exceptional orbits E and fixed point set F. Then:

- M^* is a simply-connected 3-manifold with $\partial M^* \subset F^*$.
- The set $F^* - \partial M^*$ is finite, and F^* is nonempty.
- The closure of E^* is a collection of polyhedral arcs and simple closed curves in M^*. The components of E^* are open arcs on which orbit types are constant, and these arcs have closures with distinct endpoints in $F^* - \partial M^*$. (continued on next slide)
- If, in addition, M is a homotopy 4-sphere, then: F is either S^2 or S^0 (in the former case, $E = \emptyset$ and M^* is a homotopy 3-cell with boundary F^*; in the latter case, M^* is a homotopy 3-sphere). In the latter case, if there is only one type of exceptional orbit, E^* is an arc and F^* its endpoints; if there are two types of exceptional orbits, then $E^* \cup F^*$ is a scc separated by F^* into two arcs, on each of which the orbit type is constant.

Conjecture. There is no proper, closed map defined on S^4 such that each of its point preimages is a [shape] circle.

Theorem. *Suppose $\pi : S^4 \to B$ is a proper, closed surjection such that $\tilde{b} = \pi^{-1}(b)$ has the shape of S^1, for all $b \in B$; dim$B < \infty$; and one more hypothesis stated later in this talk. Then B is not a generalized manifold (over $\mathbb{Q}$).*

Assuming the existence of the map, we begin a catalogue of facts regarding B:

- $\dim B = 3$;
- B is simply connected (π is a π_1-epimorphism);
- and LC^1 (Dydak).

The Leray sheaves of π.

We let $\mathcal{H}^j = \mathcal{H}^j[\pi] = \mathcal{H}^j[\pi; \mathbb{Q}]$ denote the Leray sheaf in dimension j, where $j = 0, 1$

For each $b \in B$ and $j = 0, 1$, the stalk $\mathcal{H}^j_b$ is isomorphic to $\check{H}^j(\tilde{b}; \mathbb{Q}) \cong H^j(S^1; \mathbb{Q}) \cong \mathbb{Q}$.

The topology on $\mathcal{H}^j$ is discrete when restricted to any stalk.

Aside: a crash course in sheaf topology.

For $b \in B$, let $\widetilde{U}$ be a *saturated* nbhd of $\tilde{b}$. Then there is a saturated nbhd $\widetilde{V} \subseteq \widetilde{U}$ that shape deformation retracts to $\tilde{b}$ in $\widetilde{U}$. For any $b_1 \in V$, there is a map $H^j(\tilde{b}; \mathbb{Q}) \to H^j(\widetilde{b_1}; \mathbb{Q})$, the *$j$-winding function* of b_1 about b. Note that this function is either an isomorphism or the zero map. Given a section σ of $\mathcal{H}^j$ at b defined on V, the section evaluated at b_1 will naturally correspond to the value of the j-winding function of b_1 around b evaluated at $\sigma(b)$. This defines the topology on $\mathcal{H}^j$.

Clearly, then $\mathcal{H}^0$ is sheaf isomorphic to the constant sheaf $\mathbb{Q} \times B$.

More items for the catalogue ...

Theorem. [Dydak and Walsh] There is an open, dense subset C (the *continuity set*) of B on which $\mathcal{H}^1$ is locally constant.

Definition. Let $K = B - C$, the *degeneracy set.*

Corollary. Then K is nowhere dense in B.

Theorem. [Daverman and Snyder; Snyder] C is a generalized 3-manifold, *i.e.* C is an ANR with local (co)homology of a manifold:

$$H^i(B, B - b; \mathbb{Q}) \cong \mathbb{Q}$$

for $i = 0, 3$ and is trivial for all other i.

Theorem. [Walsh] Via a pseudo-isotopy, we may assume that π is also an open map and, hence, that $\widetilde{K}$ is nowhere dense in S^4.

Theorem. [Shaw] K does not locally separate B and $\dim K \leq 1$.

Aside: the Leray Spectral Sequence.

$H^p(A; \mathcal{H}^q[\pi|_A]) \Rightarrow H^{p+q}(\widetilde{A})$

Since our Leray spectral sequence is lacunary, applied to $\pi|_A$ for any $A \subset B$ (and closed supports) we get:

$$\cdots \to H^i(A) \to H^i(\widetilde{A}) \to H^{i-1}(A; \mathcal{H}^1[\pi|_A]) \to \cdots$$

There is also a relative version, for compact A contained in a subset U of B:

$$\cdots \to H^i(U, U-A) \to H^i(\widetilde{U}, \widetilde{U}-\widetilde{A}) \to H^{i-1}(U, U-A; \mathcal{H}^1[\pi|_A]) \to \cdots$$

Aside: the Fary Spectral Sequence.

Let $B = B_0 \supset B_1 \supset B_2 \supset \cdots$ be a filtration of B by *closed* subsets of B. Let $A_t = B_t - B_{t-1}$. Then

$$\bigoplus_t H^{p+t}_{\Phi|_{A_t}}(A_t; \mathcal{H}^{q-t}[\pi]|A_t) \Rightarrow H^{p+q}(S^4)$$

We note here that $\mathcal{H}^{q-t}[\pi]|A_t$, (in our context) when restricted to A_t is the Leray sheaf $\mathcal{H}^{q-t}[\pi|\widetilde{A_t}]$.

We apply this spectral sequence here using $B_1 = K$ and $B_p = 0$ for $p > 1$. Note then that $C = A_0$ and $K = A_1$.

Continuing ...

Proposition. *$\mathcal{H}^1|_C$ is isomorphic to the constant sheaf $\mathbb{Q} \times C$.* [Proof snapshot: over C, π corresponds to a rational circle bundle over C.]

Proposition. *The sheaf $\mathcal{H}^1$ splits.* We abuse notation and say $\mathcal{H}^1 = \mathcal{H}^1|_C \oplus \mathcal{H}^1|_K$.

Let $A = B$, so $\widetilde{B} = S^4$, and apply the exact sequence (absolute version) from the Leray spectral sequence to get the following for our catalogue:

- $H^1(B)$ is trivial (trivially, since $\pi_1(B)$ is trivial)
- $H^2(B) \cong H^0(B; \mathcal{H}^1) \cong H^0(B; \mathcal{H}^1|_C) \oplus H^0(B; \mathcal{H}^1|_K)$
- $0 \cong H^2(B; \mathcal{H}^1) \cong H^2(B; \mathcal{H}^1|_C) \oplus H^2(B; \mathcal{H}^1|_K)$
- $H^3(B) \cong H^1(B; \mathcal{H}^1|_C) \oplus H^1(B; \mathcal{H}^1|_K)$
- $H^2(B; \mathcal{H}^1) \cong \mathbb{Q}$

NB: If coefficients are not shown, they are $\mathbb{Q}$. Supports for the cohomology are taken to be Ψ, the closed subsets of B. Note, for later, that the support $\Psi|_C$ is the collection of compact subsets of C.

Proposition. *$K \neq \emptyset$.*

Proof sketch: If $K = \emptyset$, then $C = B$ implies that B is a compact generalized (co)homology 3-sphere. Thus, $H^2(B)$ is trivial, which, by the previous list, implies $H^0(B; \mathcal{H}^1)$, the group of global sections, consists of only the zero section. But, as noted before, $\mathcal{H}^1|_{B=C}$ is the constant sheaf.

Since $K \neq \emptyset$, it has an open, dense (non-empty) subset K_1 on which $\mathcal{H}^1|_{K_1}$ is locally constant.

Lemma. *K_1, and hence K, is 1-dimensional.*

Proof sketch: Suppose K_1 is 0-dimensional at $b \in K_1$. Let V be an open set in B, with $b \in V$ such that $\mathcal{H}^1|_{V \cap K_1}$ is constant. Find a nbhd W of b contained in V such that $W \cap K_1 = \emptyset$. Then W admits a section that extends to B. Impossible!

Corollary. *K has no 'totally degenerate' points (and, so, no isolated points).*

We say b is totally degenerate if its 1-winding funtion is identically 0 on its punctured nbhd $V - \{b\}$.

Let $K_2 = K - K_1$ (the *second degeneracy set*), which is nowhere dense in K.

We will add as a simplifying assumption that $K_2 = \emptyset$, *i.e.* $\mathcal{H}^1_K$ is locally constant.

Notice that $\mathcal{H}^1_K$ can*not* be constant, for otherwise a global section on B exists.

Now, we move to add information from the Fary spectral sequence ...

What our Fary spectral sequence looks like in the E_2 term (similar to an S^1-bundle with singularities' - but with complicating differences):

$$\begin{array}{llll} H^1(K; \mathcal{H}^1|K) & & & \\ H^0_c(C; \mathcal{H}^1|_C) \oplus H^1(K) & H^1_c(C; \mathcal{H}^1|_C) & H^2(C; \mathcal{H}^1|_C) & H^3_c(C; \mathcal{H}^1|_C) \\ \quad H^0_c(C) & \quad H^1_c(C) & \quad H^2(C) & \quad H^3_c(C) \end{array}$$

Using this, and the relative cohomology sequence of the pair (B, K) (with coefficients $\mathbb{Q}$), we can deduce that K is connected. Moreover, $H^1(K) \cong 0$ (from the FSS) and $H^1(K; \mathcal{H}^1|_K) \cong \mathbb{Q}$ (from the LSS). (This latter fact tells us then that $H^3_c(C) \cong \mathbb{Q}$).

Having established these facts, we move to looking at the relative version of the LSS, and leverage the fact that B is assumed to be a 3-gm.

We are then able to prove (this still has the flavor of transformation group theory):

Lemma. *K is a homology 1-manifold.*

But for $n \leq 2$, a homology n-manifold is an n-manifold. Thus $K \cong S^1$.

This last statement is clearly impossible, since $H^1(K) \neq H^1(S^1)$

Question: *Is* there an example of such a map where its image is *not* a generalized manifold?

PLETHORA OF ONE-SIDED COBORDISMS

by F. C. Tinsley

(based on joint work with C. R. Guilbault)

This result is used in our paper, 'Manifolds with non-stable fundamental groups at infinity, III'. However, the proof here is rather different may may be useful in other settings.

A cobordism is a *one-sided h-cobordism* if the inclusion of one of the boundary components into the ambient manifold is a homotopy equivalence.

Theorem: Suppose (R, M, N) is a cobordism such that $inc_{\#} : \pi_1(N) \to \pi_1(R)$ has perfect kernel. Then there exists a nicely embedded, closed manifold, $P \subset int(R)$, such that the cobordism (Y, P, N) is a one-sided h-cobordism where Y is the closure of the component of $R \backslash P$ that contains N and $inc_{\#} : \pi_1(N) \to \pi_1(Y)$ has the identical kernel to $inc_{\#} : \pi_1(N) \to \pi_1(R)$.

Proof: Since R and N are compact, then the perfect kernel is the normal closure of a finite set of elements. Let $\{l_1, \cdots, l_r\}$ be a collection of loops in N representing those elements. Let (Z, Q, N) be the cobordism obtained by attaching r 2-handles, Θ_i, to N with the cores attaching to the loops $\{l_1, \cdots, l_r\}$. None of the construction below involves points of $W \backslash Z$.

We define a special finite 2-complex, K, that lives in N (see work of Daverman-and Tinsley).

Step 1: Let $B \subset N$ be a bouquet of the loops $\{l_1, \cdots, l_r\}$

Step 2: Each loop, l_i, bounds in N a disk with m_i handles, G_i. Moreover, each handle curve must represent an element of $ker(\{inc : N \to W\}_{\#}) < \pi_1(N)$. This is the first stage of a "grope" that must exist in N by the perfectness of the kernel.

Step 3: Each handle curve bounds in N a disk-with-holes where each other boundary component is one of the l_i's. For each G_i, there are $2 \cdot m_i$ such disks-with-holes. Denote this collection by $\left\{A_{i(2j-1)}, A_{i(2j)} : 1 \leq j \leq m_i\right\}$ where $A_{i(2j-1)}$ and $A_{i(2j)}$ are disks-with-holes attached to handle curves from the same handle. This geometry follows from the fact that the handle curves also represent elements of the kernel and, thus, are in the normal closure of the elements of $\pi_1(N)$ represented by the l_i's.

Step 4: We define $K \subset N$ as follows. Let

$$K = B \bigcup \left(\bigcup_{i=1}^{r} \left(G_i \bigcup \left(\bigcup_{j=1}^{m_i} \left\{ A_{i(2j-1)} \bigcup A_{i(2j)} \right\} \right) \right) \right)$$

where the unions are along the loops described in steps 1-3.

It is a straight forward application of Mayer-Vietoris to show that $H_2(K) \cong 0$. Moreover, K contains "all the action" as far as the perfectness of the kernel is concerned.

We perform plus constructions as follows. Fix i and j for a moment and consider the disk-with-holes, $A_{i(2j)}$. Assume there are n_{ij} boundary components of $A_{i(2j)}$ other than the handle curve of G_i. For each k, $1 \leq k \leq n_{ij}$, let $\phi(i,j,k)$ be the function defined so that $l_{\phi(i,j,k)}$ is k'th boundary component of $A_{i(2j)}$.

Define disks, E_i, for $1 \leq i \leq r$, by

$$E_i = \left(G_i \bigcup \left(\bigcup_{j=1}^{m_i} \left\{ \left[A_{i(2j)} \bigcup \left(\bigcup_{k=1}^{n_{ij}} D^2_{\phi(i,j,k)} \right) \right]^- \bigcup \left[A_{i(2j)} \bigcup \left(\bigcup_{k=1}^{n_{ij}} D^2_{\phi(i,j,k)} \right) \right]^+ \right\} \right) \right)$$

The $-$ and $+$ labels refer to two algebraically cancelling copies of the disk used to surger the j'th handle of G_i. For geometric reasons (if nothing else), E_i is homologous ($rel\ \partial$) to G_i in $N \bigcup \left(\bigcup_{i=1}^{r} H_i^2 \right)$ where H_i^2 is a 2-handle with core E_i. Morover, $G_i \subset K \subset N$.

We perform r plus constructions using the E_i's as the cores of the 2-handles, H_i^2. The r cancelling 3-handles, H_i^3, that complete the plus constructions require additional descriptions.

Define disks F_i, $1 \leq i \leq r$, as follows:

$$F_i = \left(G_i \bigcup \left(\bigcup_{j=1}^{m_i} \left\{ \left[A_{i(2j)} \bigcup \left(\bigcup_{k=1}^{n_{ij}} E_{\phi(i,j,k)} \right) \right]^- \bigcup \left[A_{i(2j)} \bigcup \left(\bigcup_{k=1}^{n_{ij}} E_{\phi(i,j,k)} \right) \right]^+ \right\} \right) \right)$$

Again, for geometric reasons, F_i is homologous ($rel\ \partial$) to the 2-cycle, C_i:

$$C_i = G_i \bigcup \left(\bigcup_{j=1}^{m_i} \left\{ \left[A_{i(2j)} \bigcup \left(\bigcup_{k=1}^{n_{ij}} G_{\phi(i,j,k)} \right) \right]^- \bigcup \left[A_{i(2j)} \bigcup \left(\bigcup_{k=1}^{n_{ij}} G_{\phi(i,j,k)} \right) \right]^+ \right\} \right)$$

where $C_i \subset K \subset N$. By construction, $R_i \bigcup C_i$ is a 2-cycle in the complex, K. Since $H_2(K) = 0$, $R_i \bigcup C_i$ is null-homologous in K.

