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Representable E-Theory for $C_0(X)$-Algebras

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Let X be a locally compact space, and let A and B be $C_0(X)$-algebras. We define the notion of an asymptotic $C_0(X)$-morphism from A to B and construct representable E-theory groups $\mathscr{R}E(X; A, B)$. These are the universal groups on the category of separable $C_0(X)$-algebras that are $C_0(X)$-stable, $C_0(X)$-homotopy-invariant, and half-exact. If A is $\mathscr{R}KK(X)$-nuclear, these groups are naturally isomorphic to Kasparov's representable KK-theory groups $\mathscr{R}KK(X; A, B)$. Applications and examples are also discussed. © 2000 Academic Press

1. INTRODUCTION

Let X be a locally compact space. A C^*-algebra A is called a $C_0(X)$-algebra if it is equipped with a nondegenerate action of $C_0(X)$ as commuting multipliers of A. These algebras have many important applications since they can be represented as the sections of a noncommutative bundle of C^*-algebras over X [KW95, EW97]. In particular, any continuous-trace C^*-algebra with spectrum X is a $C_0(X)$-algebra [RW98]. There is a substantial amount of literature on $C_0(X)$-algebras; see the general treatments in [Bla96, Nil96].

In 1988, Kasparov [Kas88] defined representable KK-theory groups $\mathscr{R}KK(X; A, B)$, which extend his bivariant KK-theory groups $KK(A, B)$ (where $X = \bullet$) and the representable topological K-theory $RK^0(X)$ of Segal

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[Seg70]. These groups (and their equivariant versions) have found many applications in geometry, topology and index theory, most notably in Kasparov's proof of the Novikov Conjecture for closed discrete subgroups of finite component Lie groups [Kas88]. However, these groups are difficult to compute since they have six-term exact sequences only for split-exact sequences of $C_0(X)$-algebras [Bau98].

In 1989, Connes and Higson [CH89] defined E-theory groups $E(A, B)$ based on asymptotic morphisms from A to B. An asymptotic morphism $\{\phi_t\}: A \to B$ is a one-parameter family of maps which satisfy the properties of a $*$-homomorphism in the limit as $t \to \infty$. There is a natural notion of homotopy for asymptotic morphism for which a composition product is well-defined. If A is separable, the functor $B \mapsto E(A, B)$ is the universal functor on the category of separable C^*-algebras B which is stable, homotopy-invariant and half-exact. Thus, there are six-term exact sequences in both variables for any exact sequence of separable C^*-algebras. There is a natural transformation $KK(A, B) \to E(A, B)$, which is an isomorphism if A is K-nuclear.

In this paper, we extend these constructions to the category of separable $C_0(X)$-algebras by defining a notion of asymptotic $C_0(X)$-morphism between two $C_0(X)$-algebras A and B. There is a corresponding notion of $C_0(X)$-homotopy for an asymptotic $C_0(X)$-morphism for which composition is well-defined. This leads us to define representable E-theory groups $\mathscr{R}E(X; A, B)$. If X is second countable then these groups have a composition product

$$\mathscr{R}E(X; A, B) \otimes \mathscr{R}E(X; B, C) \to \mathscr{R}E(X; A, C),$$

and we prove that these are the universal such groups on the category of separable $C_0(X)$-algebras that are $C_0(X)$-stable, $C_0(X)$-homotopy-invariant, and half-exact. We also prove that there is a natural transformation

$$\mathscr{R}KK(X; A, B) \to \mathscr{R}E(X; A, B),$$

which is an isomorphism if A is $\mathscr{R}KK(X)$-nuclear (in the sense of Bauval [Bau98].) If $X = \bullet$ is a point, we recover the original E-theory groups of Connes–Higson $E(A, B) \cong \mathscr{R}E(\bullet; A, B)$.

