21st Workshop in Geometric Topology – 2004, University of Wisconsin-Milwaukee

Problem Session – Saturday, June 12

Bob Edwards. [rde90272@gmail.com] A subset $K \subset \mathbb{R}^n$ has the *isotopy extension property* if every isotopy $f_t : K \to \mathbb{R}^n$, $0 \le t \le 1$, $f_0 = \mathrm{id}_K$, extends to an ambient isotopy $F_t : \mathbb{R}^n \to \mathbb{R}^n$, $0 \le t \le 1$, $F_0 = \mathrm{id}_{\mathbb{R}^n}$, $F_t \mid K = f_t$ for $0 \le t \le 1$.

QUESTION (Paul Fabel): Suppose $K \subset \mathbb{R}^2$ is compact and path connected. Does K have the *isotopy extension property*?

The answer is "no" for a knotted simple closed curve in $\mathbb{R}^3$: "pull out" the knot. It is known that the answer is "yes" for Peano continua in $\mathbb{R}^2$, "no" for Cantor sets in $\mathbb{R}^2$, and "yes" in other special cases like the cone on a Cantor set in $\mathbb{R}^2$.

David Wright. [wright@math.byu.edu] A Cantor set C in $\mathbb{E}^n$ is *strongly homogeneously embedded* if every homeomorphism from C to itself extends to a homeomorphism from $\mathbb{E}^n$ to itself.

Daverman has constructed examples of wild strongly homogeneously embedded Cantor sets in $\mathbb{E}^n$ for $n \ge 5$.

QUESTION: Is every strongly homogeneously embedded Cantor set in $\mathbb{E}^3$ tame?

Jim Cannon. [jmcnnn@gmail.com] Let $J_1, J_2, \cdots, J_k$ be disjoint simple closed curves in the boundary Σ of an orientable handlebody H. Perform a right-handed Dehn twist about each J_i to obtain a homeomorphism $\phi : \Sigma \to \Sigma$. Let $M^3 = H \cup_\phi H$.

QUESTION 1: Can this process yield $M^3 = T^3$?

QUESTION 2: Can this process yield all orientable 3-manfolds?

QUESTION 3: In a triangular billiard table, is there always a closed orbit?

REMARK: This is always true if all the angles are acute. What about if there is an obtuse angle?

Jim Conant. [jim.conant@gmail.com] Let $\mathcal{FD}$ denote the algebra of Feynman diagrams as described in [CT]. Then, $\mathcal{FD} = \cup_{k>0}\mathcal{FD}_k$.

QUESTION: If k is odd, it $\mathcal{FD}_k = 0$?

REMARK: This would imply that the Vassiliev invariants can't detect orientation reversal of knots.

[CT] J. Conant and P. Teichner, Grope cobordism and Feynman diagrams, *Math. Ann.* **328** (2004), 135-171.

Bob Daverman. [rjdaverman@gmail.com] Let Γ be a finite group.

QUESTION 1: Does there exist a compact manifold M^n and a regular covering $\Theta : M^n \to M^n$ such that $\Gamma = \pi_1(M^n)/\Theta_*(\pi_1(M^n))$? What about in the case that n = 3?

QUESTION 2: Given a crumpled 3-cube C, can $\mathbb{R}^3$ be tiled with mutually congruent copies of C?

REMARK: The answer to Question 2 is "yes" if C is locally collared at some point.

Ric Ancel. [ancel@uwm.edu] In [B], Bing proved that every Peano continuum has a convex metric. A *Peano continuum* is a compact connected locally connected metrizable space (or, equivalently, a compact path connected, locally path connected metrizable space). A *convex metric* on a metrizable space X is a metric d on X such that for all x, y $\in$ X, there is an isometric embedding $([0,d(x,y)],0,d(x,y)) \to (X,x,y)$ (or, equivalent, a metric in which any two points have a midpoint). A metric on a space X is *locally rectifiable* if for every for every $\epsilon > 0$, there is a $\delta > 0$ such that any two δ-close points in X can be joined by a path of length $< \epsilon$.

FACT: If a space X has a locally rectifiable metric, then the corresponding path-length metric on X is a convex metric.

QUESTION: Can Bing's result be given an alternate proof and generalization by showing that if X is a Peano continuum, then any embedding of X in the Hilbert cube $\mathbb{I}^\infty$ with a given metric d can be approximated by an embedding so that the restriction of d to the new image of X is locally rectifiable?

[B] R. H. Bing, Partitioning a set, *Bull. AMS* **55** (1949), 1101-1110.

REMARK: Bing's proof in [B] is accomplished by directly modifying the given metric on the Peano continuum on a set-theoretic level. This question seeks a "more geometric" and less set-theoretic proof of Bing's result.

Greg Conner [conner@math.byu.edu] **and Jack Lamoreux.** A Peano continuum X has the *double midpoint property* if for all x, y $\in$ X such that x $\neq$ y, there are exactly two elements z of X such that d(x,z) = d(y,z).

CONJECTURE: If a Peano continuum X has the double midpoint property, then X is homeomorphic to S^1.

Tom Thickstun [tt04@txstate.edu]

QUESTION: What open 3-manifolds embed in S^3?

An open manifold U is *1-acyclic at ∞* if for every compact subset C of U, there is a compact subset D of U such that C $\subset$ D and the inclusion induced homomorphism $H_1(U – D) \rightarrow H_1(U – C)$ is zero.

The following relevant theorem is proved in [T].

THEOREM: Modulo the Poincare Conjecture, a 1-acyclic at ∞ open 3-manifold U embeds in S^3 if and only if there is a degree one map from S^3 to the endpoint compactification of U.

[T] T. L. Thickstun, Strongly acyclic maps and homology 3-manifolds with 0-dimensional singular set, *Proc. London Math. Soc.* **55** (1987), 378-432.