Description of Lectures

<u>Lecture 1:</u> **Elliptic curves and $\mathbf{SL_2}$.**
This lecture will be an overview of the classical theory of elliptic curves and automorphic forms from Hodge-theoretic and representation-theoretic perspectives. This is the basic example of the *classical case* of Shimura varieties and has provided a model for the development of the subject. However, this model has to be significantly modified in the non-classical case, which will be the main topic of these lectures.

Supporting lecturer: Matt Kerr

<u>Lecture 2:</u> **Hodge structures and Mumford-Tate groups.**
This lecture will introduce, and give the basic properties and examples of, the main concepts of *polarized Hodge structures* (PHSs), *Mumford-Tate groups*, and *variations of Hodge structure* (VHS). The *classical case* is PHSs of weight one (abelian varieties). The main difference between the *non-classical* higher weight case and the classical case is that in the former a general PHS is *non-motivic*, i.e. it does not arise from algebraic geometry. However, the *period domains*, and more general *Mumford-Tate domains*, do have a rich complex geometric and arithmetic structure. They also have a significant relation to representation theory, and possibly to the arithmetic aspects of representation theory. An important and central arithmetic aspect of Hodge theory, namely *complex multiplication* (CM) PHSs, will be introduced and discussed.

<u>Lecture 3:</u> **Mumford-Tate domains.**
Mumford-Tate domains parametrize PHSs of a fixed Hodge type whose generic member has a given reductive Q-algebraic group as its Mumford-Tate group. This class of homogenous complex manifolds, and the natural vector bundles over them, have very special complex analytic and differential geometric properties that will be introduced and illustrated in this lecture. Of particular importance will be the duality (Matsuki duality) between the compact and non-compact orbit structures in the associated flag domains.

Supporting lectures: Jim Carlson

<u>Lecture 4:</u> **Hodge representations and Hodge domains.**
This lecture will focus on two main questions:

Which Q-algebraic groups may be realized as the Mumford-Tate groups of a polarized Hodge structure?

For a given such group, in how many ways can it be realized as a Mumford-Tate group of a polarized Hodge structure?

These questions have recently been answered. The key concept is that of a Hodge representation, which basically inverts the two questions in that one begins with the group and not with the Hodge structures. The development of this theory, which involves new analysis of some properties of finite-dimensional representations of semi-simple, real Lie groups, leads to the fundamental concept of a Hodge domain. Mumford-Tate domains are then Hodge domains associated to a particular Hodge representation. There is also now a complete classification of Hodge domains and this will be explained in the lecture.

Supporting lectures: Mark Green

Lecture 5: Discrete series and automorphic cohomology.
Schmid showed that *discrete series representations* of semi-simple real Lie groups may be realized as L^2-cohomology of homogeneous bundles over Hodge domains. This result will be discussed in the lecture, together with related topics such as the geometric description of the K-type. *Automorphic cohomology* will be introduced and its expression in terms of $\mathbf{n}$-*cohomology* due to Schmid and Williams presented, including the adelic version due to Carayol. The discussion will include the extension of the above to the important case of *non-degenerate limits of discrete series*, as well as some of what at this time is understood about degenerate limits of discrete series, the work of Carayol being the one non-classical case that seems to be well understood.

Lecture 6: Cycle and correspondence spaces.
Associated to Mumford-Tate domains and their compact duals are a remarkable class of related complex manifolds. These cycle and correspondence spaces have a universality property arising from Matsuki duality. These spaces will be introduced and their basic properties explained and illustrated. They are of particular importance in the non-classical case where there are familites of positive-dimensional compact subvarieties and the enhanced flag varieties lying over the individual compact subvarieties; these both have a very rich geometric structure.

Lecture 7: The Eastwood-Gindikin-Wang [EGW] theorem.
This result gives a general way of realizing sheaf cohomology over complex manifolds by global, holomorphic data over an associated complex manifold. In the homogenous case the associated complex manifolds are the correspondence spaces and this realization has very special features, essentially a holomorphic harmonic theory. Explaining these leads again to n-cohomology and its relation to the Borel-Weil-Bott

[BWB] theorem.

<u>Lecture 8:</u> Penrose transforms.
Penrose transforms give a general method, using correspondence spaces, of relating different cohomology groups on the same or on different complex manifolds. In the compact case they give a geometric way of realizing the isomorphisms in the BWB theorem. In the non-compact case, as illustrated first by Carayol, they may be used to relate classical and non-classical automorphic cohomology groups over quotients of Mumford-Tate domains.

<u>Lecture 9:</u> The cases of $\mathrm{SU}(2,1)$ and $\mathrm{Sp}(4)$.
The general theory has an especially rich geometric, arithmetic and representation-theoretic interpretation in these two special cases. They will have been discussed several times earlier in the lectures. Here we shall give the proofs of the main results on Penrose transforms in the non-compact case and the arithmetic aspect of non-classical automorphic cohomology, including the deep results of Carayol for $\mathrm{SU}(2,1)$ and its extension to $\mathrm{Sp}(4)$.

<u>Lecture 10:</u> Summary and potential areas for further work.
This lecture will summarize and fill in missing points from the earlier ones. I will also offer some speculations on where, and perhaps how, the theory might be further developed. Perhaps most importantly, already at the early stage in the preparation of the material for these proposed lectures, a significant number of interesting, specific open questions have arisen. For the most part, the theory needed to address these questions is largely available; it just has not yet been brought to bear on the issues at the interface of Hodge theory and representation theory that are presented in these lectures. Especially for students and young mathematicians, these questions should provide a potential set of topics to work on.