In addition to these properties, which generalize the basic properties of E-theory, we show that $\mathscr{R}E$-theory also has the following properties in common with $\mathscr{R}KK$. For $C_0(X)$-algebras A, B, C, D, there is a tensor product operation

$$\mathscr{R}E(X; A, B) \otimes \mathscr{R}E(X; C, D) \to \mathscr{R}E(X; A \otimes_X C, B \otimes_X D),$$

where $A \otimes_X B$ denotes the balanced tensor product over $C_0(X)$ [EW97, Bla95]. If $p\colon Y \to X$ is a continuous map, the pullback construction $A \mapsto p^*A$ of Raeburn–Williams [RW85] (which converts a $C_0(X)$-algebra A to a $C_0(Y)$-algebra p^*A) induces a natural transformation of functors

$$p^*\colon \mathscr{R}E(X; A, B) \to \mathscr{R}E(Y; p^*A, p^*B).$$

In Section 2 we review some definitions and constructions involving $C_0(X)$-algebras that we will need later. We define $\mathscr{R}E$-theory in Section 3. The category-theoretic tools for comparing $\mathscr{R}E$-theory and Kasparov's $\mathscr{R}KK$-theory are developed in Section 4. Finally, in Section 5 we give some examples and applications. For unital $C_0(X)$-algebras, we define the notion of a "fundamental class" in $\mathscr{R}E$-theory. Also, $\mathscr{R}E$-elements are associated to families $\{D_x\}_{x \in X}$ of elliptic differential operators parametrized by the space X. We also discuss invariants of central bimodules, which have found recent applications in noncommutative geometry and physics.

2. REVIEW OF $C_0(X)$-ALGEBRAS

Let X be a locally compact Hausdorff topological space.

DEFINITION 2.1. A C^*-algebra A is called a *central Banach $C_0(X)$-module* if A is equipped with a $*$-homomorphism $\Phi\colon C_0(X) \to ZM(A)$ from $C_0(X)$ into the center of the multiplier algebra of A. If $\Phi(C_0(X)) \cdot A$ is dense in A, we say that Φ is *nondegenerate*, and we call A a *$C_0(X)$-algebra*. We shall usually write $f \cdot a$ for $\Phi(f)\, a$.

Note that if A is a central Banach $C_0(X)$-module then $A^X =_{\text{def}} \overline{\Phi(C_0(X)) \cdot A}$ is the maximal subalgebra of A that is a $C_0(X)$-algebra. Compare the following lemma to Definition 1.5 [Kas88], which assumes σ-compactness of X.

LEMMA 2.2. *The following are equivalent*:

(i) *A is a $C_0(X)$-algebra*;

(ii) *There is a $*$-homomorphism $\Phi\colon C_0(X) \to ZM(A)$ such that for any approximate unit $\{f_\lambda\}$ in $C_0(X)$ we have $\lim_{\lambda \to \infty} \|\Phi(f_\lambda)\, a - a\| = 0$ for all $a \in A$.*

Examples of $C_0(X)$-algebras include C^*-algebras with Hausdorff spectrum X; in particular, any continuous-trace C^*-algebra is a $C_0(X)$-algebra [RW98]. Also, the proper algebras occurring in the Baum–Connes Conjecture are examples [BCH94, GHT]. In general, a $C_0(X)$-algebra can be realized as the algebra of sections of an upper-semicontinuous C^*-bundle over X which vanish at infinity [KW95, EW97]. See the general discussions in [Bla96, Nil96].

DEFINITION 2.3. Let A and B be $C_0(X)$-algebras (or central Banach $C_0(X)$-modules). A $*$-homomorphism $\psi: A \to B$ is called $C_0(X)$-*linear* if $\psi(f \cdot a) = f \cdot \psi(a)$ for all $f \in C_0(X)$ and $a \in A$. We call such a homomorphism a $C_0(X)$-*morphism*. We define $\mathbf{SC^*(X)}$ to be the category of all separable $C_0(X)$-algebras and $C_0(X)$-morphisms, and let $\mathrm{Hom}_X(A, B)$ denote the set of $C_0(X)$-morphisms from A to B.

