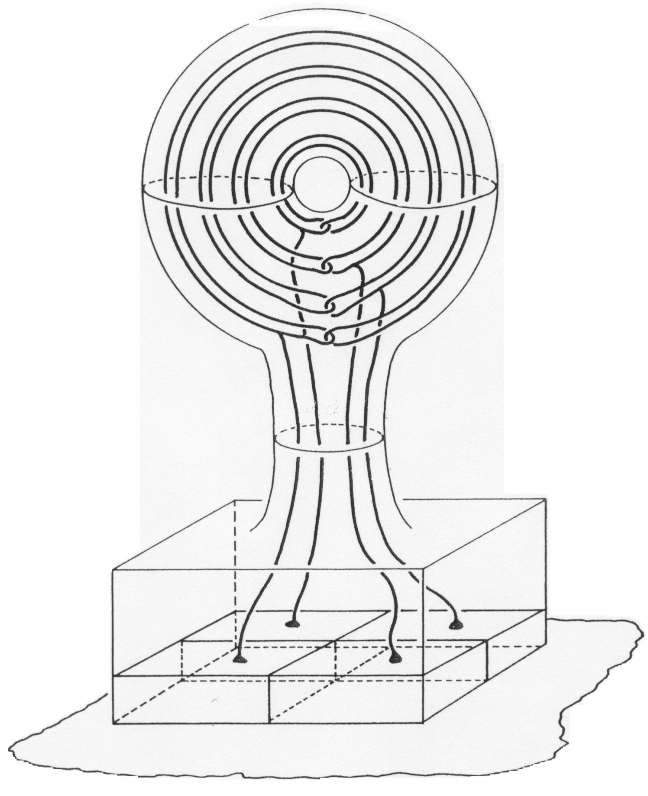
Proceedings

Nineteenth Annual Workshop in Geometric Topology



Hosted by Calvin College
June 13–15, 2002

Proceedings
of the
Nineteenth Annual
Workshop in Geometric Topology

Calvin College

Grand Rapids, Michigan

June 13–15, 2002

2000 *Mathematics Subject Classification.* 57-06

Dedicated to Bob Daverman
on the occasion of his 60th birthday

Preface

The workshop. The Nineteenth Annual Workshop in Geometric Topology was held at Calvin College, in Grand Rapids, Michigan, on June 13–15, 2002. A list of the participants may be found elsewhere in these proceedings.

The principal speaker for the workshop was Professor Alexander Dranishnikov of the University of Florida. Professor Dranishnikov presented a series of three one-hour lectures titled "Dimension Theory: local and global." A written transcript of those talks is included in these proceedings. In addition, Professor Jerzy Dydak of the University of Tennessee gave an invited one-hour talk on a related topic. His title was "The algebra of dimension theory."

As always, the workshop included a number of shorter contributed talks by participants and concluded with a problem session. Summaries of several of the contributed talks are included in these proceedings as is a summary of the problems discussed at the problem session.

The special session. Robert J. Daverman has been closely associated with the Workshops in Geometric Topology since their inception. He has also played an important role in the mathematical development of the individual organizers of the series. In recognition of Daverman's many contributions, and on the occasion of his 60th birthday, the organizers planned a special session in his honor. The special session was held after the conclusion of the regular workshop and took place on Saturday, June 15. The following mathematicians gave one-hour talks at the special session; John Bryant, James Cannon, Craig Guilbault, and William Jaco. There was a dinner in Bob's honor on Saturday evening. The dinner was held in the Hauenstein and Pfeiffer Rooms at the Frederik Meijer Gardens.

Support. The workshop received its primary financial support from the National Science Foundation under grant number DMS-0104325. In addition, Calvin College provided support for the workshop and the University of Tennessee supported the special session on Saturday.

Organizers. The workshops are organized by Fredric Ancel, University of Wisconsin-Milwaukee; Dennis Garity, Oregon State University; Craig Guilbault, University of Wisconsin-Milwaukee; Frederick Tinsley, Colorado College; Gerard Venema, Calvin College; and David Wright, Brigham Young University. The organizers serve as editors of these proceedings.

History of the Workshops in Geometric Topology

Year	Workshop Location	Principal Speaker
2003	Park City, Utah (BYU)	Martin Bridson
2002	Calvin College	Alexander Dranishnikov
2001	Oregon State University	Abigail Thompson
2000	Colorado College	Robert Gompf
1999	University of Wisconsin-Milwaukee	Robert Edwards
1998	Park City Utah (BYU)	Steve Ferry
1997	Oregon State University	James Cannon
1996	Colorado College	Michael Freedman
1995	University of Wisconsin-Milwaukee	Shmuel Weinberger
1994	Brigham Young University	Michael Davis
1993	Newport, Oregon (OSU)	John Bryant
1992	Colorado College	Mladen Bestvina
1991	University of Wisconsin-Milwaukee	Andrew Casson
1990	Oregon State University	Robert Daverman
1989	Brigham Young University	John Luecke
1988	Colorado College	John Hempel
1987	Oregon State University	Robert Edwards
1986	Colorado College	John Walsh
1985	Colorado College	Robert Daverman
1984	Brigham Young University	none

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Utah Valley State College

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Mercer Engineering Research Center

Bedenikovic, Tony
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Wright, David
Brigham Young University

Contents

ROBERT J. DAVERMAN:
A SHORT MATHEMATICAL TRIBUTE

Robert Jay Daverman was born on September 28, 1941 in Grand Rapids, Michigan. He earned a B.A. from Calvin College in 1963, after which he entered graduate school at the University of Wisconsin. In 1967 he was awarded a Ph.D. for his thesis *Locally Fenced 2-spheres in* S^3, written under the direction of R. H. Bing. Shortly thereafter he moved, with wife Lana and their children Kurt and Lara, to Knoxville to join the faculty at the University of Tennessee. There he quickly rose to the rank of Full Professor—a position he continues to hold today.

A world renowned expert in the topology of manifolds, Daverman has authored or co-authored more than a hundred original research articles. He co-edited *The Collected Papers of R. H. Bing*, and more recently the *Handbook of Geometric Topology.* Best known for his work in embedding and decomposition theory, Davermans book *Decompositions of Manifolds* has become "the bible" of that subject and may be found in research libraries and in offices of geometric topologists worldwide.

To his many friends in the field of geometric topology, "Bob" is best known for his energy, enthusiasm and generosity. Students at the University of Tennessee have benefited greatly from his willingness to offer courses and seminars on a remarkable range of topics. Among these students, eleven have earned Ph.D.s under his direction. Many other young topologists have adopted him as an unofficial mentor. The lively and collegial atmosphere fostered by Bob has attracted long- and short-term visitors from across the globe to Knoxville. His outgoing style of doing mathematics is illustrated by his publication list which contains nearly fifty collaborative papers written with no fewer than twenty-five different co-authors.

In recent years, Bob has expanded the reach of his work through his involvement in the American Mathematical Society. He served as Secretary of the Southeast Region from 1993 through 1999, after which he became the ninth Secretary of the AMS in the 114-year history of the organization. Here his broad view and unselfish attitude have served the greater mathematical community well. Through all of this, his own research has continued to thrive.

The past and present organizers of the summer Workshops in Geometric Topology wish to make a special acknowledgement of Bobs contributions to this series of conferences. It is very appropriate that an event commemorating Bob's 60th birthday is associated with the 19th Annual Workshop in Geometric Topology. Twice Bob has served as the principal speaker at a summer workshop. More importantly, his regular involvement with the workshop series has benefited all participants. His warmth, friendship and unfailing eagerness to do some real mathematics are to many of us the highlight of these annual gatherings.

Craig Guilbault
Milwaukee, May 2002

DIMENSION THEORY LOCAL AND GLOBAL

A. N. DRANISHNIKOV

Dedicated to Bob Daverman on the occasion of his 60th birthday.

These survey lectures are devoted to a new subject of the large scale dimension theory which was initiated by Gromov as a part of asymptotic geometry. We are going to enter the large scale world and consider some new concepts, results and examples which are parallel in many cases to the corresponding elements of the standard (local) dimension theory. We start our presentation with the motivations.

Lecture 1. MOTIVATONS and CONCEPTS

1.1. **Big picture of the Novikov Conjecture.** The Novikov Conjecture (NC) states that *the higher signatures of a manifold are homotopy invariant.* The *higher signatures* are the rational numbers of the type $\langle L(M) \cup \rho_M^*(x), [M]\rangle$, where $[M]$ is the fundamental class of a manifold M, L is the Hirzebruch class, $\Gamma = \pi_1(M)$, $\rho_M : M \to B\Gamma = K(\Gamma, 1)$ is a map classifying the universal cover of M and $x \in H^*(B\Gamma; \mathbb{Q})$ is a rational cohomology class. The name 'higher signature' is due to the Hirzebruch signature formula $\sigma(M) = \langle L(M), [M]\rangle$. It is known that the higher signatures are the only possible homotopy invariant characteristic numbers. It is convenient to formulate the NC for groups Γ instead of manifolds. We say that *the Novikov Conjecture holds for a discrete group* Γ *if it holds for all manifolds* M *(closed, orientable) with the fundamental group* $\pi_1(M) = \Gamma$. One of the reason for this is that the conjecture is verified for many large classes of groups. The other reason is that the Novikov Conjecture for the group can be reformulated in terms of the surgery exact sequence: *The rational Wall assembly map*

$$l_*^\Gamma : H_*(B\Gamma; \mathbb{Q}) \to L_*(\pi) \otimes \mathbb{Q}$$

is a monomorphism [Wa], [FRR],[KM].

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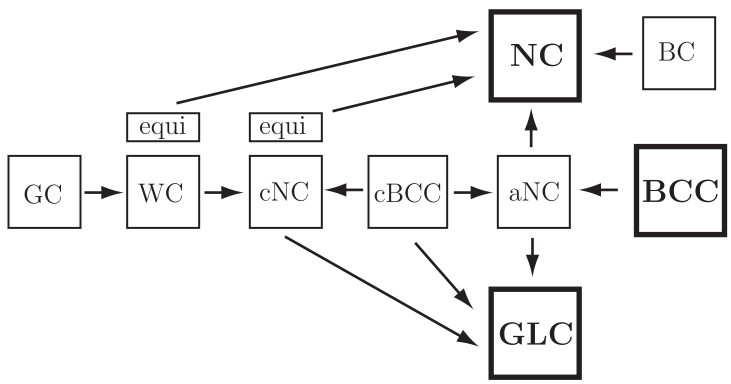
The case when $B\Gamma$ is a finite complex is of a particular interest here. In this case the group Γ taken as a metric space in the word metric is coarsely equivalent to the universal cover $E\Gamma$. This makes the methods of asymptotic geometry more natural. According to Davis' trick every finite aspherical complex K is a retract of a closed aspherical orientable manifold M [D]. Then the diagram formed by the surgery exact sequence and this retraction implies that if the NC holds for $\pi_1(M)$, then it holds for $\pi_1(K)$. Having that in mind from this moment we will stick to the case when $B\Gamma = M$ is a closed orientable manifold. Since M is aspherical, without loss of generality we may assume that the universal cover X of M is homeomorphic to a euclidean space.

There are several famous conjectures about aspherical manifolds. We arrange them in the following picture.



Here we assume that Γ is fixed and $B\Gamma = M$ is a closed manifold of dimension n. We note that almost all these conjectures are stated in more general form. With the above restriction they form this picture where every arrow is a theorem.

Below we give a brief description of the conjectures.

Borel Conjecture (BC). *Every homotopy equivalence between closed aspherical manifolds is homotopic to a homeomorphism.*

The arrow BC $\rightarrow$ NC follows from the surgery exact sequence [FRR].

Gromov-Lawson Conjecture (GLC). *A closed aspherical manifold cannot carry a metric of positive scalar curvature.*

The scalar curvature of an n-dimensional Riemannian manifold M at a point x can be defined up to a constant multiple as

$$\lim_{r\to 0} \frac{Vol\ B_r(\mathbb{R}^n, 0) - Vol\ B_r(M,x)}{r^{n+2}},$$

where $B_r(X,x)$ denotes the r-ball in a metric space X centered at x.

Analytic Novikov Conjecture (aNC). *The analytic assembly map $\mu : K_*(B\Gamma) \to K_*(C_r^*(\Gamma))$ is a monomorphism.*

Here $C_r^*(\Gamma)$ is the reduced C^*-algebra of a group Γ, i.e. the completion of the group ring $\mathbb{C}\Gamma$ in the space of bounded linear operators on the Hilbert space $l_2(\Gamma)$ of complex square summable functions on Γ. Proofs of arrows from aNC can be found in [Ros],[FRR],[Con],[Ro2]. We note that the original version of aNC (due to Mischenko and Kasparov) was slightly weaker and it used the maximal C^*-algebra $C_m^*(\Gamma)$ of the group Γ.

Baum-Connes Conjecture (BCC). *The analytic assembly map μ is an isomorphism.*

Coarse Baum-Connes Conjecture (cBCC). *The coarse index map $\mu : K_*^{lf}(X) \to K_*(C^*(X))$ is an isomorphism, where $X = E\Gamma$ and $C^*(X)$ is the Roe algebra* [Ro2].

The connection of cBCC with BCC is based on the facts that the K-theory homology group $K_*(B\Gamma)$ is a Γ-equivariant K-homology of X and the reduced C^*-algebra of Γ is Morita equivalent to the algebra $C^*(X)^\Gamma$ of fixed elements of $C^*(X)$ under the action of Γ. The arrow cBCC $\to$ cNC is trivial. The arrow cBCC $\to$ aNC can be found in [Ro2].

Coarse Novikov Conjecture (cNC). *The coarse index map μ is a monomorphism.*

The arrow cNC $\to$ GLC is proven in [Ro1]. Here we consider a coarse analog of the analytic Novikov conjecture. For the L-theoretic coarse Novikov conjecture we refer to [DFW1] and [J].

Equivariant cNC. *The coarse index map μ is a Γ-equivariant split monomorphism.*

A proof of the arrow equi-cNC $\to$ NC is contained in [Ro2]. We give more attention to the following two conjectures.

Weinberger Conjecture (WC). *Let $\bar{X} = X \cup \nu X$ be the Higson compactification of X. Then the boundary homomorphism $\delta : \check{H}^{n-1}(\nu X) \to H_c^n(X) = \mathbb{Z}$ in the exact sequence of the pair $(\bar{X}, \nu X)$ is an epimorphism.*

We recall that for a smooth manifold X the *Higson compactification* $\overline{X}$ can be defined as the closure of the image of X under the diagonal embedding $\Phi : X \to I^{C_h(X)}$ into the Tychonov cube defined by means of all smooth functions $\phi : X \to I = [0,1]$ whose gradient tends to 0 as x goes to infinity. The set of all such ϕ is denoted by $C_h(X)$. the remainder $\nu X = \overline{X} \setminus X$ of the Higson compactification is called *the Higson corona*. The arrow WC $\to$ cNC was established in [Ro1]. The Weinberger Conjecture has the rational version (when coefficients are rational). The rational WC implies the Gromov Conjecture (actually after a stabilization when n is odd) [Ro1], [DF] and hence the Gromov-Lawson conjecture. There is an equivariant version of WC which states that δ is Γ-equivariant split epimorphism for cohomology L-theory. Weinberger noted that the rational equivariant WC implies NC [DF].

Gromov Conjecture (GC). *The manifold $X = E\Gamma$ is hypereuclidean.*

Gromov called this a 'problem' rather than a 'conjecture'. We use here GC instead of GP to make the picture more homogeneous. We recall that an n-dimensional manifold X is called *hypereuclidean* if it admits a proper 1-Lipschitz map $p : X \to \mathbb{R}^n$ of degree one. A manifold X is called *rationally hypereuclidean* if there exists a map p as above with $deg(p) \neq 0$. The arrow *rational* GC $\to$ GLC was proved in [GL], [G3]. The arrow GC $\to$ WC was proved by Roe [Ro1].

We note that the stable version of GC implies the Gromov-Lawson Conjecture as well. Also in [G3] there was an announcement of the implication (*stable*) GC $\to$ NC. Previously it was known that the equivariant version of GC implies the Novikov Conjecture [CGM]. The equivariant version of GC states that $X = E\Gamma$ is equivariantly hypereuclidean. the latter means that there is a equivariant map $p : X \times X \to R^n \times X$ which is 1-Lipschitz and essential on every fiber. The main example here is the universal cover of a closed manifold of nonpositive curvature. Then the map p is defined by the formula $p(x,y) = ln_y(x)$ where $ln_x : X \to T_x$ is the inverse of the exponential map at $x \in X$.