Again by construction, the 2-sphere $S_i^2 = D_i \bigcup E_i$ is homologous in $N \bigcup \left(\bigcup_{i=1}^{r} H_i^2 \right)$ to $R_i \bigcup C_i$. Since $\pi_1 (K)$ includes trivially into $\pi_1 (Z)$, the Hurewicz Theorem says that S_i^2 bounds a 3-cell Z in the same relative homology class. These r 3-cells become the cores of the three handles, H_i^3, in the plus construction. It is easy to check that each algebraically cancels the corresponding 2-handle, H_i^2. The result of these plus constructions is the desired cobordism, (Y, P, N). Note that $P \subset int(Z)$. This completes the proof of the theorem. $\square$

PLANES IN 3-MANIFOLDS OF FINITE GENUS AT INFINITY

Bobby Neal Winters

1. Introduction and Definitions

In the study of 3-manifolds, there is a long tradition of cutting up a 3-manifold along embedded 2-manifolds in order to obtain pieces that are simpler in some sense than the original manifold. One can cite the classical examples of Heegaard splittings and the Prime Decomposition Theorem.

More recently, Jaco and Shalen in [**8**] and Johannson in [**9**] proved what is widely known as the Characteristic Pair Theorem. This theorem states a Haken manifold that is closed or has incompressible boundary can be split along embedded tori and annuli into unique pieces that are of three kinds: Seifert fibered spaces, I-bundles over surfaces, and "simple."

A manifold is said to be simple if it contains no essential annulus or torus. On the other hand, in most Seifert fibered spaces and I-bundles, one can construct lots of essential annuli and tori.

In this paper, we will create a decomposition theorem for noncompact 3-manifolds of finite genus at infinity that is analogous to the Jaco-Shalen-Johannson Theorem. In the current decomposition, noncompact 3-manifolds that contain no nontrivial places correspond to the simple pieces of the Jaco-Shalen-Johannson Decomposition, and a family of manifolds christened "nearnodes" correspond to the Seifert fibered pieces.

This paper is organized into two parts. The first part deals with preliminaries and climaxes in Theorem 5.1, which is an analog of the well-known Haken Finiteness Theorem. The second part deals with the main result of the paper. Most of the vocabulary needed in this paper is defined in the remainder of the current section. However, definitions of some terms have been postponed until Part II in hopes this will be more convenient for the reader.

In the rest of this section, I will define the vocabulary required to state the Theorem 5.1 precisely.

Let X and Y be topological spaces. A map $f : X \to Y$ is said to be *proper* if $f^{-1}(K)$ is compact for every compact $K \subset Y$. If X is a subset of Y, we say that X is *proper in* Y if the inclusion map is proper; this occurs exactly when $X \cap K$ is compact for every compact $K \subset Y$.

We let $\sharp(X)$ denote the number of components of X.

Given a map $f : X \times I \to X$, let $f_t : X \to X$ denote the map $f_t : x \mapsto f(x,t)$ for every $t \in I$. In the case f_t is a homeomorphism for every $t \in I$ and $f_0 = 1_X$, then we say that f (or f_t) is *an isotopy.*

We say that a topological space P is a *plane* if P is homeomorphic to $\mathbf{R}^2$.

Let V be a 3-manifold.

Suppose T is a 2-manifold in ∂V that is proper in V. If F is a 2-manifold that is proper in V and $F \cap \partial V = \partial F$ and $\partial F \subset T$, then we say that F is *properly embedded in* (V, T). If $T = \partial V$, then we say that F is *properly embedded in* V.

Suppose F is properly embedded in (V, T). We say that F is *parallel in* V *into* T provided there is a proper embedding (*i.e.* an embedding that is a proper map) $f : F \times I \to V$ with $f(F \times 0) = F$ and $f((\partial F \times I) \cup (F \times 1)) \subset T$. In the case $T = \partial V$, we say that F is *boundary parallel.*

Suppose that $K \subset V$ such that there is no proper homotopy $h : F \times I \to V$ with $h(F \times 0) = F$ and $h(F \times 1) \subset V - K$, then we say that K *traps* F.

Suppose that F is properly embedded in (V, T). If F is incompressible in V, and F is not parallel into T, then we say that F *is essential in* (V, T). If F is essential in (V, T), and there is a compact $K \subset V$ that traps F, then we say that F is *strongly essential in* (V, T).

Suppose that P is a plane that is proper in V. We say that P is *nontrivial in* V if there is a compact $K \subset V$ that traps P. If P is not nontrivial, then we say that P is *trivial.* In the case that V is irreducible, it follows by Lemma 4.1 of [**12**] that P is nontrivial in V iff no component of $\mathrm{cl}(V - P)$ is homeomorphic to $\mathbf{R}^2 \times [0, \infty)$. If every nontrivial plane in V is boundary parallel, we say that V is *aplanar.* If S is a 2-manifold in V that has a finite number of components each of which is a plane that is nontrivial in V, then we say that S is a *squadron in* V.

Let S be a squadron in V, and suppose K is a subset of V. If there is a compact subset T of S such that $S - T \subset K$, then we say that K *swallows the ends of* S.

Let V be a noncompact 3-manifold.

If V has no compact components and $\partial V = \emptyset$, then we say that V is *open.* Note that if V is open and $M \subset V$ is a compact 3-manifold, then $\mathrm{Fr}(M) = \partial M$.

If for every compact $K \subset V$ there is a compact 3-manifold $M_K \subset V$ such that $K \subset M_K - \mathrm{Fr}(M_K)$ and $\mathrm{Fr}(M_K)$ is incompressible in V, then we say that V is *end-irreducible.* If V is a 3-manifold that contains a compact subset K such that $\mathrm{cl}(V - K)$ is end-irreducible, then we say that V is *eventually end-irreducible.*

If exactly one component of $V - K$ has noncompact closure for every compact $K \subset V$, then we say that V has *one end.* Note that if V has one end and $F \subset V$ is a compact 2-manifold that separates V, then there is a compact 3-manifold $M_F \subset V$ such that $\partial M_F = F \cup F'$, where F' is a compact union of components of ∂V. In particular if no component of ∂V is compact, then F bounds a compact 3-manifold in V.

If X is a function from the nonnegative integers to the set of compact submanifolds of V such that

1) $X(n) \subset X(n+1) - \mathrm{Fr}(X(n+1))$ and
2) $V = \displaystyle\bigcup_{n=0}^{\infty} X(n)$,

then we say that X is an *exhausting sequence or an exhaustion* for V. We write $X_n = X(n)$ and $X = \{X_n\}$.

Suppose that V is a noncompact, connected 3-manifold.

Let $g \geq 0$ be an integer. Let X be an exhaustion for V. We say that X is a *genus g exhaustion* if ∂X_n is connected and of genus g for every $n \geq 0$. We say that V is *of at most genus g at infinity* if there exists a genus g exhaustion for X.

We say that V is *of at least genus g at infinity* if there is no genus $g-1$ exhaustion for V. In other words, there is a compact subset K of V such that whenever M is a compact 3-manifold with $K \subset M - \partial M$ and ∂M is connected, it follows that genus $(\partial M)) \geq g$. If V is of at most genus g at infinity and of at least genus g at infinity, then we say that V is *of genus g at infinity.*

We make the following observations about a 3-manifold V that is of most genus g at infinity:

1) V is open and has one end.
2) V is of genus k at infinity for some $0 \leq k \leq g$.

Note that if V is of at least genus g at infinity, then there is a compact $L \subset V$ such that if $L^* \subset V$ is a compact 3-manifold with connected boundary which contains L in its interior, then genus $(\partial L^*) \geq g$. In this case we say that V is *of at least genus g at infinity rel L.* If V is also of genus g at infinity, we say that V is of *genus g rel L.*

Note that if V is of at least genus g at infinity it does not follow that V is open or has only one end. For example, let A be Antoine's Necklace in S^3 and let M be a solid torus in S^3 that contains A in its interior. Then $V = M - A$ is of at least genus g at infinity for all $g \geq 0$.

However, the terminology "genus at least g at infinity" will be used mostly when V is open.

Suppose that V is of genus g at infinity rel L, where $L \subset V$ is compact. Let M be a compact 3-submanifold of V such that $\mathrm{cl}(V - M)$ is connected, irreducible, and end-irreducible, such that the inclusion induced map $\pi_1(M) \to \pi_1(V)$ is onto, such that ∂M is connected, and such that $L \subset M - \partial M$. Then we say that M is *regular in V with respect to L.* When there exists some compact $L \subset V$ such that M is regular in V with respect to L, then we will say that M is *regular in V.*

Suppose N is a 3-manifold such that each component of ∂N is a plane. Suppose that for every compact $K \subset N$, there is a closed 3-cell $B_K \subset N$ such that $K \subset B_K - \mathrm{Fr}(B_K)$ such that $B_K \cap P$ is either a disk or empty for every component P of ∂N. Then we say that N is a *nearnode with $\sharp(\partial N)$ faces.* (It is possible that $\sharp(\partial N)$ is infinite.) If N is a missing boundary manifold as well as well as a nearnode, then we say that N is a *node with $\sharp(\partial N)$ faces.* (Recall that N is a missing boundary manifold if $N = M - C$ where M is a compact manifold with boundary and C is a closed subset of ∂M.) Note that nearnodes are irreducible and contractible.

The author has seen references to manifolds that are nearnodes in [**4**] and [**12**]. Both of these examples were of nearnodes with two faces and neither source gives this class of manifold a name.

Observe that $\mathbf{R}^2 \times [0, \infty)$ is a node with one face and that $\mathbf{R}^2 \times I$ is a node with two faces. A characterization of nearnodes with two faces that are not nodes will be given in Lemma 4.4

In general nearnodes with two faces contain a lot of nontrivial planes. For example, we prove the following in Section 4.

Theorem (The Nonparallel Plane Theorem)
A nearnode with two faces that is not a node contains a collection of pairwise disjoint, pairwise nonparallel nontrivial planes with the cardinality of the Cantor set.

Suppose that V is a noncompact 3-manifold. Suppose that N is a nearnode with two faces that is proper in V. If P and P' are the components of ∂N, then P and P' are said to be *nearly parallel in* V, and N is called a *near parallelism between* P *and* P'. If N is a node, by Lemma 4.4, it follows that P and P' are parallel in V, and N is a parallelism between P and P' in V.

Suppose that H is a noncompact 3-manifold containing a squadron P such that the result N of splitting H along P is a nearnode (node) with $\nu+2\sharp(P)$ faces. Then we say that H is a *nearnode (node) with* ν *faces and* $\sharp(P)$ *handles.*

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Part I: Basic Results

2. Surfaces in Noncompact 3-manifolds

Lemma 2.1. (The Target Lemma)
Suppose that V is a noncompact, irreducible, connected 3-manifold. Suppose that S is a squadron in V and suppose that $K \subset V$ is a compact, connected set which traps each component of S. For each component P of S, let $D_P \subset P$ be a disk such that $K \cap P \subset D_P - \partial D_P$. Let D_S be the union of all the D_P's. Suppose that $M \subset V$ is a compact 3-manifold such that $K \cup D_S \subset M - \mathrm{Fr}(M)$ and $\mathrm{Fr}(M)$ is incompressible in $V - K$.

Also suppose that S meets $\mathrm{Fr}(M)$ transversely and that

$$\sharp(S \cap \mathrm{Fr}(M)) \leq \sharp(h_1(S) \cap \mathrm{Fr}(M))$$

for any isotopy h_t of $(V, \partial V)$ that is the identity on $K \cup D_S$ and has compact support.

1) *If J is a component of $P \cap \mathrm{Fr}(M)$ for some component P of S, then J bounds a disk $E_J \subset P$ such that $D_P \subset E_J - J$, and J is noncontractible in $\mathrm{Fr}(M)$.*
2) *If A is a component of $S \cap \mathrm{cl}(V - M)$, then A is either an annulus or a half open annulus, and A is incompressible in $\mathrm{cl}(V - M)$.*
3) *If there are no essential annuli in $(\mathrm{cl}(V - M), \mathrm{Fr}(M))$ and P is a component of S, then $P \cap \mathrm{cl}(V - M)$ is a half open annulus.*

Proof. To prove (1), let E_J be the disk in P that is bounded by J. We claim that $D_P \subset E_J - J$. Otherwise we may choose a component J' of $E_J \cap \mathrm{Fr}(M)$ which bounds a disk $E' \subset E_J$ such that $E' \cap \mathrm{Fr}(M) = J'$ and $J' = \partial E'$. Since E_J does not contain D_P, it follows that $E' \cap D_P = \emptyset$; consequently $E' \cap (K \cup D_S) = \emptyset$.

Since $\mathrm{Fr}(M)$ is incompressible in $V-K$, it follows that there is a disk $D' \subset \mathrm{Fr}(M)$ with $\partial D' = J'$. Now $D' \cup E'$ is a 2-sphere and therefore must bound a closed 3-cell $B' \subset V$ by irreducibility. Since K is connected and $K \cap \partial B' = \emptyset$, then either $K \subset B'$ or $K \cap B' = \emptyset$. However, K is not contained in B' because K traps each component of S and any component of S can be homotoped off a closed 3-cell. Hence $K \cap B' = \emptyset$. We may therefore use B' to reduce $\sharp(S \cap \mathrm{Fr}(M))$ by an isotopy with compact support that is fixed on $K \cup D_S$. This is a contradiction. Therefore $D_P \subset E_J - J$.

We claim that J is contractible in $\mathrm{Fr}(M)$. To get a contradiction, suppose it is not. Let $D \subset \mathrm{Fr}(M)$ be a disk with $\partial D = J$. We may choose P and J so that $(D - \partial D) \cap S = \emptyset$. Therefore $E_J \cap D = J$. It follows that $E_J \cup D$ is a 2-sphere which bounds a closed 3-cell $B \subset V$ because V is irreducible. We may use B to isotop P to the plane $(P - E_J) \cup D$. Note $[(P - E_J) \cup D] \cap K$ is empty because $K \cap P \subset E_J$. This is a contradiction because K traps P.

To prove part (2), let P be a component of S. It follows that each component of $P \cap \mathrm{cl}(V-M)$ is either an annulus or a half open annulus because each component J of $P \cap \mathrm{Fr}(M)$ bounds a disk E_J which contains D_P. We will show that each component is incompressible.

Suppose that A is a component of $P \cap \mathrm{cl}(V-M)$. Suppose that D is a compressing disk for A in $\mathrm{cl}(V-M)$. Let J be a component of ∂A. Then J is parallel in A to ∂D. It follows J is contractible in $\mathrm{cl}(V-M)$. However J is noncontractible in $\mathrm{Fr}(M)$ by part (1). Since $\mathrm{Fr}(M)$ is incompressible in $V-K$, it follows that J is noncontractible in $\mathrm{cl}(V-M)$. This is a contradiction.

