40th Workshop in Geometric Topology – 2023, Colorado College

Problem Session – Saturday, June 17

Ethan Duglie. [ethan_dlugie@brown.edu] In his paper *Factor Groups of the Braid Group*, Coxeter considered the quotients of braid groups $B_n(d) \coloneqq B_n/\langle\langle \sigma_i^d \rangle\rangle$. Coxeter showed that these quotients are finite if and only if the pair (n,d) falls in one of two infinite families or is one of five exceptional cases. Coxeter also gave a remarkable formula for the order of the quotient group in terms of the combinatorics of a *Platonic solid* associated to the pair (n,d). Coxeter proves this striking formula by computing the orders of the finite $B_n(d)$ using various finite group theory techniques (e.g. coset enumeration) and checking that the formula works in each of the cases (the two infinite families and the five exceptional cases).

QUESTION: Is there a proof of Coxeter's incredible formula that actually makes clear the surprising appearance of the Platonic solids?

Ric Ancel. [ancel@uwm.edu] 1) Let M be a compact topological n-manifold. Make the homeomorphism group Homeo(M) of M with the compact open topology into a moduli space to answer:

QUESTION: Is Homeo(M) a Hilbert space manifold?

This is known to be true for $n \leq 2$. It is also known that Homeo(M) is a Hilbert space manifold if and only if it is an absolute neighborhood retract.

2) It is known that that in every dimension $n \geq 3$, every embedding of the $(n-1)$-sphere S^{n-1} in the n-sphere S^n can be approximate by tame embeddings. More precisely, let S^n be the unit sphere in Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$, let $S^{n-1} = S^n \cap (\mathbb{E}^n \times \{0\})$, let $\mathrm{Emb}(S^{n-1},S^n)$ denote the space of all topological embeddings of S^{n-1} in S^n, with the compact open topology, and let $\mathrm{Homeo}(S^n)$ denote the homeomorphism group of S^n. Let α denote the action of $\mathrm{Homeo}(S^n)$ on $\mathrm{Emb}(S^{n-1},S^n)$ by composition on the left. An embedding of $e : S^{n-1} \to S^n$ is *tame* if it is an element of the orbit of the inclusion $i : S^{n-1} \to S^n$ under the action α. The first sentence of this problem can be paraphrased by saying the orbit of $i : S^{n-1} \to S^n$ under α is dense in $\mathrm{Emb}(S^{n-1},S^n)$.

QUESTION: Is the orbit of every embedding of S^{n-1} in S^n dense in $\mathrm{Emb}(S^{n-1},S^n)$ under α?

For instance, can every embedding of S^2 in S^3 be approximated by copies of the Alexander Horned Sphere? The answer to the last question (about the Alexander Horned Sphere) is “yes”. In fact, it can easily be shown that the orbit under α of an embedding $e : S^{n-1} \to S^n$ is dense in $\mathrm{Emb}(S^{n-1},S^n)$ if e has a *flat spot;* i.e., there is a non-empty open subset U of S^{n-1} and an embedding $e' : U \times \mathbb{R} \to S^n$ such that $e'(x,0) = e(x)$ for every $x \in U$. So a positive answer to this question is equivalent to a negative answer to the question: Is there an embedding of S^{n-1} in S^n that has no flat spot whose orbit under α is not dense in $\mathrm{Emb}(S^{n-1},S^n)$?

3) Let K be an abstract simplicial complex with vertex set V. In other words, K is a set of finite subsets of V satisfying: 1) $\{v\} \in K$ for every $v \in V$, and 2) $\sigma \subset \tau \in K \Rightarrow \sigma \in K$. The *polyhedron underlying K* is the set

$$|K| = \{ \phi \in \mathbb{R}^V : \phi(V) \subset [0,\infty),\ \phi^{-1}(0,\infty) \in K \text{ and } \Sigma_{v \in V}\phi(v) = 1 \}.$$