Example. All conjectures hold true when $\Gamma = \mathbb{Z}^n$. Then $B\Gamma$ is the n-dimensional torus and $X = \mathbb{R}^n$. Even in this toy case some of the above conjectures are not obvious.

We conclude the motivation part by a theorem of G. Yu [Yu1] (see also [Yu2], [HR2], [H], and [STY]).

Theorem 1.1. *If the asymptotic dimension* $\operatorname{asdim}\Gamma$ *of a finitely presented group* Γ *taken as a metric space with the word metric is finite, then the cBCC, and hence the NC, holds for* Γ.

This theorem was extended to cover the integral versions of the L- and K-theoretic Novikov conjectures in [CG], [CFY], [Ba], [DFW2].

1.2. **Coarse category and coarse structures.** The coarse category was defined by Roe in [Ro1]. He starts with the category whose objects are proper metric spaces. The morphisms are coarsely uniform, metric proper maps. Here are the definitions. A metric space X is called *proper* if every closed ball $B_r(x)$ in X is compact. We recall that a map $f : X \to Y$ is called *proper* if the preimage $f^{-1}(C)$ is compact for every compact set C. Then a metric space X is proper if and only if the distance to any fixed point is a proper function on X. A map $f : X \to Y$ is called *metric proper* if the preimage $f^{-1}(C)$ is bounded for every bounded set $C \subset Y$. A map $f : X \to Y$ is *coarsely uniform* if there is a tending to infinity function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d_Y(f(x), f(x')) \le \rho(d_X(x, x'))$ for all $x, x' \in X$. We consider the following equivalence relation on morphisms. Two maps $f, g : X \to Y$ are coarsely equivalent (bornotopic in terminology of [Ro1]) if there is a constant D such that $d_Y(f(x), g(x)) < D$ for all x. The coarse category is the quotient of the above category under this equivalence relation on the morphisms. Two metric spaces X and Y are coarsely equivalent if there are two morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is coarsely equivalent to 1_X and $f \circ g$ is coarsely equivalent to 1_Y.

Example. $\mathbb{Z}$ is coarsely equivalent to $\mathbb{R}$ with the metric $d(x, y) = |x - y|$.

More generally, if $B\Gamma$ is a finite complex, then Γ is coarsely equivalent to $E\Gamma$. Here the metric on $E\Gamma$ is lifted from one on $B\Gamma$ and the group Γ is equipped with the word metric with respect to a finite set of generators. We recall that if $S = S^{-1}$ is a finite symmetric set of generators of a group Γ then the *word metric* d_S is defined as $d_S(x, y) = \|x^{-1}y\|_S$, where the S-norm $\|a\|_S$ of an element $a \in \Gamma$ is the shortest length of presentation of a in the alphabet S. We note that if S' is another finite symmetric generating set of Γ, then the metric spaces (Γ, d_S) and $(\Gamma, d_{S'})$ are coarsely equivalent.

We call a metric space X *ϵ-discrete* if $d_X(x, x') \ge \epsilon$ for all $x, x' \in X$, $x \ne x'$. We call it *discrete* if it is ϵ-discrete for some ϵ.

Proposition 1.2. *Every metric space X is coarsely equivalent to a discrete metric space.*

Proof. By transfinite induction one can construct a 1-discrete subset $S \subset X$ with the property $d_X(x, S) \le 1$ for all $x \in X$. The inclusion $S \subset X$ is a coarse equivalence whose inverse is any map $g : X \to S$ with the property $d(x, g(x)) \le d(x, S) + 1$. $\square$

We are going to study a coarse invariant dimension on metric spaces. Before giving the definitions we will sketch an approach to an extension of the coarse category beyond the metric spaces which is due to Higson and Roe [HR].

A set X is given a *coarse structure* if for every set S there is a fixed equivalence relation on the set of maps X^S called being *close* and satisfying the following axioms:

(1) If $p_1, p_2 : S \to X$ are close, then $p_1 \circ q$ and $p_2 \circ q$ are close for every $q : S' \to S$;
(2) If $p_1, p_2 : S \to X$ are close and $q_1, q_2 : S' \to X$ are close, then $p_1 \coprod q_1$ and $p_2 \coprod q_2$ are close maps of $S \coprod S'$ to X;
(3) any two constant maps are close.

A subset $C \subset X$ is called bounded (with respect to the coarse structure on X) if the inclusion map $i : C \to X$ is close to a constant map. A map $f : X \to Y$ between two coarse spaces is called *coarse proper* if the preimage of every bounded set is bounded. Then morphisms between coarse spaces are coarse proper maps $f : X \to Y$ satisfying the condition:

$$p_1, p_2 : S \to X \ \text{ are close } \ \Rightarrow \ f \circ p_1, f \circ p_2 : S \to Y \ \text{ are close.}$$

Examples. (1) When X is a metric space one sets for being close the property to be in a finite distance.

(2) If a locally compact topological space X is embedded in its compactification $\bar{X}$, one can define two maps $p_1, p_2 : S \to X$ to be close if for every subset $S' \subset S$ the corresponding limit sets coincide: $\overline{p_1(S')} \setminus X = \overline{p_2(S')} \setminus X$.

We denote by $\mathbb{R}^n_r$ the coarse structure on $\mathbb{R}^n$ defined by the radial compactification.

In parallel with the bounded and continuous control in controlled topology [FP], the coarse structure on X defined in (1) is called *bounded* and the coarse structure defined in (2) is called *continuous*.

Proposition 1.3. *The bounded coarse structure on a proper metric space X coincides with the continuous coarse structure generated by the Higson compactification.*

The proof can be easily derived from the following description of the Higson compactification. According to Smirnov's theorem every compactification on X is defined by some proximity (and vice versa). The Higson corona of X is defined by the proximity δ_X given by the condition $A\bar{\delta}_X B$ if and only if $\lim_{r\to\infty} d_X(A \setminus B_r(x_0), B \setminus B_r(x_0)) = \infty$. It means that the closures of diverging sets in X (and only them) do not intersect in the Higson corona.

Lecture 2. THEOREMS

2.1. **Definitions.** There are several equivalent definitions of dimension of compact metric spaces. The equivalence of corresponding coarse analogs for proper metric spaces in some cases is still an open question.

We recall the terminology. Let $\mathcal{U}$ denote an open cover of a metric space X. Then $ord(\mathcal{U})$ is the *order* of the cover, i.e. the maximal number of elements of $\mathcal{U}$ having nonempty intersection. The *mesh* of a cover $\mathcal{U}$, $mesh(\mathcal{U})$, is the maximal diameter of the elements of $\mathcal{U}$. The *Lebesgue number* of a cover $\mathcal{U}$ is defined as $L(\mathcal{U}) = \inf_{y \in Y} \sup_{U \in \mathcal{U}} d(y, Y \setminus U)$. A family $\mathcal{U}$ of subsets of X is called *uniformly bounded* if there is an upper bound on the diameter of its elements.

We consider the following comparison table:

Dimension $\dim X \leq n$	**Asymptotic dimension** $\operatorname{asdim} X \leq n$
(1) $\forall\, \mathcal{V}$, open cover of X, $\exists\, \mathcal{U}$, an open cover of X, with $ord(\mathcal{U}) \leq n+1$ and $\mathcal{U} \prec \mathcal{V}$.	(1) $\forall\, \mathcal{V}$, uniformly bounded cover of X, $\exists\, \mathcal{U}$, a uniformly bounded cover of X, with $ord(\mathcal{U}) \leq n+1$ and $\mathcal{V} \prec \mathcal{U}$.
(2) $\forall\ \epsilon > 0\ \exists\ \mathcal{U}^0, \dots, \mathcal{U}^n$, disjoint families of sets in X with $mesh(\mathcal{U}^i) < \epsilon$ such that $\cup_i \mathcal{U}^i$ is a cover of X.	(2) $\forall\ \lambda\ \exists$ uniformly bounded λ-disjoint families $\mathcal{U}^0, \dots, \mathcal{U}^n$ such that $\cup_i \mathcal{U}^i$ is a cover of X.
(3) $\forall\ \epsilon > 0\ \exists$ an ϵ-map $f : X \to K$ to an n-dimensional polyhedron K.	(3) $\forall\ \lambda\ \exists$ uniformly cobounded 1-Lipschitz map $f : X \to K$ to a uniform polyhedron K with $\dim K = n$ and $mesh(K) = \lambda$.
(4) X admits a Čech approximation by n-dimensional polyhedra.	(4) X admits an anti-Čech approximation by n-dimensional polyhedra.
(5) $\forall\ f : A \to S^n$, $A \subset_{Cl} X$, $\exists$ an extension $\bar{f} : X \to S^n$.	(5) $\forall\ f : A \to \mathbb{R}^{n+1}_r$, $A \subset_{Cl} X$, $\exists$ an extension $\bar{f} : X \to \mathbb{R}^{n+1}_r$.
(6) $\operatorname{Ind} X \leq n$	(6) $\operatorname{asInd} X \leq n$.

In the column on the left we have equivalent definitions of dimension for compact metric spaces. In the right column there are asymptotic counterparts. It is likely that they all are equivalent for metric spaces with bounded geometry.

We still owe some definitions for the asymptotic part of this table. A map $f : X \to Y$ between metric spaces is called *uniformly cobounded* if for every $R > 0$ the diameter of the preimage $f^{-1}(B_R(y))$ is uniformly bounded from above. A *Čech approximation* of a compact metric space X is a sequence of finite covers $\{\mathcal{U}_n\}$ such that, $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ for all n, and $\lim_{n\to\infty} mesh(\mathcal{U}_n) = 0$. An anti-Čech approximation [Ro1] of a metric space X is a sequence of uniformly bounded locally finite covers $\mathcal{U}_n$ such that $\mathcal{U}_n$ is a refinement of $\mathcal{U}_{n+1}$, and $\lim_{n\to\infty} L(\mathcal{U}_n) = \infty$. In both cases the approximation of metric space X is given by polyhedra which are nerves of corresponding covers. We say that a simplicial complex K is given a *uniform metric* of $mesh(K) = \lambda$, if it is realized as a subcomplex in the standard λ-simplex Δ_λ in the Hilbert space l_2

$$\Delta_\lambda = \{(x_i) \mid \sum x_i = \lambda,\ x_i \geq 0\}$$

and it's metric is induced from l_2.

In condition (5), $\mathbb{R}^{n+1}_r$ stands for the continuous coarse structure on $\mathbb{R}^{n+1}$ defined by the radial compactification. Since every coarse morphism $f : A \to \mathbb{R}^{n+1}_r$ defines a continuous map between coronas $f : \nu A \to S^n$ and vice versa, the asymptotic condition (5) (in view of the classical condition (5)) can be reformulated as follows:

(5′) $\dim \nu X \leq n$.

We note that the dimension dim of a nonmetrizible compact space can be defined by the condition (5).

Finally we recall the definition of inductive dimensions. A closed subset C of a topological space X is called a *separator* between disjoint subsets $A, B \subset X$ if $X \setminus C = U \cup V$, where U, V are open subsets in X, $U \cap V = \emptyset$, $A \subset U$, $V \subset B$. We set $\operatorname{Ind} \emptyset = -1$. Then $\operatorname{Ind} X \leq n$ if for every two disjoint closed sets $A, B \subset X$ there is a separator C with $\operatorname{Ind} C \leq n - 1$ [En].

It is known that the Higson corona is a functor from the category of proper metric spaces and coarse maps into the category of compact Hausdorff spaces and continuous maps. In particular, if $X \subset Y$, then $\nu X \subset \nu Y$. For any subset A of X we denote by A' its trace on νX, i. e. the intersection of the closure of A in $\bar{X}$ with νX. Obviously, the set A' coincides with the Higson corona νA. Let X be a proper metric space. Two sets A, B in a metric space are called *asymptotically disjoint* if the traces A', B' on νX are disjoint. A subset C of a metric space X is an *asymptotic separator* between asymptotically disjoint subsets $A, B \subset X$ if the trace C' is a separator in νX between A' and B'. By the definition, $\operatorname{asInd} X = -1$ if and only if X is bounded. Suppose we have defined the class of all proper metric spaces Y with $\operatorname{asInd} Y \leq n-1$. Then $\operatorname{asInd} X \leq n$ if and only if for every asymptotically disjoint subsets $A, B \subset X$ there exists an asymptotic separator C between A and B with $\operatorname{Ind} C \leq n-1$. The dimension functions asInd is called the *asymptotic inductive dimension* .

As it was mentioned, all conditions (1)–(6) in the left column are equivalent for compact metric spaces [HW],[En]. The condition (1) is Lebesgue's definition of dimension. The equivalence (1) $\Leftrightarrow$ (2) is a theorem of Ostrand. The equivalence of the conditions (3) and (5) to the inequality $\dim X \leq n$ is due to Alexandroff. The equivalence $\dim X \leq n \Leftrightarrow$ (4) is called the Froudenthal theorem.

In the column on the right Gromov proved the equivalence of conditions (1),(2),(3) and (4) [Gr1] (see [BD2] for details). These conditions give a definition of the asymptotic dimension asdim. In [Dr1] it was shown that the condition (5′) is equivalent to the inequality $\operatorname{asdim} X \leq n$ provided $\operatorname{asdim} X < \infty$. Under the same condition the equality $\operatorname{asInd} X = \operatorname{asdim} X$ was proven in [Dr7], [DZ]. Here we exclude the case of bounded X. We note that there are implications (1) $\Rightarrow$ (5′) [DKU], (1) $\Rightarrow$ (6) [DZ]. The status of the remaining implications is unknown.

Examples.
(1) $\operatorname{asdim} \mathbb{Z} = 1$;
(2) $\operatorname{asdim} \mathbb{R}^n = n$ [DKU];
(3) $\operatorname{asdim} T = 1$ where T is a tree (with the natural metric).

We note that all asymptotic conditions (1)–(6) are coarse invariant. All of them can be stated in the setting of general coarse structures. To do that one needs a notion of a uniformly bounded family of sets in a general coarse space. A family $\mathcal{U}$ in a coarse space X is *uniformly bounded* if the maps $p_1, p_2 : S \to X$ are close, where $S = \cup_{U \in \mathcal{U}} U \times U \subset X \times X$ and $p_1, p_2 : X \times X \to X$ are the projections onto the first and the second factors respectively.

2.2. **Embedding Theorems and Applications.** A coarse morphism $f : X \to Y$ is a *coarse embedding* if there the inverse morphism defined for $f : X \to f(X)$, i.e. a morphism $g : f(X) \to X$ such that $g \circ f$ and 1_X are close in X and $f \circ g$ and $1_{f(X)}$ are close in $f(X)$ with respect to the induced coarse structure. If a coarse morphism f is injective in the set theoretic sense, then it is a coarse embedding if and only f^{-1} is a coarse morphism. In our metric setting a map $f : X \to Y$ is a coarse embedding if there are tending to infinity functions $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$

such that

$$\rho_1(d_X(x,x')) \leq d_Y(f(x),f(x')) \leq \rho_2(d_X(x,x')).$$

We recall that a metric space (X, d_X) is called *geodesic* if for every pair of its points x and y there is an isometric embedding of the interval $[0, d(x,y)]$ into X with the end points x and y. Clearly for a geodesic metric space X the function ρ_2 can be taken linear. Thus, up to a rescaling, a coarse embedding of a geodesic metric space is an 1-Lipschitz map.

The question about embeddings into nicer spaces in the coarse category is very important for applications. In [Yu2] Goulang Yu proved the Novikov Conjecture for groups Γ that admit a coarse imbedding in the Hilbert space (see aloso [H] and [STY]). It was noticed in [HR2] that a metric space with finite asymptotic dimension is coarsely imbeddable in l_2. Thus this theorem of Yu implies his Theorem 1.1.

In a geometric approach to Theorem 1.1 the need for a coarse analog of the classical Nobeling-Pontryagin embedding theorem arose. We recall that the classical Nobeling-Pontryagin embedding theorem states that every compactum X of dimension $\dim X \leq n$ can be embedded in $\mathbb{R}^{2n+1}$. It is easy to see that this statement does not have a direct asymptotic analog. Indeed, a binary tree being asymptotically 1-dimensional cannot be coarsely embedded in R^N for any N because the tree has an exponential volume growth function and a euclidean space has only the polynomial volume growth. Moreover, we show in [DZ] that there is no metric space of bounded geometry that contains in a coarse sense all asymptotically n-dimensional metric spaces of bounded geometry. Here the bounded geometry condition serves as an asymptotic analog of compactness.