Part (3) follows from part (2) and the minimality of $\sharp(S \cap \mathrm{Fr}(M))$. ♠

Lemma 2.2. *Suppose that V is a 3-manifold that has one end, is open, and is of genus at least $g \geq 1$ at infinity rel L, where L is a compact subset of V. If $F \subset \mathrm{cl}(V-L)$ is a connected, compact 2-manifold with $\partial F = \emptyset$ and* genus $(F) < g$, *then either F is nonseparating or F bounds a compact 3-manifold in* $\mathrm{cl}(V-L)$.

Proof. Suppose that F separates V. Since V has one end, there is a compact 3-manifold $M \subset V$ such that $\partial M = F$. Since V is of genus at least g at infinity rel L and genus $(F) < g$, then L is not contained in M. Therefore $M \subset \mathrm{cl}(V-L)$. ♠

It follows immediately from Lemma 2.2 that if V is irreducible and of genus at least one at infinity rel L, then $\mathrm{cl}(V-L)$ is irreducible. Note that if V is irreducible and of genus zero at infinity, then V is homeomorphic to $\mathbf{R}^3$.

Lemma 2.3. *Let V be an orientable, irreducible, connected, open, eventually end-irreducible 3-manifold with one end such that $\pi_1(V)$ is finitely generated. Also assume that there is a compact 3-manifold L in V such that V is of at least genus 1 at infinity rel L. For every compact $K \subset V$, there is an M_K which is regular in V with respect to L and such that $K \subset M_K - \partial M_K$.*

Proof. Suppose that $K \subset W$ is a compact set with $L \subset K$. Let M be a compact, connected 3-manifold in V with $K \subset M - \partial M$. This M will be gradually made larger with the promise that at some point it will be left fixed.

We may assume that $\mathrm{cl}(V-M)$ is end-irreducible because V is eventually end-irreducible. Since $\pi_1(V)$ is finitely generated, we may assume that the inclusion induced map $\pi_1(M) \to \pi_1(V)$ is onto.

Since V has one end, we may apply Lemma 2.2 and assume that $\mathrm{cl}(V-M)$ and ∂M are connected by adding components of $\mathrm{cl}(V-M)$ to the original M. Suppose that $B \subset V$ is a closed 3-cell with $\partial B \subset V-M$. Since M and $\mathrm{cl}(V-M)$ are connected and $L \subset M$, it follows that $B \subset V-M$. Therefore $\mathrm{cl}(V-M)$ is irreducible. ♠

Lemma 2.4. *Let V be an orientable, irreducible, connected, open, eventually end-irreducible 3-manifold with one end such that $\pi_1(V)$ is finitely generated. Also assume that there is a compact 3-manifold $L \subset V$ such that V is of at least genus g at infinity rel L for some nonnegative integer g. Suppose that $M \subset V$ is a 3-manifold that is regular in V with respect to L.*

If $F \subset V-M$ is a connected 2-manifold with empty boundary that is proper in V, then F separates V. Furthermore if the genus of F is less than g, then F bounds a compact 3-manifold in $V-M$.

Proof. The fact that F separates follows by a $\mathbf{Z}_2$-intersection number argument. See the proof of Lemma 2.2 of [**3**], for example. This uses only the fact that $\pi_1(M) \to \pi_1(V)$ is onto.

Suppose that the genus of F is less than g. Since F separates and V has only one end, the rest follows by Lemma 2.2.♠

Lemma 2.5. *Let V be a noncompact, irreducible 3-manifold and let $S \subset V$ be a 3-manifold that is proper in V and such that each component of ∂S is incompressible in V. Suppose that Q is a squadron in V such that S swallows the ends of Q. Then Q is isotopic in V into S.*

Proof. Let $D_Q \subset Q$ be a 2-manifold such that $Q' \cap D_Q$ is a disk for each component Q' of Q and such that $Q - D_Q \subset S$. By an isotopy that is fixed on $Q - D_Q$ isotop Q so that $\sharp(Q \cap \partial S)$ is minimal. Since each component of Q is a plane and since ∂S is incompressible in V, the irreducibility of V makes it possible to reduce $\sharp(Q \cap \partial S)$ by an isotopy fixed on $Q - D_Q$ whenever $Q \cap \partial S \neq \emptyset$. It follows that $Q \cap \partial S = \emptyset$. Since $Q - D_Q$ is not moved and is contained in S, it follows that $Q \subset S$. ♠

3. Irreducible, end-irreducible 3-manifolds

In this section, we will assume that W is an orientable, irreducible, end-irreducible 3-manifold. The first result shows us there are no non nontrivial planes in W, the second gives sufficient conditions for a 3-submanifold of W to inherit end-irreducibility, while the rest of the section is devoted to proving that if W has one end, if ∂W is an open annulus that is incompressible in W, and if the normal closure of $\pi_1(\partial W)$ is $\pi_1(W)$, then $W = \partial W \times [0, \infty)$.

Lemma 3.1. *If W is an orientable, irreducible, end-irreducible 3-manifold, then every plane in W is trivial.*

Proof. Let $P \subset W$ be a plane. We may assume that P is proper in W. Let K be a compact subset of W and suppose that $M \subset W$ is a compact 3-manifold with $K \subset M - \mathrm{Fr}(M)$ such that $\mathrm{Fr}(M)$ is incompressible in W. Since every simple closed curve in P is contractible in P, the usual arguments involving the incompressibility of $\mathrm{Fr}(M)$ in W and the irreducibility of W show that we may isotop P to be disjoint from $\mathrm{Fr}(M)$. Since P is proper, $P \cap M = \emptyset$. Therefore, $P \cap K = \emptyset$. It follows that P must be trivial. ♠

Lemma 3.2. *Suppose that W is an orientable, irreducible, end-irreducible 3-manifold. Let U be a connected 3-manifold that is proper in W such that ∂U is an open annulus that is incompressible in W. Then U is irreducible and end-irreducible.*

Proof. It is clear that U is irreducible. To show U is irreducible, let A be an annulus in ∂U such that the inclusion induced map $\pi_1(A) \to \pi_1(\partial U)$ an isomorphism. Let L be a compact subset of U that contains A and is such that $L \cap \partial U$ is an annulus.

Let M be a compact connected 3-manifold with $L \subset M - \mathrm{Fr}(M)$ such that $\mathrm{Fr}(M)$ is incompressible in W. Isotop $\mathrm{Fr}(M)$ by an isotopy of compact support fixed on L such that $\sharp(\mathrm{Fr}(M) \cap \partial U)$ is minimal. We aim to show $\mathrm{Fr}(M \cap U; U)$ is incompressible in U.

Let J be a component of $\mathrm{Fr}(M) \cap \partial U$. Then J is a circle. We claim that J is noncontractible on $\mathrm{Fr}(M)$. To get a contradiction, suppose that J bounds a disk D in $\mathrm{Fr}(M)$. We may choose J so that $D \cap \mathrm{Fr}(M) = J$. Since ∂U is incompressible, there is a disk $E \subset \partial U$ with $\partial E = J$. Since $\pi_1(A) \to \pi_1(U)$ is nontrivial, A is not a subset of E. Therefore, $A \cap E = \emptyset$. So $E \cap L = \emptyset$. Therefore $E \cup D$ is a sphere in $W - L$.

By the irreducibility of W, there is a 3-ball B that is bounded by $E \cup D$. Since $\pi_1(A) \to \pi_1(W)$ is nontrivial, L is not contained in B. Therefore, $B \cap L = \emptyset$. So we may isotop $\mathrm{Fr}(M)$ along B and reduce $\sharp(\mathrm{Fr}(M) \cap \partial U)$. This is a contradiction, so J is noncontractible on $\mathrm{Fr}(M)$.

Note that $\mathrm{Fr}(M \cap U; U) = \mathrm{Fr}(M) \cap U$. Suppose D is a compressing disk for $\mathrm{Fr}(M \cap U; U)$ in U. Then ∂D bounds a disk $E \subset \mathrm{Fr}(M)$ that must contain a component of $\mathrm{Fr}(M) \cap \partial U$. Since these curves are noncontractible on $\mathrm{Fr}(M)$, this is a contradiction. It follows that $\mathrm{Fr}(M \cap U; U)$ is incompressible in U. Therefore, we may conclude that U is end-irreducible. ♠

Lemma 3.3. *Let W be an orientable, irreducible, end-irreducible 3-manifold that has only one end and is such that ∂W an open annulus that is incompressible in W. Suppose that there is an annulus $A \subset \partial W$ such that the inclusion induced map $\pi_1(A) \to \pi_1(\partial W)$ is an isomorphism. The following are equivalent.*

1) *Every loop in W is freely homotopic in W into A.*
2) *Given any compact $K \subset W$, there is a compact, connected, irreducible 3-manifold $M \subset W$ with $K \cup A \subset M - \mathrm{Fr}(M)$ such that*
    a) $\mathrm{Fr}(M)$ *is incompressible in W,*
    b) $\mathrm{Fr}(M)$ *is connected, and*
    c) *any closed path in M is freely homotopic in M into $A \cup \mathrm{Fr}(M)$.*

3) *Given any compact* $K \subset W$, *there is a compact, connected 3-manifold* $M \subset W$ *with* $K \cup A \subset M - \mathrm{Fr}(M)$ *such that the triple* $(M, A, \mathrm{Fr}(M))$ *is homeomorphic as a triple to* $(A \times I, A \times 0, A \times 1)$.
4) *The pair* (W, A) *is homeomorphic as a pair to* $(A \times [0, \infty), A \times 0)$.

Proof. We first prove ($1 \Longrightarrow 2$). Let $K \subset W$ be compact. Since W is end-irreducible, there is a compact, connected 3-manifold $M \subset W$ with $K \cup A \subset M - \mathrm{Fr}(M)$ such that $\mathrm{Fr}(M)$ is incompressible in W. Since W has one end, we may argue that $\mathrm{cl}(W - M)$ is connected and noncompact. It follows that M is irreducible because W is irreducible. Since $A \subset M$ and W is irreducible, it follows that $\mathrm{cl}(W - M)$ is irreducible.

Let F be a component of $\mathrm{Fr}(M)$. If $\mathrm{Fr}(M) - F \neq \emptyset$, there is a loop λ in W that meets F in exactly one point because $\mathrm{cl}(W - M)$ is connected. Now λ is freely homotopic in W into A. By a $\mathbf{Z}_2$-intersection number argument, we get a contradiction. See the proof of Lemma 2.3 of [**3**] for example. So $F = \mathrm{Fr}(M)$.

Let $\lambda : S^1 \to M$ be a map. We may assume that $\lambda(S^1) \subset M - \partial M$. Since every closed path in W is freely homotopic in W into A, there exists a map $\Lambda : S^1 \times I \to W$ such that $\Lambda(z, 0) = \lambda(z)$ for every $z \in S^1$ and $\Lambda(S^1 \times 1) \subset A$.

We may assume that Λ is transverse to F and that $\sharp(\Lambda^{-1}(F))$ is minimal.

Of course if $\Lambda^{-1}(F) = \emptyset$, then λ is freely homotopic in M into A.

Suppose that $\Lambda^{-1}(F) \neq \emptyset$ and that J is a component of $\Lambda^{-1}(F)$. Since $\Lambda(S^1 \times \partial I) \cap F = \emptyset$, it follows that J is a simple closed curve. Since F is incompressible and $\sharp(\Lambda^{-1}(F))$ is minimal, it can be argued that J is isotopic in $S^1 \times I$ to a curve $S^1 \times t$ for some $t \in (0, 1)$. Alter the product structure of $S^1 \times I$ so that $J = S^1 \times t$. We may choose J so that $\Lambda(S^1 \times [0, t)) \cap F = \emptyset$. It now follows that λ is freely homotopic in M into F.

We now prove ($2 \Longrightarrow 3$). By Theorem 3.1 of [**1**], there is a component C of $\mathrm{Fr}(M) \cup A$ such that the map induced by inclusion on fundamental group is onto. Note that $\pi_1(\mathrm{Fr}(M)) \to \pi_1(M)$ is injective because $\mathrm{Fr}(M)$ is incompressible in M, and the inclusion induced map $\pi_1(A) \to \pi_1(W)$ is injective. Therefore, by Lemma 10.2 of [**6**], it follows that there is a homeomorphism $h : (M, C) \to (C \times I, C \times 0)$.

Suppose that $C = A$. Then (M, A) is homeomorphic to $(A \times I, A \times 0)$. Since $\mathrm{Fr}(M)$ is connected and incompressible in $A \times I$, it follows that $\mathrm{Fr}(M)$ is either a disk or an annulus.

Suppose that $\mathrm{Fr}(M)$ is a disk. Then M must be a closed 3-cell because W is irreducible and boundary-irreducible and $\mathrm{cl}(W - M)$ is connected and noncompact. Since $\pi_1(A) \to \pi_1(M)$ is nontrivial, this is absurd.

So we may assume that $\mathrm{Fr}(M)$ is an annulus. Observe that $\mathrm{cl}(\partial M - A)$ is an annulus. Since $\mathrm{Fr}(M) \subset \mathrm{cl}(\partial M - A)$ and is incompressible in M, it follows that $(M, A, \mathrm{Fr}(M))$ is homeomorphic to $(A \times I, A \times 0, A \times 1)$.

Suppose that $C = \mathrm{Fr}(M)$. Then $h^{-1}((\partial C \times I) \cup (C \times 1)) = M \cap \partial W$. Since $\mathrm{Fr}(M)$ and ∂W are incompressible in W, it can be argued that the inclusion induced map $\pi_1(M \cap \partial W) \to \pi_1(M)$ is injective. Recall ∂W is an open annulus, so $M \cap \partial W$ must be an annulus. Since $A \subset M \cap \partial W$ and A is incompressible in M, it follows that $M \cap \partial W$ is an annulus and is a regular neighborhood of A in ∂W. It follows that there is a homeomorphism from $(M, A, \mathrm{Fr}(M))$ to $(A \times I, A \times 0, A \times 1)$.

We now prove (3$\Longrightarrow$ 4). Let X be an end-irreducible exhaustion for W such that, for $n \geq 0$, $A \subset X_n - \mathrm{Fr}(X_n)$ and $\mathrm{cl}(W - X_n)$ is connected and noncompact, and $(X_n, A_n, \mathrm{Fr}(X_n))$ is homeomorphic to $(A \times I, A \times 0, A \times 1)$ for $n \geq 0$. By Lemma IX.1 of [**7**], it follows that, for $n \geq 1$, there is a homeomorphism $h_n : \mathrm{cl}(X_n - X_{n-1}) \to A \times [n-1, n]$ such that $h_n|\mathrm{Fr}(X_k) : \mathrm{Fr}(X_k) \to A \times k$ is a homeomorphism for $k = n-1$ and n.