LEMMA 2.4. *Let A be a $C_0(X)$-algebra and B be a Banach $C_0(X)$-module. If $\psi: A \to B$ is a $C_0(X)$-morphism, then $\psi(A) \subset B^X$, i.e., $\mathrm{Hom}_X(A, B) = \mathrm{Hom}_X(A, B^X)$.*

Proof. Since $\psi(f \cdot a) = f \cdot \psi(a) \in B^X$ for all $f \in C_0(X)$ and $a \in A$, it follows that $\psi(C_0(X) \cdot A) \subset B^X$. Since the range of ψ is closed and $C_0(X) \cdot A$ is dense in A, the result follows. Q.E.D

An easy to prove, but important, result is the following:

LEMMA 2.5. *If A is a $C_0(X)$-algebra, then under the maximal tensor product, $A \otimes B$ is a $C_0(X)$-algebra for any C^*-algebra B, with the $C_0(X)$-action given by the formula $f \cdot (a \otimes b) = (f \cdot a) \otimes b$.*

In this paper, we will only use the maximal tensor product, since it is the appropriate tensor product when working with asymptotic morphisms.

Let B be any C^*-algebra and let $\phi: B \to A$ be a $*$-homomorphism into a $C_0(X)$-algebra A. By the universal property of the maximal tensor product, the map $f \otimes a \mapsto f \cdot \phi(a)$ defines a unique $C_0(X)$-morphism $\phi_X: C_0(X) \otimes B \to A$.

Let $\mathscr{K}$ be the C^*-algebra of compact operators on separable Hilbert space. A $C_0(X)$-algebra A is called $C_0(X)$-*stable* if $A \cong A \otimes \mathscr{K}$ as $C_0(X)$-algebras (see the example following Corollary 6.10 of [RW98] for two $C_0(X)$-algebras with the feature that $A \otimes \mathscr{K}$ and $B \otimes \mathscr{K}$ are isomorphic as C^*-algebras, but not as $C_0(X)$-algebras.)

PROPOSITION 2.6. *The category $\mathbf{SC^*(X)}$ is closed under the operations of taking ideals, quotients, direct sums, suspensions, and $C_0(X)$-stabilizations.*

Let A and B be $C_0(X)$-algebras. If $i_A: M(A) \to M(A \otimes B)$ and $i_B: M(B) \to M(A \otimes B)$ denote the natural inclusions, then $A \otimes B$ has two natural $C_0(X)$-algebra structures. Typically these $C_0(X)$-algebra structures do not agree. However, they can be made to agree by taking a quotient by an appropriate ideal:

Definition 2.7 [EW97, Bla95]. Let I_X be the closed $C_0(X)$-ideal of $A \otimes B$ generated by elementary tensors

$$\{f \cdot a \otimes b - a \otimes f \cdot b \colon f \in C_0(X), a \in A, b \in B\}.$$

Define $A \otimes_X B = (A \otimes B)/I_X$ to be the quotient C^*-algebra. It has a natural $C_0(X)$-structure given on the images $a \otimes_X b$ of elementary tensors $a \otimes b$ by

$$f \cdot (a \otimes_X b) = (f \cdot a) \otimes_X b = a \otimes_X (f \cdot b).$$

$A \otimes_X B$ is called the (maximal) $C_0(X)$-balanced tensor product of A and B.

Note that if A is a $C_0(X)$-algebra, then $C_0(X) \otimes_X A \cong A$ via the map $f \otimes_X a \mapsto f \cdot a$.

We note that there are other notions of a balanced tensor product, specifically, the minimal balanced tensor product $\otimes_X^m$. However, this tensor product is not associative (see 3.3.3 [Bla95]). In addition, the maximal balanced tensor product enjoys a very useful property that we describe below. We first observe that if C is a $C_0(X)$-algebra, then the multiplier algebra $M(C)$ is canonically a central Banach $C_0(X)$-module (but with a possibly degenerate $C_0(X)$-action).

Proposition 2.8 (Compare Prop 2.8 [EW97]). *Let A, B, and C be $C_0(X)$-algebras. If $\psi \colon A \to M(C)$ and $\phi \colon B \to M(C)$ are commuting $C_0(X)$-morphisms, then there exists a unique $C_0(X)$-morphism*

$$\psi \otimes_X \phi \colon A \otimes_X B \to M(C)$$

such that $(\psi \otimes_X \phi)(a \otimes_X b) = \psi(a)\, \phi(b)$.