We recall that the *ϵ-capacity* $c_\epsilon(W)$ of a subset $W \subset X$ of a metric space X is the maximal cardinality of ϵ-discrete set in W. A metric space X has *bounded geometry* if there are $\epsilon > 0$ and a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that $c_\epsilon(B_r(x)) \leq c(r)$ for all $x \in X$. Finitely generated groups give us one of the main sources of examples of metric spaces of bounded geometry.

Neveretheless in asymptotic topology there is an embedding theorem which turnes out to be sufficient for the purpose of Theorem 1.1.

Theorem 2.1 ([Dr4]). *Every metric space of bounded geometry X with* $\operatorname{asdim} X \leq n$ *can be coarsely embedded in a* $2n+2$*-dimensional manifold of nonpositive curvature.*

The proof is based on the following embedding theorem.

Theorem 2.2 ([Dr4]). *Every metric space of bounded geometry X with* $\operatorname{asdim} X \leq n$ *can be coarsely embedded in the product of $n+1$ locally finite trees.*

In the classical dimension theory there is a theorem [Bow] analogous to Theorem 2.2 which states that an n-dimensional compact metric space can be imbedded in the product of $n+1$ dendrits (= 1-dimensional AR).

We recall that in the classical dimension theory for every n there is the universal Menger compactum μ^n which is n-dimensional and contains a copy of every n-dimensional compactum. As we mentioned, there is no similar object in the coarse category for $n > 0$ [DZ]. Using an embedding $X \to \prod T_i$ into the product of trees as in Theorem 2.2 we built a coarse analog of the Menger space $M(\{T_i\})$, $\operatorname{asdim} M(\{T_i\}) = n$, out of this product and get an embedding of X into $M(\{T_i\})$. This construction leads to the universal space for asdim $\leq n$ but we lose the bounded geometry condition.

For $n = 0$ a universal object with bounded geometry does exist. It is a literal generalization of the Cantor set: M^0 is the subset of all reals that do not use 2 in their ternary expansion. The classical Cantor set is $M^0 \cap [0, 1]$.

We proved a stable version of the Gromov Conjecture (see §1) for a group Γ with $\operatorname{asdim} \Gamma < \infty$.

Theorem 2.3 ([Dr2]). *Let M be a closed aspherical manifold with* $\operatorname{asdim} \pi_1(M) < \infty$ *and let X be its universal cover. Then the manifold $X \times \mathbb{R}^m$ is hypereuclidean for some m.*

A weaker theorem states that $X \times \mathbb{R}^m$ is integrally hyperspherical [Dr4]. This theorem enables us to prove the GLC. There is a relatively short proof of this which is based on the Theorem 2.1. We recall that an n-dimensional manifold Y is integrally hyperspherical [GL] if for arbitrary large r there is an n-submanifold with boundary $V_r \subset Y$ and an 1-Lipschitz degree one map $p_r : (V_r, \partial V_r) \to (B_r(0), \partial B_r(0))$ to the euclidean ball of radius r. If X is embedded in a k-dimensional nonpositively curved manifold W^k, the R-sphere $S_R(x_0)$ in X for large enough R is linked with a manifold M which has a sufficiently large tubular neighborhood N in W^k also linked with $S_R(x_0)$ and with an 1-Lipschitz trivialization $\pi : N \to B_r(0)$. Then we take a general position intersection $X \cap N$ as V_r and the restriction $\pi|_{V_r}$ as p_r. Crossing with $\mathbb{R}^m$ helps to achieve the above properties of the tubular neighborhood N.

When Gromov defined the asymptotic dimension [G1] he already suggested to consider the asymptotic behavior of some natural functions that appeared in the definition as legitimate asymptotic invariants of dimension type. Here we consider one of such functions defined as

$$asd_X(\lambda) = \min\{ord(\mathcal{U}) \mid L(\mathcal{U}) \geq \lambda\} - 1,$$

where $\mathcal{U}$ is a uniformly bounded cover of X. We note that taking the limit gives the equality:

$$\lim_{\lambda\to\infty} asd_X(\lambda) = \operatorname{asdim} X.$$

So we will refer to the function $asd_X(\lambda)$ as to the asymptotic dimension of X in the case when $\operatorname{asdim} X = \infty$. Clearly, for a space of bounded geometry the function $asd_X(\lambda)$ is at most exponential. The following is a generalization of the theorem of Yu (Theorem 1.1).

Theorem 2.4 ([Dr5], [Dr8]). *If $asd_\Gamma(\lambda)$ has the polynomial growth, then the Novikov Conjecture holds for Γ.*

This theorem holds for all finitely presented groups Γ. In contrast with Theorem 2.3, the proof here relies heavily on the results of [Yu2], [STY], and [H].

2.3. **Finite dimensionality theorems.** Finite dimensionality results for groups are important for the application to the Novikov Conjecture. The first finite dimensionality result in the asymptotic dimension theory is due to Gromov who proved that $\operatorname{asdim}\Gamma < \infty$ for hyperbolic groups [Gr1], [Ro3]. Then we proved in [DJ] that $\operatorname{asdim}\Gamma < \infty$ for all Coxeter groups. In [BD1] we proved that the asymptotic finite dimensionality is preserved by the amalgamated product and by the HNN extension. We gave a general estimate.

Theorem 2.5 ([BD2]). *Suppose that Γ is the fundamental group of a finite graph of groups with all vertex groups G_v having $\operatorname{asdim} G_v \leq n$. Then $\operatorname{asdim}\Gamma \leq n+1$.*

A *graph of groups* is a graph in which every vertex v and every edge e have assigned group G_v and G_e such that for the endpoints $e^\pm$ of e there are fixed monomorphisms $\phi_{e^\pm} : G_e \to G_{e^\pm}$. The fundamental group of a graph of groups can be viewed as the fundamental group of a complex built out of the mapping cylinders of the maps between Eilenberg-Maclane complexes $f_{e^\pm} : K(G_e, 1) \to K(G_{e^\pm}, 1)$ defined by the homomorphisms $\phi_{e^\pm}$. Clearly, this is a generalization of the amalgamated product and the HNN extension which correspond to the graphs with one edge.

By Bass-Serre theory the fundamental groups of graphs of groups are exactly the groups acting on trees (without inversion). We used this action to obtain our estimate. We proved the following theorem.

Theorem 2.6 ([BD2]). *Suppose that a group Γ acts by isometries on a metric space X with* $\operatorname{asdim} X \leq k$ *in such a way that for every r, the r-stabilizer $W_r(x_0)$ of a fixed point $x_0 \in X$ has* $\operatorname{asdim} W_r(x_0) \leq n$. *Then* $\operatorname{asdim} \Gamma \leq n + k$.

We define the *r-stabilizer* $W_r(x_0)$ as the set

$$\{g \in \Gamma \mid d_X(g(x_0), x_0) \leq r\}.$$

Thus, to prove Theorem 2.5 it suffices to show that $\operatorname{asdim} W_r(x_0) \leq n$ for the Serre action of the group Γ on a tree. It is not an easy task by any means even in the simplest case of the free product of groups. the difficulties were overcome by further development of the asymptotic dimension theory. We proved the following union theorem.

Theorem 2.7 ([BD1]).

(1) *Suppose $X = A \cup B$ is a metric space. Then* $\operatorname{asdim} X \leq \max\{\operatorname{asdim} A, \operatorname{asdim} B\}$;
(2) *Suppose $X = \cup_i A_i$ is a metric space and let* $\operatorname{asdim} A_i \leq n$ *for all i. Then* $\operatorname{asdim} X \leq n$ *provided the following condition is satisfied: $\forall r\ \exists\ Y_r \subset X$ with* $\operatorname{asdim} Y_r \leq n$ *such that the family of sets $\{A_i \setminus Y_r\}$ is r-disjoint.*

We note that these union theorems differ from their classical analogs.

Using the asymptotic inductive dimension asInd we managed to get an exact formula in the case of the nondegenerate amalgamated product.

Theorem 2.8 ([BDK]). *There is a formula*

$$asdim A * B = \max\{asdim A, asdim B, 1\}$$

for finitely generated groups A and B.

For the amalgamated product, the best what we have is the inequality [BD3]

$$asdim A *_C B \leq \max\{asdim A/C, asdim B/C, asdim C + 1\}.$$

Lecture 3. COUNTEREXAMPLES

3.1. **Coarse Alexandroff Problem.** We recall that the classical Alexandroff problem was about coincidence of the integral cohomological dimension of a compact metric space with its dimension. Since the 1930s the problem was reduced to the question whether there is an infinite dimensional compactum with a finite cohomological dimension. The problem was solved negatively [Dr6]. We recall that the cohomological dimension of X is defined in terms of Čech cohomology as the maximal number n such that the relative cohomology group is non-trivial, $\check{H}(X, A; \mathbb{Z}) \neq 0$, for some closed subset $A \subset X$. The Čech cohomology is defined by means of a Čech approximation of a compactum X (or of a pair (X, A)) and the ordinary (simplicial) cohomology. Similarly one can define the anti-Čech homology called *coarse homology* of a metric space (or pair) [Ro2], [Dr1] by means of an anti-vCech approximation of a metric space X (or of a pair (X, A)) and the simplicial homology with infinite chains. Roe denoted the coarse homology as HX_* [Ro1]. Then using the coarse homology one can define an asymptotic homological dimension in a similar fashion: $\operatorname{asdim}_{\mathbb{Z}} X = \max\{n | HX_n(X, A) \neq 0, A \subset_{Cl} X\}$. The homology is more preferable here than the cohomology since the latter involves the $\lim^1$ term. By analogy we can pose the coarse version of Alexandroff problem:

Coarse Alexandroff Problem. *Does there exist an asymptotically infinite dimensional metric space with a finite asymptotic homological dimension?*

In view of Yu's theorem (Theorem 1.1), it is not difficult to show (see [Dr1]) that the negative answer to this problem implies the Novikov Conjecture for the groups Γ with $B\Gamma$ a finite complex.

The following was the first counterexample to the coarse Alexandroff Problem, though it appeared as a counterexample to a general version of the Gromov Conjecture (GC) as well as to a preliminary version of the coarse Baum-Connes conjecture [Ro1].

Counterexample 3.1 ([DFW1]). *There exists a uniformly contractible Riemannian metric on $\mathbb{R}^8$ which gives a metric space X with* $\operatorname{asdim} X = \infty$ *and with the asymptotic homological dimension equal to 8.*

In our paper we proved that X is not stably hypereuclidean. This already implies that $\operatorname{asdim} X = \infty$. Higson and Roe proved [HR1] that for uniformly contractible spaces the coarse homology coincides with the locally finite homology. This gives us the required estimate for asymptotic homological dimension.

We recall that a metric space X is *uniformly contractible* if there is a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that every r-ball $B_r(x)$ in X can be contracted to a point in $B_{\rho(r)}(x)$. We note that the universal cover of a closed aspherical manifold is always uniformly contractible. This let Gromov to pose his conjecture GC for all uniformly contractible manifolds [G2]. The counterexample 3.1 disproves GC in the general setting but not the rational GC. The construction of it is based on a (dimension raising) cell-like map of a 7-dimensional sphere which has non-zero kernel in the homology K-theory. Since rationally a cell-like map is always an isomorphism, our approach did not touch the rational GC.

The drawback of this counterexample is that X is not a $E\Gamma$ and moreover X does not have bounded geometry.

Recently Gromov came with a better example.

Counterexample 3.2 ([G5]). *There is a closed aspherical manifold M with* $\operatorname{asdim} \pi_1(M) = \infty$.

We note that the universal cover X of M, as well as the fundamental group $\pi_1(M)$, has the asymptotic homological dimension $\operatorname{asdim}_{\mathbb{Z}} X$ equal to the dimension of M (=4 in the most recent version). Gromov's construction is based on use of expander. He constructed his manifold M with $\pi_1(M)$ containing an expander in a coarse sense. Then the equality $\operatorname{asdim} \pi_1(M) = \infty$ follow (see the next section).

3.2. **Expanders.** Let (V, E) be a finite graph with the vertex set V and the edge set E. We denote the cardinalities $|V|$ and $\|E\|$ by n and m. Let $l_2(V)$ and $l_2(E)$ denote complex vector spaces generated by V and E. We view an element of $l_2(V)$ as a function $f : V \to \mathbb{C}$. We fix an orientation on E and define the differential $d : l_2(V) \to l_2(E)$ as $(df)(e) = f(e^+) - f(e^-)$. The operator d is represented by $m \times n$ matrix D. We define the Laplace operator $\Delta = D^*D$ where D^* is the transpose of D. It is an easy exercise to show that Δ does not depend on orientation on E. By the definition the operator Δ is self-adjoint. Also it is positive: $\langle \Delta f, f \rangle = \langle Df, Df \rangle \geq 0$. Therefore Δ has real nonnegative eigenvalues. We denote by $\lambda_1(V)$ the minimal positive eigenvalue of the laplacian on the graph V.

Definition. A sequence of graphs (V_n, E_n) of a fixed valency d and with $|V_n| \to \infty$ is called an **expander** (or expanding sequence of graphs) if there a positive constant c such that $\lambda_1(V_n) \geq c$ for all n.

The last condition on the graphs is equivalent to the following [Lu]: *there is a constant $c_0 > 0$ such that $|\partial A| \geq c_0|A|$ for all subsets $A \subset V_n$ with $|A| \leq |V_n|/2$.*

Here the boundary of A in a graph V is defined as

$$\partial A = \{x \in V \mid dist(x, A) = 1\}.$$

It is easy to prove that the solutions of the Laplace equation $\Delta f = 0$ are exactly constant functions. The orthogonal space to the constants we denote by $l_2^0(V) = \{f \mid \sum_{v \in V} f(v) = 0\}$. We consider the restriction

Δ to $l_2^0(V)$. Let $\{v_i\}$ be a orthonormal basis of eigenvectors in $l_2^0(V)$ and let $f = \sum \alpha_i v_i$. Then

$$\frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} = \frac{\langle \sum \lambda_i \alpha_i v_i, \sum \alpha_i v_i\rangle}{\langle \sum \alpha_i v_i, \sum \alpha_i v_i\rangle} = \frac{\sum \lambda_i \alpha_i^2}{\sum \alpha_i^2} \geq \frac{\sum \lambda_1 \alpha_i^2}{\sum \alpha_i^2} = \lambda_1.$$

We apply the above inequality to a real-valued function $f : V \to \mathbb{R}$, $f \in l_2^0(V)$ to obtain the following inequality:

$$\lambda_1 \leq \frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} = \frac{\langle df, df\rangle}{\langle f, f\rangle} = \frac{\sum_E |f(e^+) - f(e^-)|^2}{\sum_V |f(x)|^2}.$$

We rearrange this inequality into the inequality $\lambda_1 \sum_V |f(x)|^2 \leq \sum_E |f(e^+) - f(e^-)|^2$. Clearly this inequality holds for any function $f : V \to l_2$ to a Hilbert space such that $\sum_V f(v) = 0$:

$$\lambda_1 \sum_V \|f(x)\|^2 \leq \sum_E \|f(x) - f(y)\|^2.$$

Since $m = |E| = dn/2$, we can change the above inequality into the following

$$\lambda_1 \frac{1}{|V|} \sum_V \|f(x)\|^2 \leq \frac{d}{2|E|} \sum_E \|f(x) - f(y)\|^2.$$

On the right we have $d/2$ times average of squares of lengths of the images under f of edges in the graph. Applying this inequality to an 1-Lipschitz map and using the estimate $\lambda_{\geq} c$ we obtain the following.

Proposition 3.3. *Let $f_n : V_n \to l_2$ be a sequence of 1-Lipschitz maps of an expander to a Hilbert space. Then*

$$\frac{1}{|V|} \sum_V \|f_n(x)\|^2 \leq \frac{d}{2c}$$

for all n.