For $n \geq 1$, let $\eta_n : A \times [n-1, n] \to A$ be the projection map. We define a homeomorphism $g : A \times [0, \infty) \to W$ as follows. Let $g(x,t) = h_1^{-1}(x,t)$ for $(x,t) \in A \times [0,1]$. Let $n \geq 2$ be given and suppose that g is defined on $A \times [0, n-1]$ and takes it homeomorphically onto X_{n-1} with $g|A \times (n-1) : A \times (n-1) \to \mathrm{Fr}(X_{n-1})$ a homeomorphism. For $(x,t) \in A \times [n-1, n]$, define $g(x,t) = h_n^{-1}(\eta_n h_n g(x, n-1), t)$. One may check that g is well-defined, continuous, and that $g|A \times [0,n] : A \times [0,n] \to X_n$ and $g|A \times n : A \times n \to X_n$ are homeomorphisms. By continuing inductively, we can extend g to a homeomorphism from $A \times [0, \infty)$ to W which takes $A \times 0$ to A.

That (4$\Longrightarrow$ 1) is clear. ♠

4. Nearnodes

Lemma 4.1. *Suppose that N is a nearnode with $\nu \geq 2$ faces. Suppose that C is a compact, connected subset of N that meets at least two components of ∂N. If B is a closed 3-cell in N with $C \subset B - \mathrm{Fr}(B)$ such that $B \cap Q$ is either a disk or empty depending or whether $C \cap Q \neq \emptyset$ or $C \cap Q = \emptyset$, respectively, for each component Q of ∂N, then* $\mathrm{Fr}(B)$ *is incompressible in $N - C$.*

Proof. Suppose that $D \subset N - C$ is a compressing disk for $\mathrm{Fr}(B)$. Let E_1 and E_2 be the closures of the components of $\partial B - \partial D$. Since ∂D is noncontractible in $\mathrm{Fr}(B)$ is follows that $E_i - \partial E_i$ contains a component of $B \cap \partial N$ for $i = 1$ and 2. Therefore $C \cap (E_i - \partial E_i) \neq \emptyset$ for $i = 1, 2$. Since N is irreducible, there is a 3-cell $B_1 \subset N$ with $\partial B_1 = E_1 \cup D$. Since $C \cap D = \emptyset$, it follows after a bit of argument that $C \cap \partial B_1 \subset E_1 - \partial E_1$. Therefore $C \subset B_1$. Since $B_1 \cap \partial N \subset E_1$, it follows that $C \cap (E_2 - \partial E_2) = \emptyset$ which is a contradiction. ♠

Lemma 4.2. *Suppose that N is a noncompact 3-manifold containing a nontrivial plane $P \subset N - \partial N$ which separates N. Let N' be the result of splitting N along P. Then N is a nearnode iff each component of N' is a nearnode.*

Proof. Let N_0 and N_1 be the components of N' and let $\eta : N' \to N$ be the quotient map of the splitting. Let $P_i = \eta^{-1}(P) \cap N_i$ for $i = 0$ and 1.

($\Longleftarrow$) Suppose that each component of N' is a nearnode. Let $K \subset N$ be compact. Since P is proper in N, it follows that η is a proper map. Therefore $\eta^{-1}(K) \cap N_i$ is compact for $i = 0$ and 1. Since N_0 is a nearnode, there is a closed 3-cell $B_0 \subset N_0$ such that $B_0 \cap Q$ is either empty or a disk for every component of ∂N_0 and $\eta^{-1}(K) \cap N_0 \subset B_0 - \mathrm{Fr}(B_0)$. We may take B_0 large enough so that $P_0 \cap B_0$ is a disk.

Note that $\eta^{-1}\eta(B_0 \cup P_0) \cap P_1$ is a disk. Let $B_1 \subset N_1$ be a closed 3-cell such that

$$\left[\eta^{-1}\eta(B_0 \cap P_0) \cup \eta^{-1}(K)\right] \cap P_1 \subset B_1$$

and so that $B_1 \cap Q$ is either a disk or empty for every component Q of ∂N_1. Note that $\eta(B_0 \cup B_1)$ is a closed 3-cell which contains K and is such that $\eta(B_0 \cup B_1) \cap Q$ is either a disk or empty for every component Q of N. Therefore N is a nearnode.

($\Longrightarrow$) Now suppose that N is a nearnode. If ∂N is connected, N is homeomorphic to halfspace which contains no nontrivial planes. Therefore, $\sharp(\partial N) \geq 2$.

Let $i = 0$ or 1 be given. Let $K_i \subset N_i$ be compact. Then $\eta(K_i) \subset N$ is compact. There is a compact, connected $K \subset N$ which traps P such that $K_i \subset K$, and such that K meets at least two components of ∂N. Let $D_P \subset P$ be a disk in P with $P \cap K \subset D_P - \partial D_P$. Let $B' \subset N$ be a closed 3-cell with $K \cup D_P \subset B' - \mathrm{Fr}(B')$ such that $B' \cap Q$ is either a disk or empty for every component Q of ∂N. Let Q^* be the union of components Q of ∂N such that $B' \cap Q \neq \emptyset$ but $K \cap Q = \emptyset$. Let U be a regular neighborhood of Q^* in $N - (K \cap D_P)$. Let $B = \mathrm{cl}(B' - U)$. We may choose U so that B is a closed 3-cell. Note that $B \cap Q$ is either a disk or is empty for each component Q of ∂N. By Lemma 4.1 it follows that $\mathrm{Fr}(B)$ is incompressible in $N - K$.

By Lemma 2.1, The Target Lemma, there is an isotopy of N fixed on $K \cup D_P$ which takes P to a plane P' such that each component J of $P' \cap \mathrm{Fr}(B)$ is noncontractible in $\mathrm{Fr}(B)$ and bounds a disk $E_J \subset P'$ such that $D_P \subset E_J - J$. It follows that each compact component of $P' \cap \mathrm{cl}(N - B)$ is an annulus that is incompressible in $\mathrm{cl}(N - B)$.

Suppose that A is an annulus component of $P' \cap \mathrm{cl}(N - B)$. Let J_1 and J_2 be the components of ∂A. Let D_1 and D_2 be properly embedded disjoint disks in B such that $\partial D_j = J_j$ for $j = 1$ and 2. Since N is irreducible, there is a closed 3-cell $C \subset N$ with $\partial C = D_1 \cup A \cup D_2$. It is not difficult to see that $B \cup C$ is a closed 3-cell. Let B^*_C be a regular neighborhood of $B \cup C$ in N. Then each compact component of $P' \cap \mathrm{cl}(N - B^*_C)$ is an incompressible annulus in $\mathrm{cl}(N - B^*_C)$ and

$$\sharp(P' \cap \mathrm{cl}(N - B^*_C)) < \sharp(P' \cap \mathrm{cl}(N - B)).$$

By continuing in this fashion, we obtain a closed 3-cell B^* such that $B^* \cap \partial N = B \cap \partial N$ and $P' \cap \mathrm{Fr}(B^*)$ is a simple closed curve, say J. Let E be the disk in P' with $\partial E = J$. Let B^*_i be the closure of the component of $B^* - E$ which contains $\eta(K_i)$.

There is a homeomorphism $h : N \to N$ that is the identity on K such that $h(P') = P$. Then $h(B^*_i)$ is a 3-cell which meets P in a single disk, namely $h(E)$. Let $B'_i = \eta^{-1}h(B^*_i)$. Then $K_i \subset B'_i - \mathrm{Fr}(B'_i)$ and $B'_i \cap Q$ is either a disk or empty for each component Q of ∂N_i. Therefore N_i is a nearnode. ♠

Lemma 4.3. *If N is a nearnode and $N' \subset N$ is a proper 3-manifold such that $\partial N'$ is a squadron in N, then N' is a nearnode.*

Proof. This follows immediately from Lemma 4.2 ♠

Let F^∞ be the 2-manifold obtained from the closed upper half plane by removing an open disk of radius $\dfrac{1}{3}$ centered at $(0, n)$ for $n = 1, 2, 3, \ldots$. Let $\Sigma^\infty = F^\infty \times S^1$.

Lemma 4.4. *Let N be a nearnode with two faces that is not a node, i.e. is not a missing boundary manifold. Then there is an embedding $\iota : \Sigma^\infty \to N$ which* *is proper and is such that exactly one component of* $\mathrm{cl}(N - \iota(\Sigma^\infty))$ *is a 2-handle and every other component of* $\mathrm{cl}(N - \iota(\Sigma^\infty))$ *is homeomorphic to the exterior of a nontrivial knot in* S^3.

Proof. Let B be an exhaustion for N of closed 3-cells which meet each component of ∂N in single disks. By Lemma 3.3 there is at least one loop λ_n in $N - B_n$ that is not freely homotopic in N into $\mathrm{Fr}(B_n)$ for each $n \geq 0$. By taking a subsequence of B, we may assume that $B_{n+1} - B_n$ contains λ_n. For $n \geq 0$, let $V_n = \mathrm{cl}(B_{n+1} - B_n)$. Then $(V_n, \mathrm{Fr}(V_n))$ is not homeomorphic to $(S^1 \times I \times I, S^1 \times I \times \partial I)$ for any $n \geq 0$. Note, however, that ∂V_n is a torus for $n \geq 0$.

For $n \geq 0$, let U_n be a regular neighborhood of ∂V_n in V_n and let $T_n = \partial U_n - \partial V_n$. Let $\Sigma = \bigcup_{n=0}^{\infty} U_n$. It is not difficult to construct a homeomorphism from Σ to Σ^∞ by taking U_n to $F_n \times S^1$, where $F_0 = \left\{(x,y) \in F^\infty | x^2 + y^2 \leq \left(\frac{3}{2}\right)^2\right\}$ and $F_n = \left\{(x,y) \in F^\infty | \left(n + \frac{1}{2}\right)^2 \leq x^2 + y^2 \leq \left(n + \frac{3}{2}\right)^2\right\}$ for $n \geq 1$. Let $V_n' = \mathrm{cl}(V_n - U_n)$.

Note B_0 is a 2-handle attached to the open annulus component of $\partial\Sigma$. Let $n \geq 0$ be given. We claim that V_n' is the exterior of a nontrivial knot in S^3. Since $V_n' \subset B_{n+1}$ and $\partial V_n'$ is a torus, it follows that V_n' is either a nontrivial knot exterior or a solid torus. To get a contradiction, suppose that V_n' is a solid torus. Let λ be the generator of $\pi_1(V_n)$. There exists ν such that λ^ν is freely homotopic in V_n to a loop contained in $\mathrm{Fr}(B_n)$. By Van Kampen's Theorem, it follows that $\pi_1(B_{n+1})$ is isomorphic to $\langle \lambda | \lambda^\nu = 1 \rangle$. Since B_{n+1} is simply connected, it follows that $|\nu| = 1$. Therefore $(V_n, \mathrm{Fr}(V_n))$ is homeomorphic to $(S^1 \times I \times I, S^1 \times I \times \partial I)$. This violates the first paragraph of this proof. Therefore V_n' is a nontrivial knot exterior. This ends the proof. ♠

The following Theorem has benefited from a discussion with Mike Starbird.

Theorem 4.5. *A nearnode with two faces that is not a node contains a collection of pairwise disjoint nontrivial planes with the cardinality of the Cantor set no two of whose members are parallel.*

Proof. As usual let I be the closed unit interval. Let $K \subset I$ be the Classical Cantor Set. Recall that K is constructed recursively by successively removing middle thirds of closed intervals. For $n \geq 0$, let $U_{n,1}, \ldots, U_{n,2^n}$ be the middle thirds removed during the nth stage of construction.

Let $i \geq 0$ and $1 \leq j \leq 2^i$ be given. For $l \geq 1$, let D_{ijl} be a round open disk in $U_{ij} \times [0, \infty)$ whose center is on $I \times (l + i)$. We assume that radii are chosen so that the disks are disjoint. Let $\mathcal{D}$ be the set of all D_{ijl}. Then $\mathcal{D}$ is countable. Observe that $I \times [0, n]$ meets only finitely many elements of $\mathcal{D}$.

Let $F_K = (I \times [0, \infty)) - (\bigcup \mathcal{D})$. Let $\Sigma_K = F_K \times S^1$. For every $x \in K$, let $A_x' = (x \times [0, \infty)) \times S^1$. Then A_x' is a half open annulus for every $x \in K$.

Let N be a nearnode with two faces. By Lemma 4.4, we may assume that Σ^∞ is contained in and proper in N and that exactly one component of $\mathrm{cl}(N-\Sigma^\infty)$ is a 2-handle and the rest are nontrivial knot exteriors in S^3.

Let H be the 2-handle component of $\mathrm{cl}(N-\Sigma^\infty)$ and let $A = H \cap \Sigma^\infty$. It is not difficult to show that F_K is homeomorphic to F^∞. It follows that Σ_K is homeomorphic to Σ^∞. Indeed there is a homeomorphism of pairs $h : (\Sigma_K, I \times 0 \times S^1) \to (\Sigma, A)$.

For each $x \in K$, let $A_x = h(A'_x)$. Make the identification $H = I \times D^2$ so that $A = I \times \partial D^2$. For every $x \in K$, let $D_x = x \times D^2$ and let $P_x = A_x \cup D_x$. Then P_x is a plane that is proper in N and for every $x \in K$, and $P_x \cap P_y = \emptyset$ whenever $x \neq y$. Let $\mathbf{P}_K = \{P_x | x \in K\}$.

We claim that $\mathbf{P}_K$ is a set of planes as in the statement of the theorem. All we need to show is that if $x \neq y$ are elements of K, then P_x is not parallel in N to P_y. Let $x \neq y \in K$. Let N_{xy} be the near parallelism in N between P_x and P_y that is guaranteed by Lemma 4.2. There exist i and j such that U_{ij} is between x and y. Then U_{ij} contains infinitely many elements of $\mathcal{D}$. Hence N_{xy} contains infinitely many of the nontrivial knot exteriors of $\mathrm{cl}(N-\Sigma^\infty)$. One may conclude that $\pi_1(\mathrm{cl}(N_{xy} - H))$ is nonfinitely generated. It follows that N_{xy} is not a node by Lemma 3.3.
♠

The result given below extends a result of Kinoshita [**10**].

Theorem 4.6. *Let V be a connected, orientable, irreducible open 3-manifold of genus 1 at infinity. Suppose that $P_0 \subset V$ is a nontrivial plane. Then V is a nearnode with one handle.*

Proof. It suffices to show that the manifold obtained by splitting along P_0 is a nearnode with two faces. Let U be a regular neighborhood of P_0 in V and let $V' = \mathrm{cl}(V-U)$. Suppose that K' is a compact subset of V'. Let P_1 and P_2 be the components of ∂U. Then P_i is parallel in V to P_0 for $i = 1$ and 2. Let P be the squadron $P_0 \cup P_1 \cup P_2$. Let $K \subset V$ be a compact, connected 3-manifold which contains K' and traps each component of P in V. By Lemma 1 of [**14**], we may assume that $P_i \cap K$ is a single disk for $i = 0, 1, 2$. Since V is of genus 1 at infinity, we may assume that K is chosen so that if $M' \subset V$ is a compact 3-manifold with connected boundary such that $K \subset M' - \partial M'$, then $\partial M'$ is of genus at least one. For $i = 0, 1, 2$, let $D_i \subset P_i$ be a disk with $K \cap P_i \subset D_i - \partial D_i$. Since V is of genus 1 at infinity, there is a compact 3-manifold $M \subset V$ with $K \cup (D_0 \cup D_1 \cup D_2) \subset M - \partial M$ such that ∂M is a torus.

