34^{th} Workshop in Geometric Topology – 2017, Brigham Young University

Problem Session – Saturday, June 10

Greg Conner. [conner@mathematics.byu.edu] Let $\mathbb{N} = \{ 1, 2, 3, \ldots \}$.

CONJECTURE: If X is a compact, connected, locally connected subset of $\mathbb{R}^3$, then either

$$H_1(X) \approx \mathbb{Z}^a \qquad \text{where } a \in \{ 0 \} \cup \mathbb{N}$$

or

$$H_1(X) \approx \left(\mathbb{Z}^{\mathbb{N}} / \bigoplus_{\mathbb{N}} \mathbb{Z} \right) \oplus \mathbb{Z}^a \qquad \text{where either } a \in \{ 0 \} \cup \mathbb{N} \text{ or } a = \mathbb{N}.$$

Craig Guilbault. [craigg@uwm.edu] QUESTION 1: Does the Mazur 4-manifold contain a pair of disjoint spines?

NOTE: The above question provides a nice entry into a collection of related questions. In all dimensions > 4 there exist compact contractible manifolds, not homeomorphic to an n-ball, which contain disjoint pairs of spines. But the following is also open:

QUESTION 2: Does there exist a compact contractible manifold in any dimension that does not contain a pair of disjoint spines?

Eric Swenson. [eric@mathematics.byu.edu] DEFINITION: Let p be a point in a continuum X. p is a *cut point* if X – { p } is not connected. p is a *weak cut point* if there are distinct points in X – { p } that are not contained in a continuum in X – { p }.

CAT(0) group boundaries are known have no cut points

EXAMPLE: The dyadic solenoid has no cut points, but it has weak cut points.

CONJECTURE: If Z is a connected boundary of a CAT(0) group, then Z has no weak cut points.

This would imply that CAT(0) groups are semi-stable at infinity.

Ric Ancel. [ancel@uwm.edu] DEFINITION: A contractible open n-manifold is *splittable (in the sense of Gabai)* if it can be written as a union $U \cup V$ of open sets U and V such that U, V and $U \cap V$ are homeomorphic to $\mathbb{R}^n$.

QUESTION 1: Is the interior of the Mazur compact contractible 4-manifold splittable?

QUESTION 2: Is every contractible open 4-manifold splittable?

CONJECTURE: For $n \geq 5$, every contractible open n-manifold is splittable.

BACKGROUND:

1) D. Gabai showed that the Whitehead contractible open 3-manifold is splittable. (The Whitehead manifold is a union of two Euclidean spaces, *J. Topology* **4** (2011), 529-534)
2) D. Garity, D. Repovs and D. Wright showed that there exist uncountably many non-homeomorphic splittable contractible open 3-manifolds and there exist uncountably many non-homeomorphic non-splittable contractible open 3-manifolds. (Contractible 3-manifolds and the double 3-space property, *Trans. AMS* **370** (2018), 2039-2055)
3) Pete Sparks showed there exist uncountably many non-homeomorphic splittable contractible open 4-manifolds. (The double n-space property for contractible n-manifolds, *Algebra. Geom. Topol.* **18** (2018), 2131-2149)
4) Let $n \geq 5$. Ancel and Guilbault (unpublished) have proved that the interior of every compact contractible n-manifold is splittable, and every n-dimensional Davis manifold is splittable Furthermore, every contractible open n-manifold can be written as a union $U \cup V$ of open sets U and V such that U and V are homeomorphic to $\mathbb{R}^n$ and $U \cap V$ is contractible. No example of a non-splittable contractible open n-manifold is known.
5) Sparks and Ancel showed that the Dunce Hat spine of the Mazur 4-manifold is *not* splittable in the sense that it can't be expressed as the union of two compact contractible proper subpolyhedra. (Most unexposed taut one-relator presentation 2-complexes are finitely unsplittable, *arXiv:1907.06742* (2019))

Remark. If a spine of the Mazur 4-manifold were splittable, then the interior of the Mazur 4-manifold would also be splittable. However, the converse doesn't hold: the non-splittability of a spine of a compact manifold does not imply the non-splittability of the interior of the manifold. Indeed, the Dunce Hat can be embedded in 3-space where it is the spine of a 3-ball; the Dunce Hat is non-splittable but the interior of a 3-ball is splittable. So the result stated in 5) doesn't resolve Question 1.

Mike Mihalik. [michael.l.mihalik@Vanderbilt.Edu] Let G be a 1-ended finitely presented group. Let X be any finite complex with $\pi_1(X) \approx G$. Let X* be the universal cover of X. A *strong end* of X* is a proper homotopy class of proper rays in X*. (A *proper ray* in X* is a proper map from $[0,\infty)$ to X*.) Let SE(G) denote the set of strong ends of X*.

OBSERVATION: If G is semistable, then SE(G) is a one-point set.

QUESTION: Is there a "natural" topology on SE(G).

NOTE: G acts on SE(G). A "natural" topology on SE(G) should make the action of G on SE(G) continuous.