Corollary 3.4. *If $K > \sqrt{d/c}$, then for maps $f_n : V_n \to l_2$ as above there is the inequality $|\{x \in V_n \mid \|f_n(x)\| \leq K\}| > |V_n|/2$ for all n.*

Proof. Assume the contrary. Then we have a contradiction

$$\frac{d}{2c} \geq \frac{1}{|V|} \sum_V \|f_n(x)\|^2 \geq \frac{1}{|V|} K^2 \frac{|V|}{2} > \frac{d}{2c}.$$

□

Nice groups cannot contain (in the coarse sense) an expander. We proved the following

Theorem 3.5 ([Dr3])**.** *Suppose that the universal cover X of a closed aspherical manifold is equivarinatly hypereuclidean, then X does not contain an expander.*

Corollary 3.6. *Gromov's example (Counterexample 3.2) is a counterexample to the equivariant Gromov Conjecture (equi-GC) and to the equivariant Weinberger Conjecture (equi-WC).*

Here we give a proof of a weaker statement which is due to Gromov and Higson. Namely we show that

A contractible Riemannian manifold with a nonpositive sectional curvature does not contain an expander.

We note that in view of this result Theorem 2.1 implies that every space containing an expander has infinite asymptotic dimension.

We present Higson's argument here.

Proof. Let $\dim X = m$ and let $\{V_n\}$ be an expander that lies in X. By Hadamard theorem the exponent $\exp_x : T_x \to X$ is a diffeomorphism for every $x \in X$. We note that the inverse map $\log_x : X \to T_x = \mathbb{R}^m$ is 1-Lipschitz.

First we show that for every n there is a point y_n such that

$$\sum_{x \in V_n} \log_{y_n}(x) = 0.$$

Assume the contrary $w_y = \sum_{x\in V_n} \log_y(x) \neq 0$ for all $y \in X$. Then the vector $-w_y$ defines a point $s_y \in S(\infty)$ in the visual sphere at infinity $S(\infty)$ of a manifold X. It is not difficult to check that the correspondence $y \to s_y$ defines a continuous map $f : X \to S(\infty)$ which is a retraction of the topological m-ball $X \cup S(\infty)$ to its boundary. This is a contradiction.

We take K as in Corollary 3.4. Then

$$|(\log_{y_n})^{-1}(B_K(0)) \cap V_n| > \frac{|V_n|}{2}.$$

Since $(\log_{y_n})^{-1}(B_K(0)) = \exp_{y_n}(B_K(0)) = B^X_K(y_n)$, where the latter is the K-ball in X, we have an estimate

$$2d^{2K} \geq 1+d+\cdots+d^{2K} \geq |B^{V_n}_{2K}(v)| = |B^X_{2K}(v)\cap V_n| \geq |B^X_K(y_n)\cap V_n| > \frac{|V_n|}{2}$$

for any $v \in B^X_K(y_n)\cap V_n$. This gives a contradiction with $|V_n| \to \infty$. □

In conclusion we note that the Novikov Conjecture holds true for this Gromov's group.

Another remark is that the Higson corona of an expander, considered metrically as a garland of finite graphs attached to a half-line, might produce by means of factorization dimensionally exotic metric compacta. It could give a clue to some long standing problems in infinite dimensional dimension theory.

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TOPOLOGICAL SINGULARITIES IN COSMOLOGY

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Introduction. By definition a space-time is a smooth four-dimensional manifold X admitting a Lorentzian metric g whose curvature tensors satisfy the Einstein field equations for some "reasonable" distribution of matter and energy.

On the other hand, the singularity theorems of Penrose and Hawking [8] assert that any such space-time must contain singular points. In other words, it can't be a smooth manifold with metric g defined at every point.

I suppose that the logical conclusion is that the universe cannot exist. Yet somehow God was able to overcome this difficulty [5]. Can we?

The best-publicized attempt is due to Hawking and Hartle [7]. The Hawking-Hartle "no boundary" theory has been popularized as a theory of "imaginary time." A better description of the geometry of the model is given by saying that in a small neighborhood of the "big bang" the metric changes signature, becoming positive definite locally. The physical interpretation is that at the moment of creation none of the four directions in space-time had yet been distinguished as "time."

The original model for this construction was the closed positively curved model characterized physically by the condition that the total mass-energy content of the cosmos is greater than the "critical value" ($\Omega > 1$). Although the no boundary concept has been extended to the flat and negatively curved standard models now favored by extragalactic observations [9], the positively curved model remains the most successful in revealing the geometry of space-time near the big bang singularity.

The reason for this is clear. In the standard closed model the space-like cross sections (all of space at a particular moment in time) are three-spheres. The entirety of space-time up to the present is viewed as an expanding family of such spheres originating in a "sphere of radius zero" at the moment of creation. Topologically, that is to say, the history of the cosmos so far is the cone on S^3. Since the cone on S^3 is just the ordinary four-disc, topologically there is nothing to distinguish the moment of the big bang from any other point of space-time. Thus the singularity at the beginning of time, whose existence is guaranteed by the Penrose-Hawking theorems, is "merely" geometrical and physical, not topological. Curvature tensors associated with the metric diverge to infinity, as does the mass-energy density. But the background structure of space-time maintains its integrity as a topological manifold even at the singular point.

Alas, the real world intrudes into our theorizing. Beginning in 1998 data from deep space studies using space-based telescopes and large array imaging techniques have effectively ruled out the positively curved model in favor of negatively curved models and (most popular currently) flat models with substantial cosmological constant (dark energy) [15]. At the same time there has been an explosion of interest in cosmological models whose space-like cross sections are not simply connected [11]. Indeed, physicists have not presented any reasons for preferring simply connected models except a vague feeling that such models are "simpler" than multiply connected ones.

In this note we present a family of topological spaces in which the requirement of simple connectivity is weakened to the condition that the "space-like" submanifolds are homology 3-spheres. These spaces have the feature that their geometric properties are underlain by exotic topological structure at the singular point.

The construction. Let $S_1, \ldots, S_k$ be a collection of 2-spheres, let Γ be an acyclic graph on vertices $v_1, \ldots, v_k$, and let $w_1, \ldots, w_k$ be integer "weights" assigned to the vertices. Denote by $E_i, i = 1, \ldots, k$, the total space of the 2-plane bundle on S_i with Euler number $-w_i$. Plumb these spaces together according to the prescription of the graph Γ. That is, locally identify the zero section of E_i with a fiber of E_j, and vice versa, whenever v_i meets v_j in Γ. Let M denote the compact three-manifold obtained by taking the union of the plumbed unit circle bundles of the E_i's, and smoothing the corners. Finally, let X be the space obtained by collapsing the zero sections to a point P. The resulting space X is homeomorphic to the cone on M, and is a smooth four-manifold except at the singular point P.

Theorem ([2])**.** *For $\Gamma, w_1, \ldots, w_k$ as above, denote by $A(\Gamma)$ the "dual intersection matrix"* $\mathrm{diag}(w_1, \ldots, w_k)$ *— adjacency matrix of Γ. Then the 3-fold M of the construction is a homology 3-sphere if and only if the determinant of $A(\Gamma) = \pm 1$.*

Moreover, if $A(\Gamma)$ is positive definite, then X admits the structure of a two-(complex)-dimensional complex algebraic variety, with a unique singular point at the origin [6]. Since the germ of the variety at the singular point determines the topology of the entire space, such singular complex surfaces provide an interesting setting in which to study the relations between the topology and the geometry, hence the physics, of big bang models in cosmology, with the compact homology 3-spheres M playing the role of the space-like submanifolds in space-time. The central question motivating this inquiry is this:

Guiding question: To what extent are the geometrical and physical properties of big bang space-time models determined by the topology of the singular point?

Egyptian fractions. One way to obtain particular examples of such spaces is as follows.

Theorem ([3])**.** *Let $n_1, \ldots, n_k$ be a solution in positive integers to one of the two unit fraction Diophantine equations*

$$(*) \qquad \Sigma 1/n_i = 1 \pm 1/\Pi n_i.$$

In the "minus" case, let Γ be the star graph with a central vertex of weight $w_0 = 1$, and with k arms of length 1, with weight $w_i = n_i$ on the single vertex of the ith arm. In the "plus" case, we take Γ to be the star-shaped graph whose central vertex has weight $w_0 = k-1$ and whose ith arm consists of $k-1$ vertices, each of weight 2. Then the 3-fold $M = M(\Gamma, n_1, \ldots, n_k)$ of the construction outlined above is a homology 3-sphere.

This raises a question in number theory, which is interesting in its own right and which enjoys a distinguished history dating back 4000 years to dynastic Egypt [4, 16, 17]: For fixed k, find all solutions in positive integers $n_1, \ldots, n_k$ to the equations $(*)$. Not only is this a fun and instructive problem, but also it is one that undergraduate students can understand and tackle. With motivation from the geometry of complex surfaces and the possible relevance of this topic to cosmological models, the Wayne State Undergraduate Research Group ("Surge" – the W in the acronym is silent) has attacked this problem with great vigor. After several semesters of work by of a total of 33 students involved in the program, the students, much to my pride and joy, succeeded in producing the complete list of all solutions through $k = 8$. There are 160 solutions to the minus equation and 598 solutions to the plus equation in this range [10, 12].

Examples. The most intensively studied example is the equation

$$1/2 + 1/3 + 1/5 = 1 + 1/30.$$

The corresponding weighted graph Γ is the Dynkin diagram (Coxeter graph) E_8 of the root system of the simple complex Lie algebra e_8.

The associated complex surface singularity is the rational double point given in complex co-ordinates x, y, z on $\mathbb{C}^3$ as the zero set

(a) $\{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$

The compact 3-fold M of the construction of this paper for this weighted graph is homeomorphic to the Poincaré 120-cell [14]. The first homotopy group is the group of rigid motions of the dodecahedron. This is a well-known finite perfect group of order 120; its Abelianization is trivial, hence M is a homology 3-sphere as required.

The simply connected covering space in this example is S^3. In fact, M is obtained by a tiling of S^3 by 120 "twisted" dodecahedra. Thus M inherits a homogeneous, isotropic line element $d\sigma$ of constant positive curvature from S^3. As above, let X denote the cone on M, and define a metric on X by $ds^2 = -t^{4/3}dt^2 + d\sigma^2$. The resultant space-time model, via the Einstein field equations, satisfies the physical requirements of spatially homogeneous distribution of matter, decreasing in density proportionally to t^{-2} from an infinitely dense big bang singularity. This model is indistinguishable locally from the matter-dominated "dust" model in standard cohomology. In principle its validity could be verified by the discovery of "ghosts"—multiple sightings of the same galaxy cluster in different directions—or by analyses of distinctive patterns of inhomogeneities in the cosmic microwave background radiation. Serious experiments are underway by astronomers seeking to detect just such heavenly anomalies (mostly working in the context of the 3-torus model), but so far without success [20].

In [1] I gave the details of a similar treatment of the complex hypersurface

(b) $\{x^2 + y^3 + z^6 = 0\} \subset \mathbb{C}^3$.

Since 1, 2, and 6, do not satisfy either of the relations $(*)$ we do not obtain a homology 3-sphere by the construction of this paper. However, if we intersect this complex variety X with a 5-sphere in $\mathbb{C}^3$, the intersection is a smooth compact 3-fold M and X is locally the cone on M [13]. Thus the topological space X is a candidate for a big bang space-time model.

This 3-fold M turns out to be homeomorpic to a non-trivial S^1-bundle on the 2-torus, with $H_1(M, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Furthermore, M admits naturally a metric that extends to a Robertson-Walker metric on the cone X and which is homogeneous on the space-like cross sections [18]. The metric is not, however, fully isotropic; the physical result is a tiny amount of universal pressure in the direction of the fiber of M, regarded as a circle bundle on T^2. See [1] for the details of the geometry and physical interpretation of this model.) This space-time begins in a singular point of infinite density and pressure, expands to a maximum size, and then contracts symmetrically to a "big crunch." Indeed, the "size" $R(t)$ of the universe at time t is given by the inverse relation

$$(**) \qquad t/(2C) = \arcsin \sqrt{R/C} - \sqrt{(R/C)(1 - R/C)}$$

where C is a constant of integration representing the maximum size of the universe at the end of the expansion phase.

Open question: Are these physical properties of the model determined by the topology at the singular point, or do they vary with choice of metric?

To complete this cycle of ideas, consider the complex variety

(c) $\{x^2 + y^3 + z^7 = 0\} \subset \mathbb{C}^3$.

Since $1/2 + 1/3 + 1/7 = 1 - 1/(2 * 3 * 7)$ (the "minus" version of equation $(*)$), we obtain a very inviting topological space X, the cone on a homology 3-sphere M, whose singularity at the origin is very well understood by algebraic geometers [20]. The fundamental group is presented by generators $\alpha, \beta, \gamma, \omega$, with relations $\alpha^2 = \beta^3 = \gamma^7 = \alpha\beta\gamma = \omega$. This group is an infinite perfect group that is a non-trivial central extension by $\mathbb{Z}$ of the group of symmetries of the tiling of the Poincaré disc by triangles with angles $\pi/2, \pi/3$, and $\pi/7$.

Open question: Does there exist a homogeneous Lorentzian metric on this singular space-time candidate, which exhibits realistic physical properties?

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REFLECTIONS ON THE BING-BORSUK CONJECTURE

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The question as to whether a homogeneous euclidean neighborhood retract (ENR) is a topological manifold goes back, at least, to the paper by Bing and Borsuk [2] in which they show that an n-dimensional homogeneous ENR is a topological manifold when $n < 3$. In this paper they discuss the question as to whether the result holds in higher dimensions and suggest that, at the least, homogeneous ENR's should be generalized manifolds (i.e., ENR homology manifolds). One of the main conjectures in [6] is that a generalized n-manifold, $n \geq 5$, satisfying the disjoint disks property is homogeneous. Thus, the spaces constructed in [6] may provide examples of homogeneous ENR's that are not topological manifolds. Another possible example was constructed by Jakobsche in [11] in dimension 3, assuming the Poincaré conjecture is false. Our first attempt to show that a homogeneous ENR is a homology manifold [5] succeeded at the expense of imposing the condition that the local homology groups of the space are finitely generated in all dimensions. This result was, in fact, already to be found in [4]. More specifically, the following theorem is known:

Theorem 1 ([4, 5])**.** *If X is an n-dimensional, homogeneous* ENR*, and $H_k(X, X - x; \mathbb{Z})$ is finitely generated for some* (*and, hence, all*) *x, then X is a homology manifold.*

In this talk we discuss attempts to prove the conjecture of Bing and Borsuk:

Conjecture 1. *If X is an n-dimensional, homogeneous* ENR*, then X is a homology n-manifold.*

Related to this conjecture is an older conjecture of Borsuk [3].

Conjecture 2. *There is no finite dimensional, compact, absolute retract.*

Definitions. A **homology n-manifold** is a space X having the property that for each $x \in X$,

Key words and phrases. generalized manifolds, homogeneity.

$$H_k(X, X-x;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=n \\ 0 & k \neq n. \end{cases}$$

A **euclidean neighborhood retract** (ENR) is a space homeomorphic to a closed subset of euclidean space that is a retract of some neighborhood of itself. A topological space X is **homogeneous** if, for any two points x and y in X, there is a homeomorphism of X onto itself taking x to y.

We will assume from now on that X is a n-dimensional homogeneous ENR and R is a PID. It's easy to get started:

Lemma 1. *For all $x \in X$, $H_0(X, X-x;R) = 0$, if $n > 0$ and $H_1(X, X-x;R) = 0$, if $n > 1$.*

One of the main problems that arises is the possibility that for some (and hence, all) $x \in X$, $H_k(X, X-x;\mathbb{Z})$ is infinitely generated for some $k \geq 2$. This difficulty could be overcome for $k < n$, if k-dimensional homology classes are carried by k-dimensional subsets of X. There are counterexamples for k-dimensional *homotopy* classes when $k \geq 2$ [7, 10], but I know of no counterexamples for carriers of homology classes.

Via Alexander duality, mapping cylinder neighborhoods provide an alternative way to view the local homology groups of X. Assume X is nicely embedded in $\mathbb{R}^{n+m}$, for some $m \geq 3$, so that X has a mapping cylinder neighborhood $N = C_\phi$ of a map $\phi\colon \partial N \to X$, with mapping cylinder projection $\pi\colon N \to X$ [12, 13]. Given a subset $A \subseteq X$, let $A^* = \pi^{-1}(A)$ and $\dot{A} = \phi^{-1}(A)$.

By a result of Daverman-Husch [8], the Bing-Borsuk Conjecture is equivalent to

Conjecture 3. *$\pi\colon N \to X$ is an approximate fibration.*

Duality shows that the local homology of X is captured in the cohomology of the fibers of this map (in the dual dimensions).