Claim 4.6.1 ∂M is incompressible in $V - K$.

Proof: This follows because adding a compressing 2-handle to or removing a 1-handle from M in the complement of K will result in a compact 3-manifold M' such that $K \subset M' - \partial M'$ and $\partial M'$ is a 2-sphere. This contradicts our choice of K.
♣

By Lemma 2.1, we may isotop P in V by an isotopy fixed on $K \cup (D_0 \cup D_1 \cup D_2)$ so that for each $i = 0, 1, 2$, each component J of $P \cap \partial M$ bounds a disk $E_J \subset P$

such that $D_i \subset E_J - J$ for the appropriate i and such that J is noncontractible in ∂M.

Claim 4.6.2 M is a solid torus, and each component of $P \cap \partial M$ is a meridian of M.

Proof: Let $i = 0, 1$ or 2, and let J be a component of $P_i \cap \partial M$ and such that $(E_J - J) \cap \partial M = \emptyset$. Let N be a regular neighborhood of E_J in M. Since J is noncontractible in ∂M, it follows that $\mathrm{cl}(\partial M - N)$ is an annulus. Hence $\mathrm{cl}(\partial M - N) \cup \mathrm{Fr}(N; M)$ is a 2-sphere which must bound a 3-cell B in V. Since N and P_0 are on the same side of ∂B, and since P_0 is noncompact and proper in V, it follows that $B \cap N = \mathrm{Fr}(N; M)$. Since N and B are 3-cells and V is orientable, it follows that $N \cup B$ is a solid torus. Note that $\partial(N \cup B) = \partial M$. Because V is noncompact, it follows that $N \cup B = M$ and so M is a solid torus. Note that E_J is a meridian disk for M. Since each component of $P \cap \partial M$ is noncontractible in ∂M, it follows that each is parallel in ∂M to J and, therefore, must be a meridian for M. ♣

Claim 4.6.3 V' is a nearnode with two faces.

Proof: First consider the case where $P_i \cap M$ is a single disk for each $i = 0, 1, 2$. Then $U \cap M$ is a regular neighborhood of $P_0 \cap M$ in M. Let $B = \mathrm{cl}(M - U)$. Then B is a 3-cell, $B \subset V'$, and $B \cap P_i$ is a single disk for $i = 1, 2$. It follows that $K' \subset B$ because $K' \subset M$ and $K' \subset V'$. Therefore V' is a nearnode with two faces. Now suppose that $P_i \cap M$ is not a single disk for some $i = 0, 1$ or 2. Then there is an annulus component A of $P_i \cap \mathrm{cl}(V - W)$. By Claim 4.6.2 there are disjoint meridian disks D' and D'' in M such that $\partial D' \cup \partial D'' = \partial A$. Note that $D' \cup A \cup D''$ is a 2-sphere that bounds a closed 3-cell $C \subset V$. It is not difficult to see that $M \cup C$ is a solid torus. Let M' be a regular neighborhood of $M \cup C$ in V. By continuing in this fashion, we may reduce to the case where $P_i \cap M$ is a single disk for $i = 0, 1, 2$. ♣

The theorem now follows from these claims. ♠

Corollary 4.7. *Let V be a connected, orientable, irreducible open 3-manifold of genus 1 at infinity. Suppose that $P_0 \subset V$ is a nontrivial plane. Then $\pi_1(V)$ is infinite cyclic.*

5. Finiteness Conditions

Theorem 5.1. *Suppose that V is a noncompact, irreducible, orientable connected 3-manifold that has one end. Assume that V has an exhaustion X such that ∂X_n is connected, of genus $g \geq 2$, and is incompressible in $V - X_0$ for $n \geq 1$.*

Let U be a 3-manifold that is proper in V such that ∂U is a squadron. Let ν be the number of components of U.

1) *If $\nu > 2g - 2$, then at least one component of U is a nearnode with two faces.*
2) *If W is a component of U, then $\sharp(\partial W) \leq 2g$.*

Proof. It is not difficult to show that there is a compact 3-manifold $M \subset V$ such that $\mathrm{cl}(V - M)$ is end-irreducible, and for every compact $K \subset V$ with $M \subset K$ there is a compact 3-manifold N_K such that $K \subset N_K - \partial N_K$ and ∂N_K is connected, of genus g and incompressible in $W - M$.

Since V is irreducible and connected and since $g \geq 1$, it follows that M may be chosen so that $\mathrm{cl}(V - M)$ is irreducible.

Suppose that $U_1, \ldots, U_{2g-1}$ are distinct components of U. Let $U' = \bigcup_{i=1}^{2g-1} U_i$. Let D be a 2-manifold in $\partial U'$ such that $D \cap P$ is a disk for every component P of $\partial U'$ and such that $\partial U' \cap M \subset D - \partial D$. Let K be a compact subset of U'. We may assume that $K \cap P \neq \emptyset$ for each component P of $\partial U'$ and that $K \cap U_i$ is connected for $1 \leq i \leq 2g - 1$. Let N be a compact 3-manifold in V such that $K \cup D \cup M \subset N - \partial N$, such that ∂N is connected, is of genus g, and is incompressible in $V - (K \cup D \cup M)$.

By Lemma 2.1, we may move $\partial U'$ by an isotopy of V fixed on $K \cup D \cup M$ so that if J is a component of $\partial U' \cap \partial N$, then J is a simple closed curve that bounds a disk $E_J \subset \partial U'$ such that $D \cap P \subset E_J - J$, where P is the component of $\partial U'$ that contains J. We may also assume that J is noncontractible on ∂N.

We first claim that no component of $\partial N \cap U'$ or $\partial N \cap \mathrm{cl}(V - U')$ is a disk. Suppose that E is such a disk. Let $J = \partial E$. Then J is a component of $\partial N \cap \partial U'$. Let P be the component of $\partial U'$ which contains J. Recall there is a disk $E_J \subset P$ such that $\partial E_J = J$ and $M \cap P \subset E_J - J$. Now $E \cup E_J$ is a 2-sphere which must bound a closed 3-cell $B \subset V$ by irreducibility. We may use B to isotop P free of M. However since $\mathrm{cl}(V - M)$ is end-irreducible, Lemma 3.1 produces a contradiction.

Let χ denote the Euler characteristic. Since neither $\partial N \cap U'$ nor $\partial N \cap \mathrm{cl}(V - U')$ has disk components, it follows that $\chi(\partial N \cap U')$ and $\chi(\partial N \cap \mathrm{cl}(V - U'))$ are non-positive. Consequently

$$\begin{aligned} \chi(\partial N) &= \chi(\partial N \cap U') + \chi(\partial N \cap \mathrm{cl}(V - U')) \\ &\leq \chi(\partial N \cap U'). \end{aligned}$$

For $1 \leq i \leq 2g - 1$, let $F_i = \partial N \cap U_i$. Then

$$\chi(\partial N) \leq \sum_{i=1}^{2g-1} \chi(F_i).$$

Since no component of F_i is a disk and each component has boundary, it follows that $\chi(F_i) \leq 0$ for $1 \leq i \leq 2g - 1$.

We claim that $\chi(F_i) = 0$ for some i. Otherwise $\chi(F_i) \leq -1$ for $1 \leq i \leq 2g - 1$, and hence $\chi(\partial N) \leq 1 - 2g$. But $\chi(\partial N) = 2 - 2g$ because ∂N is of genus g. This is a contradiction.

Choose notation so that $\chi(F_1) = 0$. Since no component of F_1 is a disk, each component of F_1 has Euler Characteristic equal to zero. Since ∂N is orientable, no component of F_1 is a mobius band. Therefore each component of F_1 is an annulus.

We claim there is a component A of F_1 such that each component of ∂A is contained in a different component of ∂U_1. In order to get a contradiction, suppose that this is not the case. Let P be a component of ∂U_1. Then $\sharp(P \cap \partial N)$ is even since $\sharp(P \cap \partial N) = 2\sharp(F_1')$, where F_1' is the union of all the components of F_1 that meet P. On the other hand, $\sharp(P \cap \partial N)$ must be odd because $\sharp(P \cap \partial N) = \sharp(\partial(P \cap N))$ and exactly one component of $P \cap N$ is a disk while the rest are annuli.

Let J_1 and J_2 be the components of ∂A and let P_i be the component of ∂U_1 that contains J_i and let E_i be the disk in P_i bounded by J_i for $i = 1, 2$. Then $E_1 \cup A \cup E_2$ is a 2-sphere which must bound a 3-cell $B_0 \subset V$ by irreducibility. Since $A \subset U_1$ and since each component of ∂U_1 is a proper plane, it follows that $B_0 \subset U_1$.

We claim that $K \cap U_1 \subset B_0$. Since $\mathrm{cl}(P_1 - E_1)$ is noncompact and proper in V, we know that $P_1 - E_1$ is not contained in B_0. Since $K \cap U_1$ and $P_1 - E_1$ are on opposite sides of ∂B_0, it follows that $K \cap U_1 \subset B_0$.

Since U' has only finitely many components, it follows that at least one of them, say U_1, has an exhaustion B such that B_n is a closed 3-cell and $B_n \cap \partial U_1$ is a pair of disks each contained in a different component of ∂U_1. Therefore U_1 is a nearnode with two faces. This ends the proof of (1)

To prove (2), let W be a component of U.

There is a component F of $\partial N \cap W$ that meets each component of ∂W. Consequently $\sharp(\partial W) \leq \sharp(\partial F)$. We will now obtain a bound on $\sharp(\partial F)$. Let g_F be the genus of the closed 2-manifold obtained from F by capping of each component of ∂F with a disk. Clearly $g_F \leq g$. As before no component of F or $\mathrm{cl}(\partial N - F)$ is a disk. So $\chi(F) \geq \chi(\partial N)$. Hence

$$\begin{aligned}(2 - 2g_F) - \sharp(\partial F) &= \chi(F) \\ &\geq \chi(\partial N) \\ &\geq 2 - 2g.\end{aligned}$$

That is

$$\begin{aligned}\sharp(\partial F) &\leq 2g - 2g_F \\ &\leq 2g.\end{aligned}$$

It follows that $\sharp(\partial W) \leq 2g$. ♠

For the last result of the section it will additionally be assumed that $\pi_1(V)$ is finitely generated of rank ρ. We will also assume that P is a squadron in V such that $V - P$ is connected.

Lemma 5.2. *Suppose that V is a noncompact connected 3-manifold that has one end and that $\pi_1(V)$ is finitely generated of rank ρ. Suppose that P is a squadron in V such that $V - P$ is connected. Then $\sharp(P) \leq \rho$.*

Proof. This follows by Van Kampen's Theorem and Grushko's Theorem.
♠

Part II: Major Results

6. More Definitions

We now make our entrance into the second half of the paper, where the results are deeper and more difficult. Before proving these results, we will state definitions of some concepts that were not needed until this point and state some results that could not be stated without this vocabulary.

One of the chief goals of this portion of the paper is to create a place in a noncompact 3-manifold into which nontrivial planes can be isotoped. This creation proceeds in stages that will be described in broad strokes below.

In [**13**], the weak characteristic pair of an end-irreducible 3-manifold was introduced. The noncompact components of a weak characteristic pair are Seifert fibered spaces over noncompact 2-manifolds and $[0, \infty)$-bundles. It was shown in that paper that essential, half-open annuli can be isotoped into the weak characteristic pair.

As we've seen, in the Target Lemma, for instance, planes in eventually end-irreducible 3-manifolds can be isotoped so as to meet the end of the ambient manifold in half-open annuli. Consequently, the weak characteristic pair of the end of the manifold catches the end of nontrivial planes of that 3-manifold.

It turns out we may alter the weak characteristic pair by adding, 2-handles, removing 1-handles, and performing other simple modifications to obtain nearnodes and blemishes. Nearnodes and blemishes, it turns out, have certain properties that enable us to prove uniqueness of a particular type of decomposition.

This particular type of decomposition is called a "hangar decomposition." Planes are moved into hangars. The creation of the hangar is a two-stage process. Modifying the weak characteristic pair of the end produces something called a "strip." Strips are sufficient for a place into which planes may be isotoped, but they are inadequate so for as having a unique structure is concerned. For this, we must enlarge them slightly so as to obtain hangars.

Let V be a noncompact, irreducible, connected 3-manifold. We say that V is a *missing boundary manifold* if there exists a compact 3-manifold M_V such that V is homeomorphic to $M_V - L$, where L is a closed subset of ∂M_V.

Suppose that N is a noncompact, irreducible, connected orientable 3-manifold that has one end and is a missing boundary manifold. Also suppose that every component of ∂N is noncompact and that N contains a nontrivial plane. Then we say that N is a *blemish.* If N is a blemish and every component of ∂N is a plane, then we say that N is a *polished blemish.*

Let H be a proper 3-submanifold of V. We say that H is a *prehangar for* V if

1) ∂H is a squadron in V,
2) each component of H is either a nearnode with two faces or a polished blemish,
3) whenever N is a component of $\mathrm{cl}(V - H)$ that is either a nearnode with two faces or a polished blemish, then N is a node with two faces, and
4) whenever N is a component of $\mathrm{cl}(V - H)$ that is a node with two faces and H' and H'' are the components of H which contain the components of ∂N, then $H' \cup N \cup H''$ is neither an nearnode with two faces nor a polished blemish.

Given a proper 3-submanifold H in V that satisfies conditions 1 and 2 in the definition of prehangar, let $\alpha(H)$ be the number of components of $\mathrm{cl}(V - H)$ that are either nearnodes with two faces or polished blemishes but are not nodes with two faces. Given a squadron P in V, there is a proper 3-submanifold H of V that satisfies 1 and 2 (a product neighborhood of P, for instance). If we choose H so that $(\alpha(H), \sharp(H))$ is minimal when taken in lexicographic order, then it is not difficult to show that H is a prehangar for V.

By combining this observation with Lemma 8.3 of the sequel, we may obtain the following.

Lemma (The Prehangar Lemma) *Suppose that V is a noncompact, irreducible, orientable, connected 3-manifold such that*

1) *V has one end,*
2) *$\pi_1(V)$ is finitely generated, and*
3) *V has an exhaustion X such that, for $n \geq 1$, ∂X_n is connected, of genus g, and incompressible in $V - X_0$.*

Then there is a prehangar H for V such that every nontrivial plane in V that is contained in $V - H$ is nearly parallel in $\mathrm{cl}(V - H)$ *to a component of ∂H.*

If H is a prehangar for V and $\mathrm{cl}(V - H)$ is aplanar, then we say that H is a *hangar for V*.

We say that a set $\mathcal{H}$ of prehangars for V is a *hangar system for V* if for every squadron $P \subset V$, there is an $H_P \in \mathcal{H}$ such that P is isotopic in V into H_P.