Let $p \colon Y \to X$ be a continuous map of locally compact spaces. The pullback $\phi(g) = g \circ p$ defines a $*$-homomorphism $\phi \colon C_0(X) \to C_b(Y) = M(C_0(Y))$ and so gives $C_0(Y)$ a central Banach $C_0(X)$-module structure. For a detailed discussion of the following, see [RW85].

Definition 2.9. Let $p \colon Y \to X$ be a continuous map, and let A be a $C_0(X)$-algebra. The balanced tensor product $C_0(Y) \otimes_X A$ is called the *pullback C^*-algebra* and is denoted p^*A. Although it is a $C_0(X)$-algebra, it also has a natural $C_0(Y)$-action given by the canonical embedding $C_0(Y) \hookrightarrow M(C_0(Y) \otimes_X A)$.

If $A = \Gamma_0(E)$ is the C^*-algebra of sections of a C^*-bundle $E \to X$ which vanish at infinity, then p^*A is the C^*-algebra of sections of the pullback bundle $p^*E \to Y$ [RW85]. Note that if $p \colon Y \to \bullet$ is the map to a point, then $p^*A = C_0(Y) \otimes A$ for any C^*-algebra A.

COROLLARY 2.10. *Let $p: Y \to X$ be a continuous map. If $\psi: A \to B$ is a $C_0(X)$-morphism, there is a canonical $C_0(Y)$-morphism $p^*(\psi): p^*A \to p^*B$. In fact, the pullback construction defines a functor $p^*: \mathbf{SC^*(X)} \to \mathbf{SC^*(Y)}$ that is both additive and multiplicative.*

Let $B[0,1] = C([0,1], B) \cong B \otimes C[0,1]$ denote the C^*-algebra of continuous functions from the unit interval $[0,1]$ into B. It has a $C_0(X)$-algebra structure by Lemma 2.5, and the evaluation maps $\mathrm{ev}_t: B[0,1] \to B$ are $C_0(X)$-morphisms. Also, the inclusion $i_B: B \to B[0,1]$ by constant functions is a $C_0(X)$-morphism. Finally, the flip map $B[0,1] \to B[0,1]$: $f(t) \mapsto f(1-t)$ is also a $C_0(X)$-morphism.

DEFINITION 2.11. Let A and B be $C_0(X)$-algebras. Two $C_0(X)$-morphisms $\psi_0, \psi_1: A \to B$ are $C_0(X)$-*homotopic* if there is a $C_0(X)$-morphism $\Psi: A \to B[0,1]$ such that $\mathrm{ev}_i \circ \Psi = \psi_i$ for $i = 0, 1$. This defines an equivalence relation on the set of $C_0(X)$-morphisms from A to B. Let $[A, B]_X$ denote the set of $C_0(X)$-homotopy equivalence classes of $C_0(X)$-morphisms from A to B. We let $[\psi]_X$ denote the equivalence class of $\psi: A \to B$, and we define $\mathbf{SH(X)}$ to be the category of separable $C_0(X)$-algebras and $C_0(X)$-homotopy classes of $C_0(X)$-morphisms.

3. REPRESENTABLE E-THEORY

Let A and B be $C_0(X)$-algebras.

DEFINTION 3.1. As *asymptotic $C_0(X)$-morphism* from A to B is an asymptotic morphism $\{\psi_t\}_{t \in [1,\infty)}: A \to B$ which is asymptotically $C_0(X)$-linear, i.e.,

$$\lim_{t \to \infty} \|\psi_t(f \cdot a) - f \cdot \psi_t(a)\| = 0$$

for all $a \in A$ and $f \in C_0(X)$.

For a review of the basic properties of asymptotic morphisms of Connes and Higson, see the papers [CH89, CH90, GHT] and the books [Con94, Bla98]. A $C_0(X)$-morphism $\psi: A \to B$ determines a (constant) asymptotic $C_0(X)$-morphism by the formula $\psi_t = \psi$.