Lemma 2. *If A is a closed subset of X, then $H_k(X, X-A;R) \cong \check{H}_c^{n+m-k}(A^*, \dot{A};R)$.*

Proof. Suppose A is closed in X. Since $\pi\colon N \to X$ is a proper homotopy equivalence,

$$H_k(X, X-A;R) \cong H_k(N, N-A^*;R).$$

Since ∂N is collared in N,

$$H_k(N, N-A^*;R) \cong H_k(\mathrm{int}N, \mathrm{int}N - A^*;R),$$

and by Alexander duality,

$$H_k(\mathrm{int}N, \mathrm{int}N - A^*; R) \cong \check{H}_c^{n+m-k}(A^* - \dot{A}; R)$$

$$\cong \check{H}_c^{n+m-k}(A^*, \dot{A}; R)$$

(since $\dot{A}$ is also collared in A^*). □

Lemma 3. $H_k(X, X - x; R) = \varinjlim H_k^{\ell f}(U; R)$, *where the limit is taken over open neighborhoods U of x.*

Proof. Again, using Lemma 2 and the fact that π is proper, we have, for each neighborhood U of x in X,

$$H_k^{\ell f}(U; R) \cong H_k^{\ell f}(U^*; R) \cong$$

$$H^{n+m-k}(U^*, \dot{U}; R) \to \check{H}^{n+m-k}(x^*, \dot{x}; R) \cong H_k(X, X - x; R).$$

□

As the next lemma shows, homogeniety, specifically microhomogeneity, implies that any finitely generated submodule of the local homology module $H_k(X, X - x; R)$ propagates naturally to all points near x.

Lemma 4. *Suppose F is a finitely generated submodule of $H_k(X, X - x; R)$, $k \geq 0$. Then there is a neighborhood U of x and a submodule $F_0 \subseteq H_k(X, X - U; R)$ such that*

(i) $F_0 = \mathrm{im}\, F$ *under inclusion,*
(ii) *for all $y \in U$, the inclusion $H_k(X, X - U; R) \to H_k(X, X - y; R)$ is one-to-one on F_0.*

Proof. Given finitely generated $F \subseteq H_k(X, X - x; R)$.

Let $a_1, \ldots, a_r$ be generators of F, represented by singular chains $c_1, \ldots, c_r$, respectively, and let $B_1, \ldots, B_r$ be the carriers of $\partial c_1, \ldots, \partial c_r$, respectively. $B_1 \cup \ldots \cup B_r$ is a compact set in $X - x$, and there is a neighborhood U_1 of x such that for every smaller neighborhood V of x,

$$F \subseteq \mathrm{im}(H_k(X, X - V; R) \to H_k(X, X - x; R).$$

By Effros Theorem [9, 1], homogeneity implies micro-homogeneity: Given $\epsilon > 0$ there is a $\delta > 0$, such that if $d(x, y) < \delta$, then there is a homeomorphism $h_y \colon X \to X$ such that $h_y(x) = y$ and h_y moves no point of $(B_1 \cup \ldots \cup B_r)$ more than ϵ.

For ϵ small, h_y is homotopic to the identity on X by a homotopy whose restriction to $(B_1 \cup \ldots \cup B_r)$ has image in $X - x$, hence, in $X - U$ for some neighborhood U of x. □

The Leray spectral sequence of the Leray sheaf $\mathcal{H}^q(\pi)$ of $\pi\colon N \to X$, with stalk $\mathcal{H}^q(\pi)_x = \check{H}^q(x^*, \dot{x}; R)$, has E_2-term

$$E_2^{p,q} = H_c^p(X; \mathcal{H}^q(f)),$$

and converges to

$$E_\infty^{p,q} = H_c^{p+q}(N, \partial N; R).$$

In [5] it is proved that the Bing-Borsuk Conjecture is equivalent to

Conjecture 4. *For all q, $\mathcal{H}^q(\pi)$ is locally constant.*

Theorem 2. *If R is a PID, then $H_n(X, X - x; R) \neq 0$. Moreover, if U is a sufficiently small neighborhood of x, $H_c^n(U; R) \neq 0$, and $H_n^{\ell f}(U; R) \neq 0$ and free.*

Proof. Since U is an ENR of dimension n, the locally finite homology of U can be computed from a chain complex (using nerves of sufficiently fine covers of U of order $n+1$) that is 0 in dimension $n+1$; hence, $H_n^{\ell f}(U; R)$ is free. Thus, $H_c^n(U; R) = 0$ implies $H_n^{\ell f}(U; R) = 0$. If $H_n^{\ell f}(U; R) = 0$ for every neighborhood U of x, then $\check{H}^m(x^*, \dot{x}; R) \cong H_n(X, X - x; R) = \varinjlim H_n^{\ell f}(U; R) = 0$, so that $\mathcal{H}^m$ is the 0 sheaf.

Restrict the map π to $(U^*, \dot{U})$, where U is an open neighborhood of x. By definition,

$$E_3^{n,q} = \ker(d_2\colon E_2^{n,q} \to E_2^{n+2,q-1})/\operatorname{im}(d_2\colon E_2^{n-2,q+1} \to E_2^{n,q}).$$

Since $\dim U = n$ implies $E_2^{n+2,q-1} = 0$, so that $E_2^{n,m}$ maps onto $E_3^{n,m}$. Similarly, $E_r^{n,m}$ maps onto $E_{r+1}^{n,m}$, for $r \geq 2$, so that $E_2^{n,m}$ maps onto $E_\infty^{n,m}$. However, if U is connected, $E_\infty^{n,m} = H_c^{n+m}(U^*, \dot{U}; R) \cong R \neq 0$. Hence, $\mathcal{H}^m$ is not 0, which, in turn, implies $H_n^{\ell f}(U; R) \neq 0$ and $H_c^n(U; R) \neq 0$, for some neighborhood U of x. $\square$

Remark. The argument in this proof can be used to see that $H_c^n(X; \mathcal{H}^m) \neq 0$; but, if $H_n(X, X - x; R)$ is not finitely generated, we cannot necessarily conclude that the ordinary cohomology of X is nonzero. If so, we would have a proof of Conjecture 2.

Suppose that F is a finitely generated submodule of $H_k(X, X-x; R)$. By Lemma 4 there is a neighborhood U of x and a constant sheaf $\mathcal{F}$ on U such that $\mathcal{F} \subseteq \mathcal{H}^q|U$, $q = n+m-k$, and $\mathcal{F}_x = F$. Since $\dim U = n$, the short exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{H}^q|U \to \operatorname{coker} \iota \to 0$$

induces a long exact sequence on Borel-Moore homology

$$\begin{aligned} 0 &\to H_n(U; \mathcal{F}) \to H_n(U; \mathcal{H}^q) \\ &\to H_n(U; \operatorname{coker} \iota) \to \cdots, \end{aligned}$$

which implies $H_n(U;\mathcal{F}) \to H_n(U;\mathcal{H}^q)$ is one-to-one.

We would like for the same to be true for inclusion in cohomology,

$$\mathrm{im}(H^n(U;\mathcal{F}) \to H^n(U;\mathcal{H}^q)),$$

since this would allow us to get a good relationship between sheaf cohomology of U and ordinary cohomology of $(U^*, \dot{U})$.

Unfortunately, there is nothing that seems to preclude the Bockstein

$$H^{n-1}(U; \mathrm{coker}\ \iota) \to H^n(U;\mathcal{F})$$

from being onto. Indeed, it is possible to construct a rather "homogeneous" looking sheaf over the interval $(0,1)$, having infinitely generated stalks, for which this Bockstein (with $n = 1$) is onto.

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ON CERTAIN I-D COMPACTA

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Abstract. Three examples of nontypical i-d compacta are presented. An application to absorbers follows.

1. Nontypical compacta

The typical property, for an i-d compactum K, is that K is homeomorphic to its square, that is,

$$K \cong K \times K.$$

Here are the properties that are stronger than the *negation* of the above:

(1) No open subset of $K \times K$ can be embedded into $K \times I^q$ for any q (I stands for $[-1,1]$).
(2) $K \times K$ cannot be embedded into $K \times \sigma$; $\sigma = \bigcup_{q=1}^{\infty} I^q \subset Q = I^\infty$.
(3) $K \times K$ cannot be embedded into $K \times I^q$ for any q.
(4) $K \times K$ cannot be embedded into K.

Definition. A map $K \times K \supset A \to Z$ is *fiberwise injective (f-i)* if restricted to every fiber $\{k\} \times K$ or $K \times \{k\}$ it is injective.

Fact 1. If K is carries either a group structure or a convex structure then $K \times K$ admits a f-i map into K. The maps

$$(x, y) \to xy$$

or

$$(x, y) \to \frac{1}{2}(x + y)$$

are easily seen to be f-i.

Here are counterparts of properties (1)-(4):

(1′) No open set U of $K \times K$ admits a f-i map into $Z = K \times I^q$ for any q.
(2′) There is no f-i map $K \times K \to Z = K \times \sigma$.
(3′) There is no f-i map $K \times K \to Z = K \times I^q$ for any q.
(4′) There is no f-i map $K \times K \to K$.

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For a compactum K, we have the following implications

$$1' \Rightarrow 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$$

and

$$1' \Rightarrow 2' \Rightarrow 3' \rightarrow 4'$$

The implications $1 \Rightarrow 2$ and $1' \Rightarrow 2'$ follow from the Baire category theorem applied to $K \times K$ (having in mind that $K \times \sigma = \bigcup_{q=1}^{\infty} K \times I^q$).

Furthermore, we have

Remark 1. Assume $K \subset Z$ and Z is a countable union of compacta embeddable in $K \times \sigma$. If K satisfies property (1) (resp., property (1′)), then Z satisfies (2) (resp., (2′)); consequently, $Z \times Z$ is not embeddable in Z (resp., there is no f-i map $Z \times Z \to Z$).

In what follows we will discuss examples that were presented in [D] (see also [BC]).

Example 1. Let C be Cook's continuum, that is, C is hereditarily indecomposable continuum and, for every continuum $A \subset C$, every map $A \to C$ is either constant or an inclusion. Every compactum of the form

$$P = \prod_{i=1}^{\infty} A_i,$$

where $A_i \subset C$ are pairwise disjoint subcontinua, satisfies property (1). Moreover, P (and every open subset of P) is strongly infinite-dimensional and contains subsets of all finite dimensions.

Example 2. Let us recall that the Smirnov Cubes S_α, $\alpha < \omega_1$, are compacta defined as follows $S_0 = \{0\}$, $S_{\beta+1} = S_\beta \times I$; and, for a limit ordinal α, $S_\alpha = \omega(\oplus_{\beta<\alpha} S_\beta)$, the one-point compactification of S_β. For, for $\alpha_0 = \omega^\omega$, the space

$$S = S_{\alpha_0}$$

satisfies (3).

Proof. This follows from the fact that $\mathrm{trind}(S_{\alpha_0} \times S_{\alpha_0}) = \alpha_0(+)\alpha_0$ and $\mathrm{trind}(S_\alpha \times I^q) \leq \alpha(+)q$, where trind stands for the small transfinite inductive dimension. □

The next example is due to J. Kulesza.

Example 3. The space

$$T = \omega((\oplus_{n\geq 1} I^n) \oplus H),$$

where H is a hereditary i-d continuum, has property (3′).

Proof. Let $f : T \times T \to T \times I^q$ be f-i. Then $f(H \times I^k) \subset H \times I^q$ for $k > q$. In particular, I^k embeds into $H \times I^q$. Since $k > q$ and the projection is closed, there exists a fiber $H \times \{x\} \subset H \times I^q$ containing a closed set with dim > 0, a contradiction. $\square$

Congesting singularities. Write L for either S or T. Pick a null-sequence $\{C_n\}$ of pairwise disjoint Cantor sets in the Cantor set C so that every open nonempty subset of C contains some C_n. Let $f_n : C_n \to L$, be a surjection. Define $\tilde{S}$ (resp., $\tilde{T}$) to be the adjoint space with S (resp., T) attached in place of each C_n via the map f_n.

Fact 2. The compactum $\tilde{S}$ satisfies property (1); moreover, it is countable-dimensional and $\operatorname{trind}(\tilde{S}) \leq \operatorname{trind}(S) + 1$. The compactum $\tilde{T}$ is **not** countable dimensional and satisfies property (1′).

Proof. This is a consequence of the facts that $\tilde{S}$ (resp., $\tilde{T}$) is a union of pairwise disjoint copies of S (resp., T) and a subset of irrationals, and that each open subset of $\tilde{S}$ (resp., $\tilde{T}$) contains a copy of S (resp., T). $\square$

2. An application to absorbers

For a compactum K, let $\mathcal{C} = \mathcal{C}(K)$ be the class of compacta embeddable in $K \times \sigma$ (notice that the class $\mathcal{C}$ is $[0,1]$-multiplicative, i.e., for $L \in \mathcal{C}$, $L \times [0,1] \in \mathcal{C}$). There exists an absorber $\Omega(K)$ for the class $\mathcal{C}$ (see [BRZ] for the definition). We will describe $\Omega(K)$, as done in [D]. Let

$$\mathcal{E} = \{(x_i) \in \ell^2 | \sum_1^\infty i^2 x_i^2 \leq 1\}$$

be the i-d convex ellipsoid in ℓ^2, a topological copy of Q, and

$$B = \{(x_i) \in \ell^2 | \sum_i^\infty i^2 x_i^2 = 1\} \subset \mathcal{E}$$

be its pseudoboundary. Embed K into B such that $K \subset B$ is linearly independent and there exists a countable, linearly independent $D \subset B \setminus K$ dense in B. Notice that $\operatorname{span}(D) \cap \mathcal{E}$ is a topological copy of σ (which is also denoted by σ). Define

$$\Omega(K) = \{tk + (1-t)x | k \in K, x \in \sigma, t \in [0,1]\}.$$

Most absorbers enjoy a regular structure, but absorbers of the form $\Omega(K)$ for nontypical K are themselves nontypical. Since $\Omega(K)$ is a countable union of elements of $\mathcal{C}$, applying Remark 1, we obtain:

Theorem. *For the absorber $\Omega(K)$, we have:*

(a) *if K satisfies property* (1)*, then $\Omega(K) \times \Omega(K) \not\cong \Omega(K)$;*

(b) *if K satisfies property $(1')$, then there is no f-i of $\Omega(K)\times\Omega(K)$ into $\Omega(K)$; in particular, there is no group or convex structure on $\Omega(K)$.*

Corollary. *None of the absorbers $\Omega(P)$, $\Omega(\tilde{S})$, and $\Omega(\tilde{T})$ is homeomorphic to its square. They are pairwise nonhomeomorphic. Moreover, $\Omega(P)$ and $\Omega(\tilde{T})$ do not carry a group structure or a convex structure.*

Proof. It is enough to show that

$$\omega(P) \ncong \omega(T').$$

To see this use the facts that: (1) every open subset of P contains a copy of P, (2) P is connected, (3) P contains closed subsets of all finite dimensions. As a consequence, no open subset of P can be embedded into $\tilde{T}\times I^q$. □

With an extra work (see [D]), we obtain:

Remark 2. For $n < m$,
a) $\Omega(\tilde{S})^m \ncong \Omega(\tilde{S})^n$;
b) $\Omega(P)^m$ does not admit a f-i map into $\Omega(P)^n$.

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ON HOMOTOPY PROPERTIES OF CERTAIN COXETER GROUP BOUNDARIES

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Abstract. There is a canonical homomorphism $\psi : \pi_1(\text{bdy } X) \to \pi_1^\infty(X)$ from the fundamental group of the visual boundary, here denoted by bdy X, of any non-positively curved geodesic space X into its fundamental group at infinity. In this setting, the latter group coincides with the first shape homotopy group of the visual boundary: $\pi_1^\infty(X) \equiv \check{\pi}_1(\text{bdy } X)$. The induced homomorphism $\varphi : \pi_1(\text{bdy } X) \to \check{\pi}_1(\text{bdy } X)$ provides a way to study the relationship between these groups.

We present a class $\mathcal{Z}$ of compacta, so-called *trees of manifolds*, for which we can show that the homomorphisms $\varphi : \pi_1(Z) \to \check{\pi}_1(Z)$ $(Z \in \mathcal{Z})$ are injective. This class $\mathcal{Z}$ includes the visual boundaries $Z = \text{bdy } X$ which arise from right-angled Coxeter groups whose nerves are closed PL-manifolds. In particular, it includes the visual boundaries of those Coxeter groups which act on Davis' exotic open contractible manifolds [2].