If N and N' are both nearnodes with two faces or both polished blemishes, then we say that N and N' *are of the same type*.

Let V be a 3-manifold and let $M \subset V$ be a compact 3-manifold. Suppose that $S \subset V$ is a 3-manifold that is proper in V such that

1) S has a finite number of components,
2) ∂S is incompressible in V,
3) if S' is a component of S, then either S' is a nearnode with two faces or a blemish, and
4) if Q is a squadron in V such that each component of $Q \cap \mathrm{cl}(V - M)$ is a half open annulus that is incompressible in $\mathrm{cl}(V - M)$, then Q is isotopic in V into S,

then we say that S is *a strip for V rel M*. If each component of ∂S is a plane, then we shall refer to S as a *polished strip for V rel M*.

Let V be a noncompact 3-manifold that contains a compact 3-manifold L such that V is of finite genus $g \geq 2$ at infinity rel L. Suppose that for every compact 3-manifold $K \subset V$ there is a compact 3-manifold $M_K \subset V$ that is regular in V with respect to L for which there is a polished strip S for V rel M_K such that if Q is a plane in V that is nearly parallel in V to a component of ∂S, then Q is isotopic in V into S. Then we say that V is *pristine*.

We prove the following in Lemma 8.4

Lemma (The System Lemma) *Let V be an orientable, irreducible, connected, open, eventually end-irreducible 3-manifold with one end such that $\pi_1(V)$ is finitely generated. Suppose that L is a compact subset of V such that V is of at least genus $g \geq 2$ at infinity rel L and that V is pristine. Let $\mathcal{M}$ be the set of compact 3-manifolds in V that are regular in V with respect to L.*

1) *For every $M \in \mathcal{M}$, let $S(V, M)$ be a polished strip such that if Q is a plane in V that is nearly parallel in V to a component of $\partial S(V, M)$, then Q is isotopic in V into $S(V, M)$. There is a prehangar $H(V, M)$ for V such that*

$S(V,M) \subset H(V,M)$ and every component of $\partial H(M,V)$ is either a component of $\partial S(V,M)$ or is parallel to a component of $\partial S(V,M)$.

2) *If P is a squadron in V, then there is a compact 3-manifold M_P that is regular in V with respect to L such that P is isotopic in V into $H(V,M_P)$. Consequently, $\mathcal{H}(V) = \{H(V,M)|M \in \mathcal{M}\}$ is a hangar system for V.*

The following two results are Theorem 8.5 and Lemma 9.1 which combine to form the main result of this paper.

Theorem (The Hangar Theorem) *Let V be an orientable, irreducible, connected, open, pristine 3-manifold with finite genus $g \geq 2$ at infinity and a finitely generated fundamental group. Then there is a hangar H for V such that if P is a squadron in V, then P is isotopic in V into H. Furthermore, if G is any hangar for V, then G is isotopic in V to H.*

In Lemma 9.1 we prove the following

Lemma (The Strip Lemma) *Let V be an orientable, irreducible, connected, open, eventually end-irreducible 3-manifold such that $\pi_1(V)$ is finitely generated. Also assume that there is a compact 3-manifold L such that V is of finite genus $g \geq 2$ at infinity rel L. Suppose that $M \subset V$ is a 3-manifold with $L \subset M$ that is regular in V with respect to L.*

Then there is a strip S for V rel M such that if Q is a plane in V that is nearly parallel in V to a component of ∂S, then Q is isotopic in V into S.

We are now able to state and prove (modulo the above results) the following theorem which summarizes the results of Theorem 8.5 and Lemma 9.1.

Theorem 6.1. (Main Theorem) *Let V be an orientable, irreducible, connected 3-manifold of finite genus $g \geq 2$ at infinity and with finitely generated fundamental group. Then V is pristine if at least one of the following holds*

1) *Whenever F is a noncompact, connected 2-manifold such that the inclusion induced map $\pi_1(F) \to \pi_1(V)$ is injective, $\pi_1(F)$ is finitely generated, and there is a compact $K \subset V$ that traps F, then F is a plane.*
2) *There is no nontrivial subgroup of $\pi_1(V)$ that is free.*
3) *$\pi_1(V)$ is trivial.*

In any of these cases there is a hangar H for V such that if P is a squadron in V, then P is isotopic in V into H. Furthermore, if G is any hangar for V, then G is isotopic in V to H.

Proof. By Lemma 9.1, for every regular $M \subset V$ there is a strip S for V rel M such that if Q is a plane in V that is nearly parallel in V to a component of ∂S, then Q is isotopic in V into S. Each of the conditions 1,2, and 3 of the theorem ensure that any blemish component of S is a polished blemish. Therefore, V is pristine. The remainder follows from Theorem 8.5. ♠

7. Blemishes

The following beautiful characterization of missing boundary manifolds is due to Thomas Tucker [**11**]. It is stated here because it will be used in the proof of Lemma 7.2

Lemma 7.1. (Tucker) *Let M be a $\mathbf{P^2}$-irreducible 3-manifold that is connected. Then M is a missing boundary manifold iff for every compact $C \subset M$, each component of $M - C$ has finitely generated fundamental group.*

Lemma 7.2. *Let N be a noncompact, irreducible connected 3-manifold. Suppose that $P \subset N$ is a nontrivial plane. Let N' be obtained from N by splitting N along P. Then N is a (polished) blemish iff each component of N' is a (polished) blemish.*

Proof. We will prove the result for blemishes; the result for polished blemishes follows immediately. Let $\eta : N' \to N$ be the quotient map of the splitting. Since P is proper in N, it follows that η is a proper map. Suppose that $K' \subset N'$ is compact. Then $\eta(K')$ is compact. Let K be a compact, connected 3-manifold that traps P, and which contains $\eta(K')$ in its interior.

Since P is a proper plane, it follows that N is irreducible iff N' is irreducible. Henceforth, we will assume that N and N' are irreducible. By Lemma 1 of [**14**], there is a compact 3-manifold $M \subset N$ with $K \subset M - \mathrm{Fr}(M)$ such that $P \cap M$ is a single disk whose boundary is noncontractible in $\mathrm{Fr}(M)$. Therefore $\mathrm{cl}(P - M)$ is a half open annulus which is incompressible in $\mathrm{cl}(N - M)$.

Let N_0 and N_1 be the components of N'. (It may be that $N_0 = N_1 = N'$.) Let $M_i = \eta^{-1}(M) \cap N_i$ for $i = 0$ and 1. Note that $\mathrm{cl}(N - M)$ has exactly one noncompact component iff $\mathrm{cl}(N_i - M_i)$ has exactly one noncompact component for $i = 0$ and 1, so N has one end iff N_i has one end for $i = 0$ and 1, so, therefore, we may assume that N and each component of N' has one end.

Note that $\pi_1(N - M)$ is isomorphic to either

$$\pi_1(N' - \eta^{-1}(M)) *_{\pi_1(A)}$$

or

$$\pi_1(N_0 - M_0) *_{\pi_1(A)} \pi_1(N_1 - M_1)$$

depending upon whether or not N' is connected. By Theorems 25 and 31 of Chapter 1 of [**5**], we may deduce that $\pi_1(N - M)$ is finitely generated iff the fundamental group of each component of $N' - \eta^{-1}(M)$ is finitely generated.

Suppose that N is a blemish. Then $\pi_1(N - M)$ is finitely generated. Let $i =$ 0 or 1 be given. Then $\pi_1(N_i - M_i)$ is finitely generated. Since $\pi_1(M_i - K')$ is finitely generated, it follows that $\pi_1(N_i - K')$ is finitely generated by Van Kampen's Theorem. By Lemma 7.1 it follows that N' is a missing boundary manifold. Hence each component of N' is a blemish.

Suppose that each component of N' is a blemish. Then $\pi_1(N_i - M_i)$ is finitely generated for $i = 0$ and 1. Therefore $\pi_1(N - M)$ is finitely generated. Since $\pi_1(M - K)$ is finitely generated, it follows by Van Kampen's Theorem that $\pi_1(N - K)$ is finitely generated. Hence N is a missing boundary manifold. Therefore, N is a blemish. ♠

Lemma 7.3. *Now suppose that N^* is a connected, proper 3-submanifold of N such that $\partial N^* \subset N - \partial N$ and ∂N^* is a squadron in N. If N is a (polished) blemish, then N^* is a (polished) blemish.*

Proof. This follows directly from Lemma 7.2. ♠

8. Prehangars and Hangar Systems

Lemma 8.1. *Let V be a noncompact, open, irreducible 3-manifold. Suppose H and G are prehangars for V such that $\partial H \subset G - \partial G$. Suppose H is chosen in its isotopy class in V with respect to this condition so H contains the fewest components of* $\mathrm{cl}(V - G)$. *Then $H \subset G$.*

Proof. Suppose that H' is a component of H. Let $\Gamma = \mathrm{cl}(V - G)$. Since $\partial H \subset G - \partial G$, a component of Γ is either contained in H' or misses H'. Let $\Gamma' = \Gamma \cap H'$. Note Γ' consists of the components of Γ that are contained in H'. It follows by Lemmas 7.3 and 4.3 and the fact G is a prehangar that each component of Γ' is a node with two faces. Let $G' = G \cap H'$. Note that $H' = G' \cup \Gamma'$, and $H' \subset G$ iff $\Gamma' = \emptyset$.

It suffices to prove that $\Gamma' = \emptyset$. Suppose that $\Gamma' \neq \emptyset$. For every component P of $\partial H'$, let G_P be the component of G' that contains P.

Let us first suppose there is a component P of $\partial H'$ such that G_P is a node with two faces. Let $P' = \partial G_P - P$ and let N be the component of Γ' that contains P'. We may use $N \cup G_P$ to reduce the number of components of Γ contained in H' by an isotopy that is fixed off a neighborhood of $N \cup G_P$.

Now suppose there is no plane P such that G_P is a node with two faces. Then every component of G that meets H' must be of the same type. Let N be a component of Γ', and let G_0 and G_1 be the components of G that meet N. Since G_0 and G_1 meet H', they are of the same type. This contradicts the assumption that G is a prehangar. ♠

Lemma 8.2. *Let V be a noncompact, open, irreducible 3-manifold. Suppose G is a prehangar for V and that G' and G'' are distinct components of G. There is no isotopy h_t of G such that $h_1(G') \subset G'' - \partial G''$.*

Proof. Suppose that h_t is such an isotopy. By Lemmas 7.3 and 4.3, it follows that $h_1(G')$ and G'' are of the same type. Hence G' and G'' are of the same type.

Let P' be a component of $\partial G'$. By Theorem 5 of [**14**], it follows there is a parallelism N' in V between P' and $h_1(P')$. Since $h_1(G') \subset G'' - \partial G''$, there is a component P'' of $\partial G''$ which is contained in $N' - \partial N'$. By Lemmas 7.3 and 4.3, it follows that there is a parallelism N'' between P' and P''. By Lemmas 7.3 and 4.3 again, we may choose P' and P'' so that $N'' \cap (G' \cup G'') = P' \cap P''$.

If N'' is a component of $\mathrm{cl}(V - G)$, this contradicts the fact that G is a prehangar for V. On the other hand, if N'' is not a component of $\mathrm{cl}(V - G)$, then N'' must contain a component of $\mathrm{cl}(V - G)$ which would likewise produce a contradiction. ♠

By Lemma 5.2, we may assume that there is a maximal squadron P such that $V - P$ is connected, *i.e.* if P' is a squadron in V and $\sharp(P') > \sharp(P)$, then $V - P'$ is not connected.

Lemma 8.3. (The Near-Parallel Lemma) *Suppose that V is a noncompact, irreducible, orientable connected 3-manifold that has one end and $\pi_1(V)$ is of rank less than ρ. Assume that V has an exhaustion X such that ∂X_n is connected, of genus $g \geq 2$, and is incompressible in $V - X_0$ for $n \geq 1$. Assume that P is a squadron with the largest number of components such that $V - P$ is connected. Suppose that H is a prehangar for V such that $P \subset H - \partial H$.*

1) *Then $\sharp(H) \leq \rho + (2g-2)(2g+1) + 1$.*
2) *Suppose that $\sharp(H)$ is as large as possible for any prehangar H for V which contains P. Then every nontrivial plane in V that is contained in $V - H$ is nearly parallel in* cl$(V - H)$ *to a component of ∂H.*

Proof. To prove (1). Let N be the union of components of H that are nearnodes with two faces and let $C = \mathrm{cl}(H - N)$. By Lemma 5.1 and the fact that H is a prehangar, it follows that $\sharp(C) \leq 2g - 2$ and $\sharp(\partial C) \leq 2g(2g-2)$.

To get a contradiction, suppose

$$\begin{aligned} \sharp(H) &> \rho + (2g-2)(2g+1) + 1 \\ &= \rho + (2g-2) + 2g(2g-2) + 1. \end{aligned}$$

There are at least $(2g-2) + 2g(2g-2) + 1$ components of H that contain no component of P and so cl$(V - H)$ has at least $2g(2g-2)+1$ components that are nearnodes with two faces. By Lemma 5.1 at most $2g(2g-2)$ of these can meet a component of C. Therefore at least one component of cl$(V - H)$ is a nearnode with two faces that meets only components of N. This contradicts that H is a prehangar.

To prove part(2), suppose that Q is a nontrivial plane in V that is contained in $V - H$. Let N be a regular neighborhood of Q in $V - H$. Then N is a node with two faces. Since $\sharp(H)$ is maximal, it follows that either a component N' of cl$(V - (H \cup N))$ is a nearnode with two faces that is not a node or N' is a node with with two faces that meets components of H of the same type. Since H is a prehangar, one component of $\partial N'$ is a component of ∂N and the other is a component of ∂H. Let N'' be the closure of the component of $N - Q$ that meets N'. By Lemma 4.2, N'' is a nearnode with two faces. So by Lemma 4.2 $N' \cup N''$ is a nearnode with two faces. This ends the proof. ♠

Lemma 8.4. (The System Lemma) *Let V be an orientable, irreducible, connected, open, eventually end-irreducible 3-manifold with one end such that $\pi_1(V)$ is finitely generated. Suppose that L is a compact subset of V such that V is of at least genus $g \geq 2$ at infinity rel L and that V is pristine. Let $\mathcal{M}$ be the set of compact 3-manifolds in V that are regular in V with respect to L.*

1) *For every $M \in \mathcal{M}$, let $S(V, M)$ be a polished strip such that if Q is a plane in V that is nearly parallel in V to a component of $\partial S(V, M)$, then Q is isotopic in V into $S(V, M)$. Then there is a prehangar $H(V, M)$ for V such that*

$S(V,M) \subset H(V,M)$ and every component of $\partial H(V,M)$ is either a component of $\partial S(V,M)$ or is parallel to a component of $\partial S(V,M)$. Consequently, If Q is a nontrivial plane in $V-H$ that is nearly parallel in $\mathrm{cl}(H-V)$ to a component of $\partial H(V,M)$, then Q is isotopic in V into $H(V,M)$.