(Here, $\mathbb{R}^V$ denotes the set of all functions from V to $\mathbb{R}$.) For $\sigma \in K$, let $|\sigma| = \{ \phi \in |K| : \phi^{-1}(0,\infty) \subset \sigma \}$. The *metric topology* on |K| is the topology determined by the metric $\rho(\phi,\psi) = \Sigma_{v \in V} | \phi(v) - \psi(v) |$. For each $\sigma \in K$, assume $|\sigma|$ has the metric topology and define the *Whitehead topology on |K|* to be

$$\{ U \subset |K| : U \cap |\sigma| \text{ is an open subset of } |\sigma| \text{ for every } \sigma \in K \}.$$

Denote the Whitehead topology on |K| by $\mathcal{T}_W$. Recall that the *box topology on $\mathbb{R}^V$* is the topology that has a basis consisting of all subsets of $\mathbb{R}^V$ of the form $\Pi_{v \in V} U_v = \{ \phi \in \mathbb{R}^V : \phi(v) \in U_v \text{ for every } v \in V \}$ where U_v is an open subset of $\mathbb{R}$ for every $v \in V$. Call the restriction of the box topology on $\mathbb{R}^V$ to |K| the *restricted box topology on |K|* and denote it by $\mathcal{T}_B$.

QUESTION: Is $\mathcal{T}_W = \mathcal{T}_B$? The answer appears to be "yes" if V is countable.

Update: Ancel's subsequent research and a literature search subsequent to the Workshop Problem Session turned up further information about this question:

The article [SY] cited below considered this question previously. One of the main results of [SY] is a proof that |K| with restricted box topology is an absolute neighborhood extensor for the class of metrizable spaces. Along the way they made discoveries that overlapped with facts that Ancel later discovered in his own researches.

1) In general, $\mathcal{T}_W \supset \mathcal{T}_B$. A simple example in which $\mathcal{T}_W \neq \mathcal{T}_B$ occurs if V is an uncountable set and K is the collection of all finite subsets of V. (This example does not appear in [SK]).

2) **Theorem.** The Whitehead and restricted box topology on |K| coincide if K satisfies the following *well-ordering condition:* for every $v \in V$, there is a well-ordering $<_v$ of $W(v) = \{ w \in V : w \neq v \text{ and } \{v,w\} \in K \}$ such that for every $w \in W(v)$, there are at most finitely many $\tau \in L(v) = \{ \tau \in K : v \notin \tau \text{ and } \{v\} \cup \tau \in K \}$ such that $\max_{<_v}(\tau) = w$. (This theorem does not appear in [SY].)

This theorem has several immediate corollaries that do appear in [SY]:

- If $\dim(K|) \leq 1$, then $\mathcal{T}_W = \mathcal{T}_B$.
- If K is *locally countable* (i.e., W(v) is a countable set for every $v \in V$), then $\mathcal{T}_W = \mathcal{T}_B$.

This verifies the conjecture in the original question: if V is countable, then $\mathcal{T}_W = \mathcal{T}_B$.

3) An example of a 2-dimensional complex K for which $\mathcal{T}_W \neq \mathcal{T}_B$ is easily described. Let $V = \{0\} \sqcup \mathbb{N} \sqcup \mathbb{R}$ where $\mathbb{N}$ denotes the positive integers and $\mathbb{R}$ denotes the real numbers. For each $(n,t) \in \mathbb{N} \times \mathbb{R}$, let $\sigma(n,t) = \{ 0, n, t \} \subset V$. Now define the 2-dimensional abstract simplicial complex K with vertex set V by $K = \{ \tau \subset \sigma(n,t) \} : (n,t) \in \mathbb{N} \times \mathbb{R}$ and $\tau \neq \emptyset \}$. A version of this example appeared first in [SY].

In view of these observations, we replace our original question by the following.

QUESTION: If K is an abstract simplicial complex with vertex set V in which $\mathcal{T}_W = \mathcal{T}_B$, does this imply that K satisfies the well-ordering condition?

[SY] K. Sakai and H. Yang, The box topology of infinite simplicial complexes, *Tsukuba Journal of Mathematics* **36** (2012), 295-309.