1. The first shape homotopy group of a metric compactum

We recall the definition of the first shape homotopy group of a pointed compact metric space (Z, z_0). Choose an inverse sequence

$$(Z_1, z_1) \xleftarrow{f_{2,1}} (Z_2, z_2) \xleftarrow{f_{3,2}} (Z_3, z_3) \xleftarrow{f_{4,3}} \cdots$$

of pointed compact polyhedra such that

$$(Z, z_0) = \varprojlim((Z_i, z_i), f_{i+1,i}).$$

The *first shape homotopy group* of Z based at z_0 is then given by

$$\check{\pi}_1(Z, z_0) = \varprojlim\left(\pi_1(Z_1, z_1) \xleftarrow{f_{2,1\#}} \pi_1(Z_2, z_2) \xleftarrow{f_{3,2\#}} \pi_1(Z_3, z_3) \xleftarrow{f_{4,3\#}} \cdots\right).$$

This definition of $\check{\pi}_1(Z, z_0)$ does not depend on the choice of the sequence
$((Z_i, z_i), f_{i+1,i})$ [8]. Let $p_i : (Z, z_0) \to (Z_i, z_i)$ be the projections of the limit (Z, z_0) into its inverse sequence $((Z_i, z_i), f_{i+1,i})$ such that $p_i = f_{i+1,i} \circ p_{i+1}$ for all i. Since the maps p_i induce homomorphisms

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$p_{i\#} : \pi_1(Z, z_0) \to \pi_1(Z_i, z_i)$ such that $p_{i\#} = f_{i+1,i\#} \circ p_{i+1\#}$ for all i, we obtain an induced homomorphism $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ given by $\varphi([\alpha]) = ([\alpha_1], [\alpha_2], [\alpha_3], \cdots)$, where $\alpha_i = p_i \circ \alpha$.

The following examples illustrate that $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ need not be injective and is typically not surjective.

Example 1. Let

$Y = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0, 0 < x \leq 1, y = \sin 1/x\} \cup (\{0\} \times [-1, 1] \times \{0\})$

be the "topologist's sine curve". Define $Y_i = Y \cup ([0, 1/i] \times [-1, 1] \times \{0\})$. Let Z and Z_i be the subsets of $\mathbb{R}^3$ obtained by revolving Y and Y_i about the y-axis, respectively, and let $f_{i+1,i} : Z_{i+1} \hookrightarrow Z_i$ be inclusion. Then Z is the limit of the inverse sequence $(Z_i, f_{i+1,i})$. If we take $z_0 = (1, \sin 1, 0)$, then $\pi_1(Z, z_0)$ is infinite cyclic, while $\check{\pi}_1(Z, z_0)$ is trivial.

Example 2. We can make the space Z of the previous example path connected, by taking any arc $a \subseteq \mathbb{R}^3$, such that $a \cap Z = \partial a = \{z_0, (0, 1, 0)\}$, and then considering $Z^+ = Z \cup a$. Notice that both $\pi_1(Z^+, z_0)$ and $\check{\pi}_1(Z^+, z_0)$ are infinite cyclic. However, the homomorphism $\varphi : \pi_1(Z^+, z_0) \to \check{\pi}_1(Z^+, z_0)$ is trivial.

Example 3. Let $Z = \bigcup_{k=1}^{\infty} C_k$ be the Hawaiian Earrings, where

$$C_k = \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - 1/k)^2 = (1/k)^2\}.$$

Put $Z_i = C_1 \cup C_2 \cup \cdots \cup C_i$ and let $z_0 = z_i = (0, 0)$. Define $f_{i+1,i} : Z_{i+1} \to Z_i$ by $f_{i+1,i}(p) = (0, 0)$ for $p \in C_{i+1}$ and $f_{i+1,i}(p) = p$ for $p \in Z_{i+1} \setminus C_{i+1}$. Then (Z, z_0) is the limit of the inverse sequence $((Z_i, z_i), f_{i+1,i})$. While this time $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ is injective [4], it is not surjective: let $l_i : (S^1, *) \to (C_i, z_0)$ be a fixed homeomorphism and consider for each i the element

$g_i = [l_1][l_1][l_1]^{-1}[l_1]^{-1}[l_1][l_2][l_1]^{-1}[l_2]^{-1}[l_1][l_3][l_1]^{-1}[l_3]^{-1} \cdots [l_1][l_i][l_1]^{-1}[l_i]^{-1}$

of $\pi_1(Z_i, z_i)$. Then the sequence $(g_i)_i$ is an element of the group $\check{\pi}_1(Z, z_0)$ which is clearly not in the image of φ.

2. Trees of manifolds

We shall call a topological space Z a *tree of manifolds* if there is an inverse sequence

$$M_1 \xleftarrow{f_{2,1}} M_2 \xleftarrow{f_{3,2}} M_3 \xleftarrow{f_{4,3}} \cdots,$$

called a *defining sequence* for Z, of distinct closed PL-manifolds M_n with collared disks $D_n \subseteq M_n$, and continuous functions $f_{n+1,n} : M_{n+1} \to M_n$ that have the following properties:

(P-1) $$Z = \varprojlim \left(M_1 \xleftarrow{f_{2,1}} M_2 \xleftarrow{f_{3,2}} M_3 \xleftarrow{f_{4,3}} \cdots \right);$$

(P-2) For each n, the restriction of $f_{n+1,n}$ to the set $f_{n+1,n}^{-1}(M_n \setminus \text{int } D_n)$, call it $h_{n+1,n}$, is a homeomorphism onto $M_n \setminus \text{int } D_n$, and $h_{n+1,n}^{-1}(\partial D_n)$ is bicollared in M_{n+1};

(P-3) For each n, $\lim_{m\to\infty} \text{diam } [f_{m,n}(D_m)] = 0$, where $f_{m,n} = f_{n+1,n} \circ f_{n+2,n+1} \circ \cdots \circ f_{m,m-1} : M_m \to M_n$ & $f_{n,n} = id_{M_n}$.

(P-4) For each pair $n < m$, $f_{m,n}(D_m) \cap \partial D_n = \emptyset$.

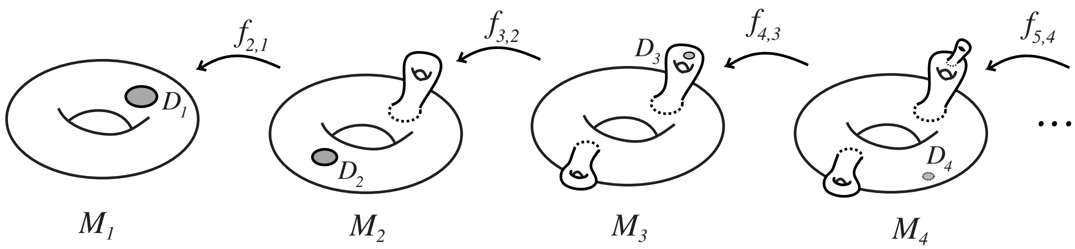

FIGURE 1. A tree of manifolds

It follows that, for $m \geq n+2$, the set

$$E_{m,n} = \text{int } D_n \cup f_{n+1,n}(\text{int } D_{n+1}) \cup f_{n+2,n}(\text{int } D_{n+2}) \cup \cdots \cup f_{m-1,n}(\text{int } D_{m-1})$$

can be written as the union of $m-n$, or fewer, open disks in M_n and that $f_{m,n}$ restricted to $f_{m,n}^{-1}(M_n \setminus E_{m,n})$ is a homeomorphism onto $M_n \setminus E_{m,n}$, which we will denote by $h_{m,n}$. Moreover, if for $n < m$ we define the spheres $S_{m,n} = h_{m,n}^{-1}(\partial D_n) \subseteq M_m$, we see that the collection $\mathcal{S}_n = \{S_{n,1}, S_{n,2}, \cdots, S_{n,n-1}\}$ decomposes M_n into a connected sum

$$M_n = [N_{n,1} \# N_{n,2} \# \cdots \# N_{n,n-1}] \# N_{n,n} \approx M_{n-1} \# N_{n,n}.$$

Hence, Z can be thought of as the limit of a growing tree of connected sums of closed manifolds. In particular, in dimensions greater than two, we have

$$\pi_1(M_n) = \pi_1(N_{n,1}) * \pi_1(N_{n,2}) * \cdots * \pi_1(N_{n,n-1}) * \pi_1(N_{n,n});$$

and in dimension two, we have

$$\pi_1(M_n) = F_{n,1} *_{\pi_1(S_{n,1})} F_{n,2} *_{\pi_1(S_{n,2})} \cdots *_{\pi_1(S_{n,n-2})} F_{n,n-1} *_{\pi_1(S_{n,n-1})} F_{n,n},$$

where $F_{n,i}$ denotes the free fundamental group of the appropriately punctured $N_{n,i}$.

Note also that each $S_{n,i} \approx \partial D_i$ naturally embeds in Z.

Definition. We will call a defining sequence $M_1 \xleftarrow{f_{2,1}} M_2 \xleftarrow{f_{3,2}} M_3 \xleftarrow{f_{4,3}} \cdots$ *well-balanced* if the set $\bigcup_{m\geq 3} E_{m,1}$ either has finitely many components or is dense in M_1, and if for each $n \geq 2$, the set $h_{n,n-1}^{-1}(M_{n-1} \setminus$

$D_{n-1}) \cup \left[\bigcup_{m \geq n+2} E_{m,n} \right]$ either has finitely many components or is dense in M_n.

Whether Z has a well-balanced defining sequence or not, will play a role only in the case when the manifolds M_n are 2-dimensional closed surfaces. Specifically, our main result is the following

Theorem. *Suppose Z is a tree of manifolds, and $z_0 \in Z$. In case Z is 2-dimensional, suppose further that Z admits a well-balanced defining sequence. Then the canonical homomorphism $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ is injective.*

Remark. In case $\pi_1(N_{n,n}) \neq 1$ for infinitely many n, an argument analogous to Example 3 shows that $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ is not surjective.

For a detailed proof of this theorem see [6]. Here, we only give a brief

SKETCH OF PROOF. Since it is known that the canonical homomorphism
$\pi_1(Y) \to \check{\pi}_1(Y)$ is injective for all 1-dimensional compacta Y [4], we will assume that $\dim Z \geq 2$.

Let $\alpha : S^1 \to Z$ be a loop such that $\alpha_n = p_n \circ \alpha : S^1 \to M_n$ is nullhomotopic for each n. We wish to show that $\alpha : S^1 \to Z$ is nullhomotopic. We will do this by constructing a map $\beta : D^2 \to Z$ with $\beta|_{S^1} = \alpha$. By assumption, we may choose maps $\beta_n : D^2 \to M_n$ with $\beta_n|_{S^1} = \alpha_n$. The difficulty of the proof, of course, is that in general $\beta_n \neq f_{n+1,n} \circ \beta_{n+1}$, so that the sequence $(\beta_n)_n$ does not even constitute a function $D^2 \to Z$ into the inverse limit, let alone a map extending α.

Although we might not be in a position to move the maps α_n the slightest bit, we can place β_n in general position with respect to the spheres of the collection $\mathcal{S}_n$ while having $\beta_n|_{S^1}$ approximate α_n with increasing accuracy as n increases. Indeed, we can arrange for each *cancellation pattern* $\beta_n^{-1}(\bigcup \mathcal{S}_n)$ to consist of finitely many pairwise disjoint straight line segments in D^2 which have their endpoints in S^1. Ideally, we would like to paste together our map β from appropriate pieces belonging to the maps of the sequence $(\beta_n)_n$, namely those pieces that cancel the elements of $\pi_1(N_{n,n})$. However, these cancellation patterns will in general not be compatible. For example, in dimensions greater than two, the cancellation pattern for an element

$$[\alpha_{n+1}] = h_1 * k_1 * h_2 * k_2 * \cdots * h_5 * k_5 = 1 \in \pi_1(M_{n+1}) = \pi_1(M_n) * \pi_1(N_{n+1,n+1})$$

might be witnessed by β_{n+1} as

$$(h_1(k_1(h_2(k_2)h_3)k_3(h_4)k_4)h_5(k_5)) = 1.$$

The induced cancellation pattern for

$$[\alpha_n] = f_{n+1,n\#}([\alpha_{n+1}]) = h_1 * 1 * h_2 * 1 * \cdots * 1 * h_5 * 1 = 1 \in \pi_1(M_n) * \{1\}$$

as obtained from $f_{n+1,n} \circ \beta_{n+1}$ would then be given by

$$(h_1((h_2h_3)(h_4))h_5) = 1.$$

On the other hand, the map β_n might cancel $[\alpha_n]$ as

$$((h_1h_2)(h_3(h_4)h_5)) = 1.$$

This is illustrated in Figure 2, which depicts the sets $\beta_n^{-1}(\partial D_n)$, $(f_{n+1,n} \circ \beta_{n+1})^{-1}(\partial D_n)$, and $\beta_{n+1}^{-1}(S_{n+1,n})$ as dashed lines.

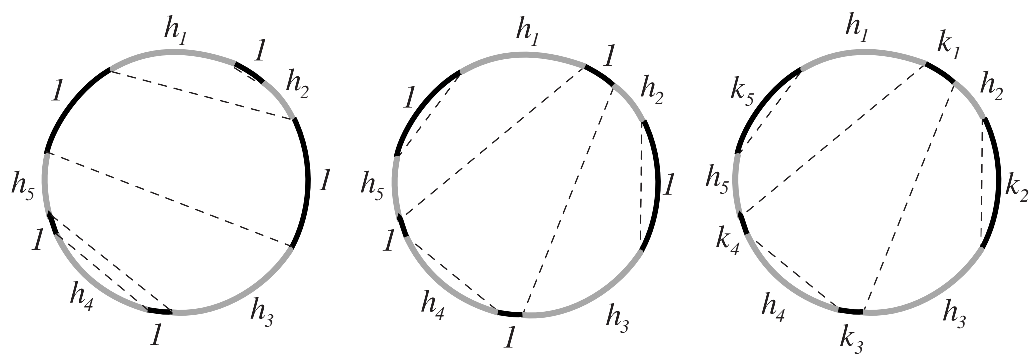

FIGURE 2.

If k_1 is not trivial and if k_3 does not cancel k_4 in $\pi_1(N_{n+1,n+1})$, then we cannot use any of the pieces of the map β_n to construct β.

As a remedy, we repeatedly select subsequences until, at least approximately, all cancellation patterns are coherent. That is, until the sets $\beta_n^{-1}(\bigcup S_n)$ are approximately nested with increasing n. Once this is achieved, the union of these cancellation patterns produce a limiting pattern $\mathcal{P}$ of possibly infinitely many straight line segments in D^2 whose interiors are pairwise disjoint and whose endpoints lie in S^1. Each segment of $\mathcal{P}$, at least approximately, maps under some β_n into some $S_{n,i}$. Note that we must allow for the possibility that α_n meets some $S_{n,i}$ in infinitely many points. This effect is accounted for by a possible increase of segments $c \subseteq \beta_m^{-1}(\bigcup \mathcal{S}_m)$ for which $\beta_m(c) \subseteq S_{m,i}$, as m increases. The map $\beta : D^2 \to Z$ can now be defined in two stages.

First, extend $\alpha : S^1 \to Z$ to a map $\beta : S^1 \cup \mathcal{P} \to Z$. If dim $Z = 2$, this can be done so that each segment of $\mathcal{P}$ maps to a local geodesic of that simple closed curve of Z which corresponds to the appropriate

∂D_i. If dim $Z \geq 3$, any coherent extension into the spheres of Z corresponding to ∂D_i will do, so long as the extension to a segment does not deviate too much from the image of its endpoints. If all this is done with sufficient care, the map $\beta : S^1 \cup \mathcal{P} \to Z$ will be uniformly continuous, so that we can extend it to the closure of its domain.