2) *If P is a squadron in V, then there is a compact 3-manifold $M_P \in \mathcal{M}$ such that P is isotopic in V into $H(V,M_P)$. Consequently, $\mathcal{H}(V) = \{H(V,M) | M \in \mathcal{M}\}$ is a hangar system for V.*

Proof. Let $M \in \mathcal{M}$ and let $S = S(V,M)$. We shall presently describe operations that will build S into a prehangar for V. Let $\mathcal{N}$ be the set of all components of $\mathrm{cl}(V-S)$ that are polished cysts or nearnodes with two faces. For each $N \in \mathcal{N}$, let $\psi(N) = \mathrm{cl}(N-U)$, where U is a regular neighborhood of ∂N in N. Let

$$\tilde{H} = S \cup \left(\bigcup_{N \in \mathcal{N}} \psi(N) \right).$$

Let $\mathcal{K}$ be the set of components of $\mathrm{cl}(V-\tilde{H})$. Let $\mathcal{L} \subset \mathcal{K}$. Put

$$H = \tilde{H} \cup \left(\bigcup_{L \in \mathcal{L}} L \right).$$

If each component of H is either a nearnode with two faces or a polished cyst, we say that H is an *amalgam* of $\tilde{H}$.

Let H be an amalgam of $\tilde{H}$ such that $\mathrm{cl}(V-H)$ has the fewest components.

Claim 8.4.1 H is a prehangar for V. Furthermore each component of ∂H is parallel in V to or equal to a component of ∂S.

Proof: One may easily check the first three parts of the definition of prehangar.

Suppose that N is a component of $\mathrm{cl}(V-H)$ that is a node with two faces and that H' and H'' are the components of H which contain the components of ∂N. If H' and H'' are of the same type, then either $H \cup N$ is a nearnode with one handle (this occurs only when $H' = H''$) or $H \cup N$ is an amalgam of $\tilde{H}$ with

$$\sharp(\mathrm{cl}(V-(H \cup N))) < \sharp(\mathrm{cl}(V-H)).$$

The latter case contradicts our assumption of minimality. On the other hand, if $H \cup N$ is a nearnode with one handle, then $V = H \cup N$ is of genus one at infinity which is also a contradiction.

To see that each component of ∂H is parallel in V or equal to a component of ∂S, one simply observes that each component of $\partial \tilde{H}$ is parallel (or equal) to a component of ∂S and $\partial H \subset \partial \tilde{H}$. ♣

Claim 8.4.2 If P is a squadron in V, then there is an $M_P \in \mathcal{M}$ such that if $M \in \mathcal{M}$ and $M_P \subset M - \partial M$, then P is isotopic in V into $H(V,M)$. Consequently $\mathcal{H}(V)$ is a hangar system for V.

Proof: By Lemma 2.3, it follows that for every compact $K \subset V$ there is an $M_K \in \mathcal{M}$ such that $K \subset M_K - \partial M_K$. By Lemma 2.1 there is an $M_P \in \mathcal{M}$ so that if $M_P \subset M - \partial M$ and $M \in \mathcal{M}$, then after an isotopy each component of $P \cap \mathrm{cl}(V-M)$ is a half open annulus. Therefore, again after an isotopy, $S(V,M)$ swallows the ends of P. Hence by Lemma 2.5, P is isotopic in V into $S(V,M)$. Therefore P is isotopic in V into $H(V,M)$. ♣

♠

Theorem 8.5. (The Hangar Theorem) *Let V be an orientable, irreducible, connected, pristine 3-manifold with finite genus $g \geq 2$ at infinity and finitely generated fundamental group. Then there is a hangar H for V such that if P is a squadron in V, then P is isotopic in V into H. Furthermore, if G is any hangar for V, then G is isotopic in V to H.*

Proof. By Lemma 2.2 of [**3**], it follows that V is eventually end-irreducible. Therefore by our own Lemma 8.4 there is a hangar system $\mathcal{H}(V)$ for V.

Let $\tilde{H}$ be a prehangar for V which contains a maximal nonseparating squadron and has the most components of any such prehangar. By Lemmas 8.1 and 8.4, there is an $H \in \mathcal{H}(V)$ such that $\tilde{H}$ is isotopic in V into H.

Claim 8.5.1 If P is a nontrivial plane in V contained in $V - H$, then P is parallel in $\mathrm{cl}(V - H)$ to a component of ∂H. Consequently, $\mathrm{cl}(V - H)$ is aplanar and H is a hangar.

Proof: Let P be such a nontrivial plane. By an isotopy, we may assume that $\tilde{H} \subset H - \partial H$. By Lemma 8.3, it follows that P is nearly parallel in V to a component $\tilde{P}$ of $\partial\tilde{H}$. Let $\tilde{N}$ be a near parallelism in V between P and $\tilde{P}$.

Since $\tilde{H} \subset H - \partial H$, it follows that $\tilde{N}$ contains a component Q of ∂H. By Lemma 4.2 the closure of each component of $\tilde{N} - Q$ is a nearnode with two faces. Let N be the near parallelism in V between Q and P. Note that Q may be chosen so that $N \subset \mathrm{cl}(V - H)$.

By Lemma 8.4, it follows that Q is parallel or equal to a component T of $\partial S(V, M)$, where $M \in \mathcal{M}$ is chosen so that $H = H(V, M)$. One may argue using Lemma 4.2 that there is a near parallelism L in V between P and T.

By Lemma 8.4, it follows that P is isotopic in V to a plane P' in $H - \partial H$. By Theorem 5 of [**14**], it follows that there is a parallelism in N' in V between P and P'. Let P'' be a component of ∂H that is contained in N'. By Lemma 4.2, there is a parallelism N'' in V between P and P''. Note that P'' can be chosen so that $N'' \cap H = P''$. This ends the proof. ♣

We shall now suppose that G is a prehangar for V such that $H \subset G - \partial G$.

Claim 8.5.2 If P is a nontrivial plane in V contained in $V - G$, then P is parallel in $\mathrm{cl}(V - G)$ to a component of ∂G.

Proof: By Claim 8.5.1 there is a parallelism N in V between P and a component Q of ∂H. Since $H \subset G - \partial G$, there is a component P' of ∂G which is contained in N. By Lemmas 7.2 and 4.2, there is a parallelism N' in V between P and P'. Note that P may be chosen so that $N' \subset \mathrm{cl}(V - G)$. ♣

Claim 8.5.3 There is an isotopy h_t of V such that $h_1(G) \subset H - \partial H$. Furthermore if G' is a component of G, then $h_1(G') \subset G'$.

Proof: By Claim 8.5.1, each component of ∂G is parallel in V to a component of ∂H. It is not difficult to construct an isotopy of V which takes ∂G into $H - \partial H$. Therefore by Lemma 8.1, it follows that G is isotopic in V into $H - \partial H$. Let h_t be this isotopy. Note that if G' is a component of G, it follows by Lemma 8.2 that $h_1(G') \subset G'$. ♣

Claim 8.5.4 G is isotopic in V to H.

Proof: By Claim 8.5.3 there is an isotopy h_t of V such that $h_1(G) \subset H - \partial H$ and such that, for each component G' of G, $h_1(G') \subset G'$.

We will first show that each component of G contains exactly one component of H. Let G' be a component of G and let H' be the component of H which contains $h_1(G')$. Then $H' \subset G'$ because $h_1(G') \subset G'$. Hence G' contains at least one component of H. Suppose that H'' is a component of H which is contained in G'. Then $h_1(H'') \subset h_1(G') \subset H'$. By Lemma 8.2 it follows that $H'' = H'$.

Let G' be a component of G and let H' be the unique component of H contained in G'. We claim that each component of $\mathrm{cl}(G' - H')$ is a parallelism in V between a component of $\partial G'$ and a component of $\partial H'$. Let P be a component of $\partial G'$. Then $h_1(P) \subset H' - \partial H'$. By Theorem 5 of [**14**] and Lemmas 7.2 and 4.2, there is a parallelism N_P in V between P and a component Q of $\partial H'$. We may choose Q so that $N_P \subset \mathrm{cl}(V - H')$. Since G is a prehangar, it can be argued that $N_P \subset \mathrm{cl}(G' - H')$; otherwise one can find components of G which are of the same type and joined by a node with two faces.

Given this, it is not difficult to see that G is isotopic in V to H. ♣

Claim 8.5.5 If P is a squadron in V, then P is isotopic in V into H.

Proof: Note that ∂H is a squadron in V. By Lemma 2.1, there is a regular M in V such that (perhaps after isotopies) each component of $P \cap \mathrm{cl}(V - M)$ and each component of $\partial H \cap \mathrm{cl}(V - M)$ is a half open annulus. Therefore P and ∂H are isotopic separately in V into some polished strip for V rel M, say $S(V, M)$. By Lemma 8.4, there is a $K \in \mathcal{H}(V)$ such that ∂H and P are isotopic into K. By Lemma 8.1, we may assume that $H \subset K - \partial K$. By Claim 8.5.4 it follows that H is isotopic in V to K. Therefore P is isotopic in V into H. ♣

Claim 8.5.6 Every hangar for V is isotopic in V to H.

Proof: Suppose that G is a hangar for V. Then ∂G is a squadron in V and so is isotopic in V into $H - \partial H$. By Lemma 8.1 we may assume that $G \subset H - \partial H$. Since $\mathrm{cl}(V - G)$ is aplanar, it follows that ∂H is isotopic in V into $G - \partial G$. Therefore by Lemma 8.1, it follows that H is isotopic in V into G. By Claim 8.5.4, it follows that G is isotopic in V to H. ♣

♠

9. The Strip for V rel M

The purpose of this section is to prove Lemma 9.1. The proof is rather complex and makes use of results concerning the weak characteristic pair of an end-irreducible 3-manifold from [**13**].

Lemma 9.1. (The Strip Lemma) *Let V be an orientable, irreducible, connected 3-manifold such that $\pi_1(V)$ is finitely generated. Also assume there is a compact 3-manifold L such that V is of finite genus $g \geq 2$ at infinity rel L. Suppose that $M \subset V$ is a 3-manifold with $L \subset M$ that is regular in V with respect to L.*

Then there is a strip S for V rel M such that if Q is a plane in V that is nearly parallel in V to a component of ∂S, then Q is isotopic in V into S.

Proof. We will now let $W = \mathrm{cl}(V - M)$. By Theorem 7.6 of [**13**], there is a Seifert pair $(\Sigma, \Phi) \subset (W, \partial W)$ such that $\mathrm{Fr}(\Sigma; W)$ is strongly essential in $(W, \partial W)$ and such that if A is a 2-manifold that is properly embedded in W each of whose components is a half open annulus that is strongly essential in $(W, \partial W)$, then A is isotopic in $(W, \partial W)$ into (Σ, Φ). Since ∂W is compact, a half open annulus is strongly essential in $(W, \partial W)$ iff it is properly embedded and incompressible in W.

Theorem 7.6 of [**13**] also considers a 2-manifold whose components are tori, annuli, and open annuli. However we will not need this strength in the sequel. Therefore we may assume that if (σ, ϕ) is a component of (Σ, Φ), then $\phi \neq \emptyset$ and σ is noncompact. We will refer to the language of [**13**].

Let (σ, ϕ) be a component of (Σ, Φ). Since $\phi \subset \partial M$, it follows that ϕ is compact. Therefore, since σ is noncompact, then (σ, ϕ) is not an I-pair. Since $\phi \neq \emptyset$, it follows that (σ, ϕ) is not an $\mathbf{R}$-pair. Therefore (σ, ϕ) is either an S^1-pair or a $[0, \infty)$-pair.

In the sequel, we will only be interested in the components of Σ that swallow the end of some plane that is nontrivial in V. Consequently, let us assume that if (σ, ϕ) is a component of (Σ, Φ), then there exists a plane P_σ that is nontrivial in V such that σ swallows the end of P_σ. We will now leave (Σ, Φ) fixed.

Using (Σ, Φ) we will construct a proper 3-submanifold $S = S(V, M)$ of V such that if Σ swallows the ends of the squadron P, then P is isotopic in V into S, and if S' is a component of S, then S' is either a nearnode with two faces or a blemish. This S will be a strip for V rel M.

For the rest of the proof, let (σ, ϕ) be a component of (Σ, Φ). Suppose that P is a plane that is nontrivial in V containing a disk D_P such that $P - D_P \subset \sigma$. Assume that $\sharp(D_P \cap \partial\sigma)$ is minimal for all isotopies of V that are fixed on $P - D_P$.

Since σ is either Seifert fibered or a $[0, \infty)$-bundle, it follows that σ is irreducible and end-irreducible.

The S^1-pair case. Let us now assume that (σ, ϕ) is an S^1-pair.

Claim 9.1.1 If U is a component of $\mathrm{cl}(V - \sigma)$, then either ∂U is incompressible in U or $M \subset U$.

Proof: Suppose that ∂U is compressible in U. Since $\mathrm{Fr}(\sigma; W)$ is incompressible in W, it follows that either U is not contained in W or ∂U is not contained in $\mathrm{Fr}(\sigma; W)$.

If U is not in W, then $M \subset U$, and we are done. Suppose ∂U is not contained in $\mathrm{Fr}(\sigma; W)$. Then ∂U meets both σ and M, but they are on opposite sides of ∂U. As U is a component of $\mathrm{cl}(V - \sigma)$, U cannot contain σ, so $\sigma \subset M$, and we are done. ♣

For the rest of the S^1-pair case, let U denote the component of $\mathrm{cl}(V - \sigma)$ which contains M.

Claim 9.1.2 ∂U is an open annulus which is compressible in U by a disk E whose boundary is a fiber in the Seifert fibration of σ.

Proof:We claim that $P \cap \partial\sigma \neq \emptyset$. Suppose otherwise. Then $P \subset \sigma$. Recall, however, that σ is irreducible and end-irreducible. By Lemma 3.1, it follows that P is trivial in σ and therefore in V which is a contradiction. Therefore $P \cap \partial\sigma \neq \emptyset$.

Let E be a disk in D_P such that ∂E is a component of $P \cap \partial\sigma$ and

$$(E - \partial E) \cap \sigma = \emptyset.$$

Since V is irreducible and $\sharp(D_P \cap \partial\sigma)$ is minimal by isotopies fixed on $P - D_P$, it is easy to argue that ∂E is noncontractible in $\partial\sigma$. Since $(E - \partial E) \cap \partial\sigma = \emptyset$, then either $E \subset \sigma$ or $E \subset \mathrm{cl}(V - \sigma)$.

We claim that $E \subset \mathrm{cl}(V - \sigma)$. In order to get a contradiction, suppose otherwise. Let $\sigma_0 \subset \sigma$ be a compact manifold which is a union of fibers of σ that contains E. Let F be the component of $\partial\sigma_0 \cap \partial\sigma$ that contains ∂E. Then either F is an annulus that is compressible in σ and σ_0, or F is an incompressible torus.

As E is a compressing disk for F, F must be a torus component of both $\partial\sigma_0$ and $\partial\sigma$. Therefore, σ_0 is a solid torus. Consequently, σ must be a solid torus. This contradicts that σ is noncompact. We must conclude that $E \subset \mathrm{cl}(V - \sigma)$.