Next, focus on the components of the subset of D^2 on which the map β is not yet defined. Call these components *holes.* The boundary, bdy H, of a hole H is a simple closed curve, which either maps to a singleton under β, in which case we extend β trivially over cl H, or $p_n \circ \beta(\text{bdy } H) \subseteq N_n^*$ for some n, where

$$N_1^* = M_1 \setminus \Big(\bigcup_{m \geq 3} E_{m,1} \Big)$$

and

$$N_n^* = M_n \setminus \Big[h_{n,n-1}^{-1}(M_{n-1} \setminus D_{n-1}) \cup \Big(\bigcup_{m \geq n+2} E_{m,n} \Big) \Big] \text{ for } n \geq 2.$$

The map $p_n \circ \beta$: bdy $H \to N_n^* \subseteq M_n$ can be extended to a map $p_n \circ \beta$: cl $H \to M_n$ so long as the hole H is sufficiently "thin", because M_n is an ANR. For the moment, assume that dim $Z \geq 3$. The map $p_n \circ \beta$: cl $H \to M_n$ can then be cut off at $S_{n,n-1} = h_{n,n-1}^{-1}(\partial D_{n-1})$ and pushed off $\bigcup_{m \geq n+2} E_{m,n}$. Hence, we may extend the map $p_n \circ \beta$: bdy $H \to N_n^*$ to a map $p_n \circ \beta$: cl $H \to N_n^*$. Since N_n^* naturally embeds in Z, we have an extension of β : bdy $H \to Z$ to β : cl $H \to Z$. For each n, there will be *finitely* many maps $p_n \circ \beta$: bdy $H \to N_n^* \subseteq M_n$ for which the hole H is not thin enough to make this argument. In those cases, some $f_{m,n} \circ \beta_m : D^2 \to M_n$, with sufficiently large m, will be witness to the fact that $p_n \circ \beta$: bdy $H \to M_n$ is nullhomotopic after all. This is due to the approximate nestedness of the cancellation patterns $\beta_n^{-1}(\bigcup \mathcal{S}_n)$. Since for sufficiently large n the subset of Z which is homeomorphic to N_n^* is arbitrarily small, this procedure guarantees continuity of the resulting map $\beta : D^2 \to Z$.

If dim $Z = 2$, the above process requires a little bit more care and is helped by the assumption that the defining tree is well-balanced. Specifically, the sets N_n^* will either be ANRs or 1-dimensional. In the former case, we can adapt the argument we just made, and in the latter case, we make use of the result in [4] mentioned at the beginning of this proof. □

3. An application to Coxeter group boundaries

We now present an application of our theorem to boundaries of certain non-positively curved geodesic spaces. Recall that a metric space is *proper* if all of its closed metric balls are compact. A *geodesic space* is a metric space in which any two points lie in a geodesic, i.e. a subset that is isometric to an interval of the real line in its usual metric. A proper geodesic space X is said to be *non-positively curved* if any two points on the sides of a geodesic triangle in X are no further apart than their corresponding points on a reference triangle in Euclidean 2-space. The *visual boundary* of a non-positively curved geodesic space X, denoted by bdy X, is defined to be the set of all geodesic rays emanating from a fixed point x_0 endowed with the compact open topology. Let some geodesic base-ray $\omega : [0, \infty) \to X$ with $\omega(0) = x_0$ be given. Under the relatively mild assumption that the pointed concentric metric spheres $(S_{x_0}(i), \omega(i))$ have the pointed homotopy type of ANRs, it is shown in [1], that

$$\check{\pi}_1(\text{bdy } X, \omega) = \pi_1^\infty(X, \omega).$$

Here, $\pi_1^\infty(X, \omega)$ is the *fundamental group at infinity* of X, that is, the limit of the sequence

$$\pi_1(X \setminus B(1), \omega(2)) \leftarrow \pi_1(X \setminus B(2), \omega(3)) \leftarrow \pi_1(X \setminus B(3), \omega(4)) \leftarrow \cdots$$

whose bonds are induced by inclusion followed by a base point slide along ω.

A class of visual boundaries to which our theorem applies, arises from non-positively curved simplicial complexes, which are acted upon by certain Coxeter groups, whose definition we now briefly recall: let V be a finite set and $m : V \times V \to \{\infty\} \cup \{1, 2, 3, \cdots\}$ a function with the property that $m(u, v) = 1$ if and only if $u = v$, and $m(u, v) = m(v, u)$ for all $u, v \in V$. Then the group $\Gamma = \langle V \mid (uv)^{m(u,v)} = 1 \text{ for all } u, v \in V \rangle$ defined in terms of generators and relations is called a *Coxeter group*. If moreover $m(u, v) \in \{\infty, 1, 2\}$ for all $u, v \in V$, then Γ is called *right-angled*. The abstract simplicial complex $N(\Gamma, V) = \{\emptyset \neq S \subseteq V \mid S \text{ generates a finite subgroup of } \Gamma\}$ is called the *nerve* of the group Γ. For a right-angled Coxeter group, the isomorphism type of the nerve $N(\Gamma, V) = N(\Gamma)$ does not depend on the Coxeter system (Γ, V) but only on the group Γ [10].

For the remainder of this discussion, let Γ be a right-angled Coxeter group whose nerve $N(\Gamma)$ is a closed PL-manifold. This includes, for example, the Coxeter groups generated by the reflections of any one of Davis' exotic open contractible n-manifolds ($n \geq 4$), for which the nerves are PL-homology spheres [2].

As described, for example, in [3], Γ acts properly discontinuously on a non-positively curved (and hence contractible) simplicial complex $X(\Gamma)$, its so-called *Davis-Vinberg complex*, by isometry and with compact quotient. In [5] it is shown that the visual boundary of $X(\Gamma)$ is a (well-balanced) tree of manifolds. (By virtue of [11], the proof given in [5] also applies to the non-orientable case.) The visual boundary of $X(\Gamma)$ is usually referred to as the *boundary* of Γ and is denoted by bdy Γ. Since Coxeter groups are semi-stable at infinity [9] and Γ is one-ended, $\pi_1^\infty(X(\Gamma),\omega) = \pi_1^\infty(\Gamma)$ is actually an invariant of the group Γ [7].

In summary, we obtain the following

Corollary. *Let Γ be a right-angled Coxeter group whose nerve $N(\Gamma)$ is a closed PL-manifold. Then the canonical homomorphism $\psi : \pi_1(bdy\ \Gamma) \to \pi_1^\infty(\Gamma)$ is injective.*

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RESOLUTIONS FOR METRIZABLE COMPACTA IN EXTENSION THEORY

LEONARD R. RUBIN AND PHILIP J. SCHAPIRO

ABSTRACT. This is a summary of research which appears in a preprint of the same title. We prove a K-resolution theorem for simply connected CW-complexes K in extension theory in the class of metrizable compacta X. This means that if $\dim X \leq K$ (in the sense of extension theory), n is the first element of $\mathbb{N}$ such that $G = \pi_n(K) \neq 0$, and it is also true that $\pi_{n+1}(K) = 0$, then there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:

(a) π is G-acyclic,
(b) $\dim Z \leq n + 1$, and
(c) $\dim Z \leq K$.

If additionally, $\pi_{n+2}(K) = 0$, then we may improve (a) to the statement,

(aa) π is K-acyclic.

To say that a map π is K-acyclic means that each map of each fiber $\pi^{-1}(x)$ to K is nullhomotopic.

In case $\pi_{n+1}(K) \neq 0$, we obtain a resolution theorem with a weaker outcome. Nevertheless, it implies the G-resolution theorem for arbitrary abelian groups G in cohomological dimension $\dim_G X \leq n$ when $n \geq 2$.

The Edwards-Walsh resolution theorem, the first resolution theorem for cohomological dimension, was proved in [Wa] (see also [Ed]). It states that if X is a metrizable compactum and $\dim_{\mathbb{Z}} X \leq n$ ($n \geq 0$), then there exists a metrizable compactum Z with $\dim Z \leq n$ and a surjective cell-like map $\pi : Z \to X$. This result, in conjunction with Dranishnikov's work ([Dr1]) showing that in the class of metrizable

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compacta, $\dim_{\mathbb{Z}}$ is distinct from dim, was a key ingredient for proving that cell-like maps could raise dimension (see [Ru1] for background). For the reader seeking fundamentals on the theory of cohomological dimension, $\dim_G$, the references [Ku], [Dr3], [Dy], and [Sh] could be helpful.

Now a map is cell-like provided that each of its fibers is cell-like, or, equivalently, has the shape of a point ([MS1]). Every cell-like compactum has trivial reduced Čech cohomology with respect to any abelian group G. This means that for every abelian group G, every cell-like map is G-acyclic, i.e., all its fibers have trivial reduced Čech cohomology with respect to the group G. This is equivalent to the statement that every map of such a fiber to $K(G,n)$ is nullhomotopic.

The latter notion may be generalized as follows. For a given CW-complex K, a metrizable compactum X is called *K-acyclic* if every map of it to K is nullhomotopic. Moreover, one should recall that when a Hausdorff compactum or metrizable space X has $\dim X \leq n$, then also $\dim_{\mathbb{Z}} X \leq n$.

With these ideas in mind, one may ask, what kind of parallel resolution theorems can be obtained under the assumption that $\dim_G X \leq n$, where G is an abelian group different from $\mathbb{Z}$? It turns out that it is not possible always to have cell-like resolutions as in the Edwards-Walsh theorem, nor can one even require in such propositions that $\dim Z \leq n$ be true (see [KY2]). So, what kind of resolution theorems can we expect? The main results of this paper go as follows.

1.1. Theorem. *Let K be a simply-connected CW-complex, n be the first element of $\mathbb{N}$ such that $G = \pi_n(K) \neq 0$, and X be a metrizable compactum with $\dim X \leq K$. Then there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:*

(a) *π is G-acyclic,*
(b) *$\dim Z \leq n+1$, and*
(c) *$\dim_G Z \leq n$.*

1.2. Theorem. *Let K be a simply-connected CW-complex, n be the first element of $\mathbb{N}$ such that $G = \pi_n(K) \neq 0$, and assume that $\pi_{n+1}(K) = 0$. Then for each metrizable compactum X with $\dim X \leq K$, there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:*

(a) *π is G-acyclic,*
(b) *$\dim Z \leq n+1$, and*
(c) *$\dim Z \leq K$.*

If in addition, $\pi_{n+2}(K) = 0$*, then we may also conclude that*

(aa) π *is* K*-acyclic.*

If $K = K(G, n)$, then $\dim X \le K$ is equivalent to $\dim_G X \le n$. Hence, as a corollary to Theorem 1.1, we get the G-acyclic resolution theorem in cohomological dimension theory.

1.3. Corollary. *Let* G *be an abelian group and* X *be a metrizable compactum with* $\dim_G X \le n$ $(n \ge 2)$*. Then there exists a metrizable compactum* Z *and a surjective map* $\pi : Z \to X$ *such that:*

(a) π *is* G*-acyclic,*

(b) $\dim Z \le n + 1$*, and*

(c) $\dim_G Z \le n$*.*

In [Le] one finds another approach to 1.3. We mention that the Edwards-Walsh theorem has been generalized to the class of arbitrary metrizable spaces by Rubin and Schapiro ([RS]) and to the class of arbitrary compact Hausdorff spaces by Mardešić and Rubin ([MR]). Corollary 1.3 was proved by Dranishnikov ([Dr2]) for the group $G = \mathbb{Z}/p$, where p is an arbitrary prime number, but with the stronger outcome that $\dim Z \le n$. Later, Koyama and Yokoi ([KY1]) were able to obtain this $\mathbb{Z}/p$-resolution theorem of Dranishnikov both for the class of metrizable spaces and for that of compact Hausdorff spaces.

In their work [KY2], Koyama and Yokoi have made a substantial amount of progress in the resolution theory of metrizable compacta, that is, towards proving Corollary 1.3. Their method relies heavily on the existence of Edwards-Walsh resolutions, which had been studied by Dydak and Walsh in [DW], and which had been applied originally, in a rudimentary form, in [Wa]. The definition of an Edwards-Walsh resolution can be found in [KY2], but we shall not use it herein.

To overcome a flaw in the proof of Lemma 4.4 of [DW], Koyama and Yokoi proved the existence of Edwards-Walsh resolutions for some groups G, but under a stronger set of assumptions on G than had been thought necessary in [DW]. It is still not known if these stronger assumptions are needed to insure the existence of the resolutions. Nevertheless, Koyama and Yokoi were able to prove substantial G-acyclic resolution theorems. Let us state two of the important theorems from [KY2] (Theorems 4.9 and 4.12, respectively), which greatly influenced the direction of the work in this paper.

1.4. Theorem. *Corollary 1.3 is true for every torsion free abelian group* G*.*

1.5. Theorem. *Let G be an arbitrary abelian group and X be a metrizable compactum with* $\dim_G X \leq n$, $n \geq 2$. *Then there exists a surjective G-acyclic map $\pi : Z \to X$ from a metrizable compactum Z where* $\dim Z \leq n+2$ *and* $\dim_G Z \leq n+1$.

In case G is a torsion group, they prove (Theorem 4.11) that Corollary 1.3 holds, but without part (c). Of course Theorem 1.5 falls short of Corollary 1.3. We observed that one of the main reasons for the relative weakness of this theorem was that Koyama and Yokoi proved it by an indirect technique, a type of "finesse." Their approach depends heavily on the Bockstein basis theorem and the Bockstein inequalities (see [Ku]), instead of the more direct method, involving Edwards-Walsh resolutions, used to prove Theorem 1.4.

We want to point out that Theorem 1.4 includes as a corollary, and therefore redeems, the $\mathbb{Q}$-resolution theorem of Dranishnikov ([Dr5]– but see also [Dr6] where a different proof is given). The Koyama and Yokoi proof shows that in the proof of Theorem 3.2 of [Dr5], the statement that $\alpha_m \circ \omega_m$ is an Edwards-Walsh resolution over $\tau_m^{(n+1)}$ is not true. This was a subtle point; to fully understand it, the interested reader may examine the text immediately following the proof of Fact 1 of the proof of Theorem 3.1 in [KY2]. Getting around the barrier naturally led to a quite complicated construction.

Our proof of Theorems 1.1 and 1.2 will be direct, using extensions which are different from Edwards-Walsh resolutions. But we will use a type of pseudo-Bockstein basis denoted $\sigma_0(G)$ (section 9). This will allow us to deal with the groups $\mathbb{Z}/p^\infty$ as well as the other groups involved. We shall employ the technique of inverse sequences both to represent our given space X and to determine the resolving space Z. The map $\pi : Z \to X$ will be obtained in a standard, yet complicated manner similar to that used in [Wa].

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PERFECT SUBGROUPS OF HNN EXTENSIONS

F. C. TINSLEY (JOINT WITH CRAIG R. GUILBAULT)

Introduction

This note includes supporting material for Guilbault's one-hour talk summarized elsewhere in these proceedings. We supply the group theory necessary to argue that Guilbault's tame ends cannot be pseudo-collared. In particular, we show that certain groups (the interated Baumslag-Solitar groups) cannot have any non-trivial perfect subgroups. The absence of non-trivial perfect subgroups, in turn, eliminates the possibility of non-trivial homotopy equivalences.

In contrast, we include an example of a pseudo-collared end based on groups (the interated Adam's groups) that are somewhat similar to the Baumslag-Solitar groups. We close with a discussion of a homotopy theoretic approach to this construction.

1. The groups

We use the following standard notation. We let $x^g = g^{-1}xg$ or the *conjugation* of x by g, and let $[s,t] = s^{-1}t^{-1}st$ or the *commutator* of s and t. Let S be a subset of elements of a group N. We denote by $\langle s_1, s_2, \cdots ; N\rangle$ the *subgroup of N generated by S* where $S = \{s_1, s_2, \cdots\}$. If N is omitted, then $\langle s_1, s_2, \cdots\rangle$ is the free group generated by the characters $s_1, s_2, \cdots$. If S and N are as above, then we denote by $ncl\{s_1, s_2, \cdots ; N\}$ the *normal closure of S in N* or the smallest normal subgroup of N containing S.

A group N is *perfect* if it is equal to its commutator subgroup. Symbolically, $N = [N,N] = N^{(1)}$. Equivalently, N is equal to the transfinite intersection of its derived series.

The construction of the pseudo-collared end is based on the one-relator group, Adam's group:

$$A_1 = \left\langle a_1, a_0 \;\middle|\; a_1^2 = a_1^{\left(a_1^{a_0}\right)} \right\rangle$$

We can repeat this pattern to obtain the iterated Adam's groups, A_k, $1 \le k < \infty$:

$$\left\langle a_0, a_1, \cdots, a_k \;\middle|\; a_1^2 = a_1^{\left(a_1^{a_0}\right)}, a_2^2 = a_2^{\left(a_2^{a_1}\right)}, \cdots, a_k^2 = a_k^{\left(a_k^{a_{k-1}}\right)} \right\rangle$$

For each $i \geq 1$, a_i is a commutator $a_i = \left[a_i, a_i^{a_{i-1}}\right]$.