Note $E \subset U$. We claim that ∂U is an open annulus. Otherwise ∂U must be a torus since σ is Seifert fibered and orientable. If ∂U were a torus, by Lemma 2.4, there is a compact 3-manifold $U' \subset W$ such that $\partial U' = \partial U$. Since σ and M are on opposite sides of ∂U, it follows that $\sigma \subset U'$ which is a contradiction because σ is proper in W. Therefore ∂U must be an open annulus.

Since ∂E is noncontractible in ∂U, it is isotopic in ∂U to a fiber in the Seifert fibration of σ. This isotopy can be extended to an isotopy of V that is fixed off of a regular neighborhood of ∂U in V. ♣

Let N be a regular neighborhood of E in U such that $N \cap \partial U$ is an annulus that is a union of fibers of σ.

Let $\mathcal{F}$ be the set of all noncompact components of $\partial\sigma - \partial U$. Let $\mathcal{T}$ be the set of compact components of $\partial\sigma - \partial U$. Then $\mathcal{T}$ is countable. For each $T \in \mathcal{T}$, let U_T be the component of $\mathrm{cl}(V - \sigma)$ which has $\partial U_T = T$. By Lemma 2.4 it follows that U_T is compact for each $T \in \mathcal{T}$. Let

$$X_\sigma = N \cup \sigma \cup \left(\bigcup_{T \in \mathcal{T}} U_T \right).$$

Then X_σ is proper in V. Note that ∂X_σ consists of the elements of $\mathcal{F}$ and two planes, say P_1 and P_2, that result from compressing ∂U with N.

Claim 9.1.3 Every compact subset of X_σ is contained in a closed 3-cell that meets P_i in a single disk for $i = 1$ and 2 and meets only a finite number of elements of $\mathcal{F}$ and each of those in a single annulus that is a union of fibers of σ.

Proof: Let Ω be the orbit manifold of σ and let $\eta : \sigma \to \Omega$ be the quotient map. We claim that Ω is planar. Otherwise Ω contains a simple closed curve J that does not separate Ω; hence $\eta^{-1}(J)$ is a torus which does not separate V. This gives us a contradiction by Lemma 2.4.

Suppose that C is a compact, connected subset of X_σ. We may assume that $N \subset C$. Let Ω_0 be a compact 2-manifold in Ω such that $\Omega_0 \cap \eta(\partial U)$ is a single arc and $\eta(C \cap \sigma) \subset \Omega_0$. We may assume that no component of $\mathrm{cl}(\Omega - \Omega_0)$ is compact and that $\Omega_0 - \mathrm{Fr}(\Omega_0; \Omega)$ contains every compact component of $\partial\Omega$ that Ω_0 meets. Hence if α is an arc component of $\mathrm{Fr}(\Omega_0; \Omega)$, then each point of $\partial\alpha$ is contained in a noncompact component of $\partial\Omega$. Observe that if λ is a noncompact component of $\partial\Omega$, then each component of $\Omega_0 \cap \lambda$ is an arc.

Let J be the component of $\partial\Omega_0$ that meets $\eta(\partial U)$. Let $\alpha = J \cap \eta(\partial U)$ and $\gamma = \mathrm{cl}(J - \alpha)$. Put $A = \eta^{-1}(\alpha)$ and $G = \eta^{-1}(\gamma)$. Then A is an annulus which contains $N \cap \partial U$ and G is an annulus which meets $\bigcup_{F \in \mathcal{F}} F$ in annuli each of which is a union of fibers of σ. Note that $G \cup \mathrm{cl}(A - N) \cup \mathrm{Fr}(N; U)$ is a 2-sphere which must bound a 3-cell B in V.

Since N and $\mathrm{cl}(\partial U - A)$ are on opposite sides of ∂B and $\mathrm{cl}(\partial U - A)$ is proper, it follows $N \subset B$.

Since C and N are on the same side of ∂B, it follows that $C \subset B$. Note that $B \cap P_i = (N \cup A) \cap P_i$ is a disk for $i = 1, 2$. This ends the proof. ♣

Recall $\mathcal{F}$ is the set of all noncompact components of $\partial\sigma - \partial U$. Then each element of $\mathcal{F}$ is an open annulus. For every $F \in \mathcal{F}$, there is an annulus A_F, properly embedded in σ (a union of fibers in fact), such that one component of ∂A_F is a fiber of σ in F, and the other is a fiber of σ in $N \cap \partial U$. We may construct the elements of $\{A_F | F \in \mathcal{F}\}$ so that $A_F \cap A_{F'} = \emptyset$ for $F \neq F'$.

For every $F \in \mathcal{F}$, let E_F^* be a disk that is properly embedded in N such that $\partial E_F^* = \partial A_F \cap \partial U$. It is clear that we may assume that $E_F^* \cap E_{F'}^* = \emptyset$ for $F \neq F'$. For every $F \in \mathcal{F}$, let $E_F = E_F^* \cup A_F$.

Given $F \in \mathcal{F}$, let $\tilde{U}_F$ be the component of $\mathrm{cl}(V - \sigma)$ which has $F = \partial \tilde{U}_F$. Let U_F be the union of $\tilde{U}_F$ and a regular neighborhood of E_F in $\sigma \cup N$. Note that ∂U_F has exactly two components each of which is a plane.

Let $\mathcal{F}'$ be the set of elements F of $\mathcal{F}$ such that U_F is a nearnode with two faces and let

$$\tilde{X}_\sigma = X_\sigma \cup \left(\bigcup_{F \in \mathcal{F}'} \tilde{U}_F \right).$$

It is easy to check, with the aid of Claim 9.1.3, every compact subset of $\tilde{X}_\sigma$ is contained in a closed ball in $\tilde{X}_\sigma$ that meets P_i in a single disk for $i = 1$ and 2 and meets only a finite number of elements of $\mathcal{F} - \mathcal{F}'$ and each of those in an annulus that is a union of fibers of σ.

Claim 9.1.4 $\mathcal{F} - \mathcal{F}'$ contains at most $2g - 2$ elements.

Proof: Suppose that $F_1, \dots, F_\nu$ are distinct elements of $\mathcal{F} - \mathcal{F}'$ and that $\nu > 2g - 2$. For $1 \leq i \leq \nu$, let U_i be the union of $\tilde{U}_{F_i}$ and a regular neighborhood of E_{F_i} in $\sigma \cup N$. Note that $U_1, \dots, U_\nu$ may be constructed to be pairwise disjoint. By Lemma 5.1, there is a k such that U_k is a nearnode with two faces. This is a contradiction. Consequently, $\nu \leq 2g - 2$. ♣

Note the set $\{E_F | F \in \mathcal{F} - \mathcal{F}'\}$ is pairwise disjoint. Let N' be a regular neighborhood of $\displaystyle\bigcup_{F \in \mathcal{F} - \mathcal{F}'} E_F$ in $\tilde{X}_\sigma$. For each $F \in \mathcal{F} - \mathcal{F}'$, let N_F be the component of N' that contains E_F. Then N_F meets F in an annulus whose core is noncontractible in F.

Let $S_\sigma = \mathrm{cl}(\tilde{X}_\sigma - N')$.

Claim 9.1.5 Each component of S_σ is a nearnode with two faces. Furthermore if σ swallows the end of a plane P', then S_σ swallows the end of P'.

Proof: Suppose that C is a compact, connected subset of S_σ. By Claim 9.1.3 there is a closed 3-cell $B \subset X_\sigma$ such that $N' \cup C \subset B - \mathrm{Fr}(B)$, such that $B \cap P_i$ is a disk for $i = 1, 2$, and each component of $B \cap \sigma$ is a union of fibers of σ for each $F \in \mathcal{F}$. Now $B \cap S_\sigma = \mathrm{cl}(B - N')$. Let S'_σ be the component of S_σ that contains C. Then $\partial S'_\sigma$ has two components each of which is a plane, and $B \cap S'_\sigma$ is a closed 3-cell which meets each component of $\partial S'_\sigma$ in a single disk. Therefore S'_σ is a nearnode with two faces.

Suppose that P' is a proper plane in V such that σ swallows the end of P'. Let $D \subset P'$ be a disk such that $P' - D \subset \sigma$. Since P' is proper, $P' \cap N'$ is compact and so there is a disk $D' \subset P'$ with $D \cup (N' \cap P') \subset D' - \partial D'$. So $P' - D' \subset S_\sigma$. ♣

The $[0,\infty)$**-pair case.** In this subsection, it will be assumed that (σ, ϕ) is a $[0,\infty)$-pair. By looking through the proof of Lemma 6.4 of [**13**], one can see that (σ, ϕ) is homeomorphic to $(\phi \times [0,\infty), \phi \times 0)$. Hence σ is a missing boundary manifold and each component of $\mathrm{cl}(\partial\sigma - \phi)$ is a half open annulus.

Let Y be the result of compressing σ completely in V, *i.e.* Y is obtained from σ by adding 2-handles and 3-handles and removing 1-handles so that the inclusion induced map $\pi_1(Y) \to \pi_1(V)$ is injective. Since $\pi_1(\partial\sigma)$ is finitely generated, this compression only requires a finite number of handle moves. Therefore Y is a missing boundary manifold. Of course ∂Y is incompressible in V. Let $Y_\sigma = Y$.

Claim 9.1.6 Y_σ is a missing boundary manifold, ∂Y_σ is incompressible in V, and if P' is a proper plane whose end is swallowed by σ, then Y_σ swallows the end of P'.

Proof: Suppose that P' is a proper plane in V such that σ swallows the end of P'. Let $D \subset P'$ be a disk such that $P' - D \subset \sigma$. Since P' is proper, there is a disk $D' \subset P'$ such that D' contains D and the intersection of P' with each of the finite number of handles used to obtain Y from σ. Then $P' - D' \subset Y_\sigma$. ♣

Let $\mathcal{T}$ be the set of compact components of ∂Y_σ. Let $T \in \mathcal{T}$. Since V has one end, there is a compact 3-manifold $U_T \subset V$ such that $\partial U_T = T$. Let

$$S_\sigma = Y_\sigma \cup \left(\bigcup_{T \in \mathcal{T}} U_T \right).$$

Therefore each component of ∂S_σ is noncompact, and S_σ is a missing boundary manifold.

Claim 9.1.7 S_σ is a blemish that is proper in V and such that $Y_\sigma \subset S_\sigma$ and $\mathrm{cl}(S_\sigma - Y_\sigma)$ is a compact 3-manifold.

Proof: Recall that there is a nontrivial plane $P \subset V$ such that σ swallows the end of P. Let D_P be a disk such that $P - D_P \subset Y_\sigma$. By Claim 9.1.6 and the fact that $Y_\sigma \subset S_\sigma$, it will do no harm to assume that $P - D_P \subset S_\sigma$. Isotop P in V by an isotopy fixed on $P - D_P$ so that $\sharp(P \cap \partial S_\sigma)$ is minimal. Since ∂S_σ is incompressible and since D_P is a disk, it follows by a minimality argument using the irreducibility of V that $P \cap \partial S_\sigma = \emptyset$. Since $P - D_P \subset S_\sigma$ and is fixed under the isotopy, it follows that $P \subset S_\sigma$. Therefore S_σ is a blemish. ♣

Back to the Mainline. Let $\sigma_1, \ldots, \sigma_\nu$ be the components of Σ. For $1 \le i \le \nu$, let $S_i = S_{\sigma_i}$. It would be nice to claim that

$$\bigcup_{i=1}^{\nu} S_i$$

is the strip for V rel M. This is in fact almost true. However, it may be that there exist $i \neq j$ such that $S_j \cap S_j \neq \emptyset$. We do claim that by judicious choices in the construction of these S_i that the set $\{S_i | 1 \leq i \leq \nu\}$ can be made pairwise disjoint.

Let $1 \leq i \leq \nu$ be given. Note that in each case S_i can be obtained from σ_i by compressing with 1- and 2-handles $H_1^i, \ldots, H_{m_i}^i$ to obtain an intermediate result σ_i' and then capping off the compact components of $\partial\sigma_i'$ with compact 3-manifolds to obtain S_i. So $\partial S_i \subset \partial\sigma_i'$.

Let $2 \leq i \leq \nu$ be given. Since ∂S_1 is incompressible, any 2-handles in $H_1^i, \ldots, H_{m_i}^i$ can be chosen to miss S_1. So $\partial S_1 \cap \partial\sigma_i' = \emptyset$. Since each component of $\mathrm{cl}(S_i - \sigma_i')$ is compact and S_1 is noncompact and proper in V, it is evident that $S_1 \subset V - S_i$, *i.e.* $S_1 \cap S_i = \emptyset$.

Therefore we may assume that $S_1 \cap S_i = \emptyset$ for $2 \leq i \leq \nu$. Note that ∂S_i is incompressible in $V - S_1$ for $2 \leq i \leq \nu$. Continuing in this way the set $\{S_i | 1 \leq i \leq \nu\}$ can be made pairwise disjoint. Let

$$S = \bigcup_{i=1}^{\nu} S_i.$$

Claim 9.1.8 If Q is a squadron in V and Σ swallows the ends of Q, then Q is isotopic in V into S.

Proof: This follows by Claims 9.1.5 and 9.1.6 and Lemma 2.5. ♣

Claim 9.1.9 If Q is a nontrivial plane in V that is nearly parallel in V to a component of ∂S, then Q isotopic in V into S.

Proof: By Claim 9.1.8, it suffices to show that Σ swallows the end of Q after an isotopy.

Let Q' be the component of ∂S to which Q is nearly parallel in V and let H be the near parallelism in V between Q and Q'. Since Q' is nontrivial, it follows from the construction of S that there is a disk $D' \subset Q'$ such that $Q' - D'$ is incompressible in Σ. Since H is proper in V, it follows that $H \cap M$ is compact. Therefore there is a 3-cell $B \subset H$ such that $B \cap Q$ and $B \cap Q'$ are both disks and $(H \cap M) \cup D' \subset B - \mathrm{Fr}(B; H)$.

Recall $W \subset \mathrm{cl}(V - M)$. Hence $\partial(B \cap Q')$ is contained in a half open annulus in $\mathrm{Fr}(\Sigma; W)$. Therefore there is an annulus $A' \subset \mathrm{Fr}(\Sigma; W)$ with one boundary component contained in ∂W and the other equal to $\partial(B \cap Q')$. Let $A'' = \mathrm{Fr}(B; H)$ and let $A = \mathrm{cl}(Q - B)$. Then A'' is an annulus, and A is a half open annulus.

Let $A^* = A \cup A' \cup A''$. Then A^* is a half open annulus. One can argue that A^* is incompressible in W by using the fact that $\mathrm{Fr}(\Sigma; W)$ is strongly essential in W. Therefore there is an isotopy h_t of $(W, \partial W)$ such that $h_1(A^*) \subset \Sigma$. We may extend h_t to an isotopy g_t of V which is fixed in M off of a regular neighborhood of ∂M. It follows that Σ swallows the end of $g_1(Q)$. This ends the proof. ♣

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