Guilbault constructs the tame end that is not pseudo-collared using the simplest Baumslag-Solitar group:

$$\begin{aligned} B_1 &= \left\langle b_1, b_0 \;\middle|\; b_1^2 = b_0^{-1} b_1 b_0 \right\rangle \\ &= \left\langle b_1, b_0 \;\middle|\; b_1^2 = b_1^{b_0} \right\rangle \end{aligned}$$

Again, we can iterate to obtain the group, B_k:

$$\left\langle b_0, b_1, \cdots, b_k \;\middle|\; b_1^2 = b_1^{b_0}, b_2^2 = b_2^{b_1}, \ldots, b_k^2 = b_k^{b_{k-1}} \right\rangle$$

For each $i \geq 1$, b_i is a commutator $b_i = \left[b_i, b_{i-1}\right]$.

2. HNN extensions

The iterations above are each of a more general form. Given $L < K$ and $\psi : L \xrightarrow{1:1} K$, we define the *HNN extension* N

$$\left\langle gen(K), t \;\middle|\; rel(K), \psi(l) = t^{-1} l t \text{ for } l \in L \right\rangle .$$

The extension is *split* if there is a retraction $r : K \to L$. The following are well-known for HNN-extensions.

Facts:

(1) K is a natural subgroup of N.

(2) N naturally retracts onto $\langle t \rangle \cong \mathbb{Z}$, called the *free part of* N. The kernel of the retraction is $ncl\{\{K\}; N\}$.

We also use the following well-known result about the structure of subgroups of HNN extensions:

Theorem 1 (see [KS], Theorem 6)**.** *Let N be the HNN group above. If H is a subgroup of N that has trivial intersection with each of the conjugates of L, then H is the free product of a free group with the intersections of H with certain conjugates of K.*

3. The main theorem

First, we list some basic propositions that easily follow from the definitions:

Proposition 1. *If N and H are any groups, $\phi : N \to H$ is a homomorphism, and $P < N$ is perfect, then $\phi(P)$ is perfect.*

First, a similarity between A and B:

Proposition 2. *For $G = A$, $G = B$, and each $k \geq 1$, there is a surjection*

$$\gamma_k : G_k \xrightarrow[g_k = 1]{} G_{k-1}$$

Then, two important *distinctions* between A and B.

Proposition 3. *The subgroup, $ncl\{a_1; A_1\}$, of A_1 is perfect. Moreover, the subgroup, $ncl\{a_1, ; A_k\}$ of A_k is perfect.*

And, now the crucial negative result for B_j:

Theorem 2. *For $j \geq 0$, the iterated Baumslag-Solitar group B_j has no non-trivial perfect subgroups.*

Proof. We induct on j. The cases $(j = 0)$ and $(j = 1)$ are handled separately. For $j = 0$, B_0 is abelian. For $j = 1$, we observe that B_1 is one of the well-known Baumslag-Solitar groups for which the kernel of the map $\psi_1 : B_1 \to B_0$ is abelian. Then, we apply the argument used for the case $j = 2$.

$(j \geq 2)$ Let

$$B_j = \left\langle b_0, b_1, \cdots, b_j \;\middle|\; b_1^2 = b_0^{-1} b_1 b_0, b_2^2 = b_1^{-1} b_2 b_1, \cdots, b_j^2 = b_{j-1}^{-1} b_j b_{j-1} \right\rangle$$

Now, B_j now can be put in the form of the HNN group. In particular, $B_j = \left\langle gen(K), t_1 \;\middle|\; rel(K), R_1 \right\rangle$ where

$$K = \left\langle b_1, b_2, \cdots, b_j \;\middle|\; b_2^2 = b_1^{-1} b_2 b_1, \cdots, b_j^2 = b_{j-1}^{-1} b_j b_{j-1} \right\rangle,$$

$t_1 = b_0$, $L_1 = \{b_1; B_j\}$, $\phi_1(b_1) = b_1^2$, and R_1 is given by $b_1^2 = b_0^{-1} b_1 b_0$. The base group, K, obviously is isomorphic to B_{j-1} with that isomorphism taking b_i to b_{i-1} for $1 \leq i \leq j-1$. Define $\psi_j : B_j \to B_{j-1}$ by adding the relation $b_j = 1$ to the group B_j. By inspection ψ_j is a surjective homomorphism. We assume that B_i contains no non-trivial perfect subgroups for $i \leq j-1$ and prove that B_j has this same property. To this end, let P be a perfect subgroup of B_j. Then, $\psi_j(P)$ is a perfect subgroup of B_{j-1}. By induction, $\psi_j(P) = 1$. Thus, $P \subset ker(\psi_j)$. By the inductive hypothesis, K has no perfect subgroups. Moreover, $b_1 \in K$ still has infinite order in both K (by induction) and B_j (since K embeds in B_j). Moreover, the HNN group, B_j, has the single associated cyclic subgroup, $L =< b_1; B_j >$, with conjugation relation $b_1^2 = b_0^{-1} b_1 b_0$. Recall that $\psi_j : B_j \to B_{j-1}$ is defined by adding the relation $b_j = 1$ to B_j. Thus, $ker(\psi_j) = ncl\{b_j; B_j\}$.

Claim. *No conjugate of L non-trivially intersects $H = ncl\{b_j; B_j\}$*

Proof of Claim: If the claim is false, then L itself must non-trivially intersect the *normal* subgroup, $ncl\{b_j; B_j\}$. This means that $b_1^m \in ncl\{b_j; B_j\} = ker(\psi_j)$ for some integer $m > 0$. Since $j \geq 2$, then $\psi_j(b_1^m) = \psi_j(b_1)^m = b_1^m = 1$ in B_{j-1}, ie, b_1 has finite order in B_{j-1}. This contradicts our observations above.

We continue with the proof of Theorem 2. Recall that P is a perfect subgroup of $ker\,(\psi_j)$. It must also enjoy the property of trivial intersection with each conjugate of L. We now apply Theorem 1 to the subgroup P to conclude that P is a free product where each factor is either free or equal to $P \cap g^{-1}Kg$ for some $g \in B_j$. Now, P projects naturally onto each of these factors so each factor is perfect. However, non-trivial free groups are not perfect. Moreover, by induction, K (or equivalently $g^{-1}Kg$) contains no non-trivial perfect subgroups. Thus, any subgroup, $P \cap g^{-1}Kg$, is trivial. Consequently, P must be trivial. This completes the proof of Theorem 2. □

4. Geometry and homotopy theory

We begin this section by emphasizing the similarities between A and B. We let G stand for *either* A or B, g stand respectively for *either* a or b, and $w(g_j)$ stand respectively for *either* $g_j^{g_{j-1}}$ or g_{j-1}. Then,

$$G_k = \left\langle g_0, g_1, \cdots, g_k \;\middle|\; g_1^2 = g_1^{w(g_1)}, g_2^2 = g_2^{w(g_2)}, \therefore, g_k^2 = g_k^{w(g_k)} \right\rangle$$

represents either A_k or B_k. For each $i \geq 1$, g_i is a commutator $g_i = \big[g_i, w(g_i)\big]$. We can summarize the relationship as follows:

SIMILARITES AND DIFFERENCES (G equals A or B)

Properties	Adam's	Baum.-Solitar
1 Relator Group	$\surd$	$\surd$
$G^{(1)} = ncl\,(g_1; G)$	$\surd$	$\surd$
Perfect $G^{(1)}$	$\surd$	$\times$
Abelian $G^{(1)}$	$\times$	$\surd$
Abelianizes to $\mathbb{Z}$	$\surd$	$\surd$
$\mathbb{Z} \xrightarrow[\text{split}]{HNN} \cdots \xrightarrow[\text{split}]{HNN} G$	$\surd$	$\surd$

Moreover, g_k is the commutator of itself with another element, $g = w(g_k)$, of G_k. So, if G_k is the fundamental group of a high-dimensional manifold, M_k, then g_k bounds a disk with one handle in M_k where one of the handle curves is homotopic (rel basepoint) to g_k. We let this be the meridianal handle curve:

We can attach a two handle to M_k along g_k. Then, the disk with handle and three copies of the core of the two handle form a 2-sphere, S_k^2, along which a three handle can be attached.

Note that this S_k^2 will algebraically cancel the 2-disk since it will be attached twice with one sign and once with the opposite sign. As a result the manifold, N_k, resulting from attaching these two handles

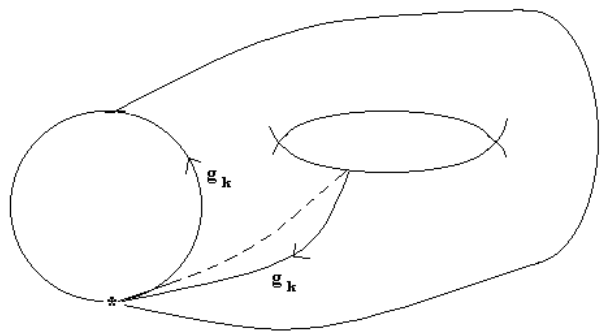

Figure 1

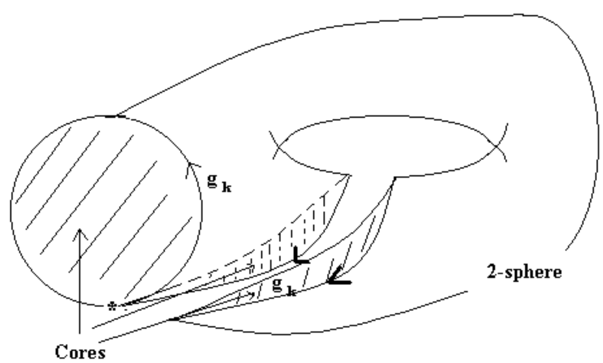

Figure 2

to M_k, will have the same homology as M_k. In fact, we obtain a cobordism, $\left(W_k, M_k, N_k\right)$, where the inclusion $N_k \rightarrow W_k$ induces an equivalence of homology groups.

This inclusion also induces an isomorphism on fundamental groups: $\pi_1\left(N_k\right) \xrightarrow{\cong} \pi_1\left(W_k\right) \cong G_{k-1}$. One might conclude that $N_k \rightarrow W_k$ induces a homotopy equivalence. However, this is the case only for Adams group.

The Hurewicz Theorem is needed to argue from data about homology to conclusions about homotopy. It requires simply connected spaces. Thus, we must pass to the universal covers, $\widetilde{W}_k$ and $\widetilde{N}_k$, of the manifolds, W_k and N_k and the cover, $\widehat{M}_k$, of the manifold, M_k, that corresponds to the π_1-kernel of the induced map, $\pi_1\left(M_k\right) \rightarrow \pi_1\left(W_k\right)$.

The key becomes the longitudinal curve, g, on the disk-with-handle (shown below in bold).

It is quite different for the A_k and B_k. For Adams group, $g = a_i^{a_{i-1}} = a_{i-1}^{-1} a_i a_{i-1}$ while for the Baumslag-Solitar group, $g = a_{i-1}$. In

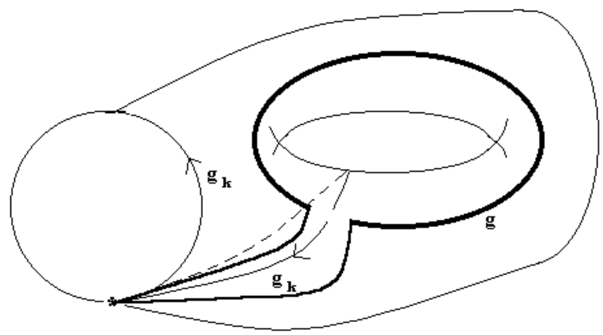

Figure 3

the first case, the element, $g = a_{i-1}^{-1} a_i a_{i-1}$, is a conjugate of a_i, and, in particular, lifts **as a loop** to $\hat{M}$. Consequently, the same cancellation in homology occurs in the universal cover as in the space itself and, thus, homology equivalences will yield a homotopy equivalence. In the second case, the conjugating element, b_{i-1}, lifts as an **arc** to $\hat{M}$. Thus, the two copies of the core of the 2-handle that cancelled as elements of $H_2(W, M)$ represent **distinct** generators of $H_2\left(\tilde{W}, \hat{M}\right)$ that have different signs but, in fact, do not cancel.

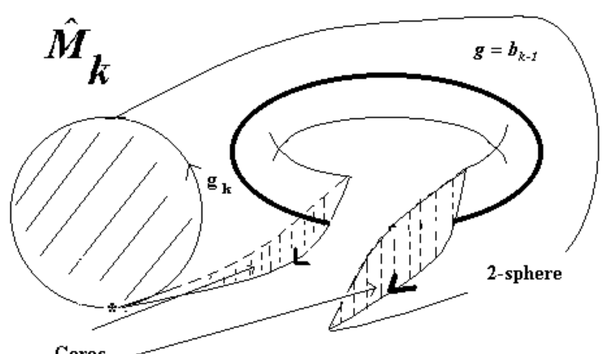

Figure 4

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PROBLEM SESSION

1. **Lawrence Brenton**
   (a) Let X be the cone on a homology 3-sphere M. Does there exist a Lorentzian metric g on X that is homogeneous on cross sections such that (X, g) satisfies the dominant energy condition?
   (b) If "no," where does the obstruction lie?
   (c) Will the spacetimes of part (a) always recollapse in a "big crunch," or does this depend on the choice of metric?

2. **Robert Daverman**
   (a) If X is a compact ANR homology 3-manifold, does there exist a real 3-manifold M such that M is homotopy equivalent to X?
   (b) If so, does X embed in $M \times \mathbb{R}$?
   (c) If so, is $X \times \mathbb{R} \cong M \times \mathbb{R}$?

3. **David Wright**
   Are there examples of compact 3-manifolds (or n-manifolds) in which every homeomorphism is isotopic to the identity?

4. **Tadek Dobrowolski**
   Let X be a contractible, locally contractible compact metric space. Does X have the fixed point property?
   The answer is known to be "yes" if there exists a function $\lambda : X \times X \times [0,1] \to X$ such that
   $$\lambda(x, y, 0) = x,$$
   $$\lambda(x, y, 1) = y, \text{and}$$
   $$\lambda(x, x, t) = x \text{ for } 0 \leq t \leq 1.$$
   Every AR has such a function.

5. **Steve Ferry**
   Is there a sequence of Riemannian manifolds, sharing a fixed contractibility function, that approach (in Gromov-Hausdorff space) an infinite dimensional space with a bound on volume?
   Definitions: A *contractibility function* on M is a function $\rho : (0, \infty) \to (0, \infty)$ such that for every $t > 0$ and for every $x \in M$

the ball of radius t in M centered at x is contractible in the ball of radius $\rho(t)$. If X and Y are compact metric spaces, the *Gromov-Hausdorff distance* $d_{\mathrm{GH}}(X,Y)$ is defined by

$$d_{\mathrm{GH}}(X,Y) = \inf\left\{ d^Z(X,Y) \;\middle|\; Z^{\text{metric space}} \supset X, Y \right\},$$

where d^Z is the usual Hausdorff distance between subcompacta of Z.

6. **Craig Guilbault**
Given a homomorphism $\mu : G \to \pi_1(M)$, with G a finitely generated group and M a closed manifold, such that $\ker(\mu)$ is perfect, does there exist a 1-sided h-cobordism that realizes μ? In other words, does there exist a triple (W, M, M^*) of manifolds such that $\partial W = M \sqcup M^*$, $M \hookrightarrow W$ is a homotopy equivalence, and

$$\begin{tikzcd} \pi_1(M^*) \arrow[r] & \pi_1(W) \\ G \arrow[u, "\approx"] \arrow[r, "\mu"] & \pi_1(M) \arrow[u, "\approx"'] \end{tikzcd}$$

commutes? [This is the reverse of Quillen's +-construction.]

7. **Sasha Dranishnikov**
   (a) Is $\operatorname{asdim}(X) = \dim(\nu X)$?
   (b) If Γ is a CAT(0) group, is $\operatorname{asdim}(\Gamma) < \infty$?
   (c) For $n \geq 2$, does there exist a Coxeter group Γ such that $\operatorname{vcd}_{\mathbb{Q}} \Gamma = 2$ and $\operatorname{vcd}_{\mathbb{Z}} \Gamma = n$?