Recall that two asymptotic morphisms $\{\psi_t\}$, $\{\phi_t\}: A \to B$ are called *equivalent* if

$$\lim_{t \to \infty} \|\psi_t(a) - \phi_t(a)\| = 0$$

for all $a \in A$. It is then easy to see that any asymptotic morphism which is equivalent to an asymptotic $C_0(X)$-morphism is also asymptotically $C_0(X)$-linear. Let $[\![\psi_t]\!]^a_X$ denote the equivalence class of the asymptotic $C_0(X)$-morphism $\{\psi_t\}$, and let $[\![A, B]\!]^a_X$ denote the collection of all equivalence classes of asymptotic $C_0(X)$-morphisms from A to B.

Let $C_b([1, \infty), B)$ denote the continuous bounded functions from the ray $[1, \infty)$ to B. It has an induced structure as a central Banach $C_0(X)$-module under pointwise multiplication (note, however, that this action is typically degenerate). The ideal $C_0([1, \infty), B) \cong C_0([1, \infty)) \otimes B$ of functions vanishing at infinity is a $C_0(X)$-algebra by Lemma 2.5. Therefore, the quotient C^*-algebra

$$B_\infty = C_b([1, \infty), B)/C_0([1, \infty), B)$$

has a canonical central Banach $C_0(X)$-module structure.

PROPOSITION 3.2 (Compare [CH89]). *There is a one-to-one correspondence between equivalence classes of asymptotic $C_0(X)$-morphisms $\{\psi_t\}: A \to B$ and $C_0(X)$-morphisms $\Psi: A \to B^X_\infty$. In other words,*

$$[\![A, B]\!]^a_X \cong \mathrm{Hom}_X(A, B^X_\infty).$$

Proof. If $\{\psi_t\}: A \to B$ is an asymptotic $C_0(X)$-morphism, we obtain a $C_0(X)$-morphism $\Psi: A \to B_\infty$ by defining

$$\Psi(a) = q(\hat{\psi}(a)),$$

where $\hat{\psi}(a)(t) = \psi_t(a)$ and $q: C_b([1, \infty), B) \to B_\infty$ is the quotient map. It depends only on the asymptotic equivalence class $[\![\psi_t]\!]^a_X$. Now invoke Lemma 2.4. Conversely, given $\Psi \in \mathrm{Hom}_X(A, B^X_\infty)$ we obtain an asymptotic $C_0(X)$-morphism $\{\psi_t\}: A \to B$ by the composition

$$A \xrightarrow{\Psi} B^X_\infty \hookrightarrow B_\infty \xrightarrow{s} C_b([1, \infty), B) \xrightarrow{\mathrm{ev}_t} B,$$

where s is any set-theoretic section of q. Different choices of s give equivalent asymptotic $C_0(X)$-morphisms. Thus, $[\![A, B]\!]^a_X \cong \mathrm{Hom}_X(A, B^X_\infty)$. Q.E.D

It follows that the asymptotic equivalence class of an asymptotic $C_0(X)$-morphism is parametrized by the choices for the section s which inverts the projection $q: C_b([1, \infty), B) \to B_\infty$.

LEMMA 3.3. *Let $\{\psi_t\}: A \to B$ be an asymptotic $C_0(X)$-morphism. Then, up to equivalence, we may assume that $\{\psi_t\}$ has either one of the following properties (but not both):*

- *$\{\psi_t\}$ is an equicontinuous family of functions; or*
- *Each map ψ_t is $*$-linear.*

Proof. The first follows from the selection theorem of Bartle and Graves [BG52]. The second is realized by using a Hamel basis. Q.E.D

DEFINITION 3.4. Two asymptotic $C_0(X)$-morphisms $\{\psi_t^0\}: A \to B$ and $\{\psi_t^1\}: A \to B$ are $C_0(X)$-*homotopic* if there is an asymptotic $C_0(X)$-morphism $\{\Psi_t\}: A \to B[0, 1]$ such that $\mathrm{ev}_i \circ \Psi_t = \psi_t^i$ for $i = 0, 1$. We denote this equivalence by $\{\psi_t^0\} \sim_{hX} \{\psi_t^1\}$, and we let $[\![A, B]\!]_X$ denote the set of $C_0(X)$-homotopy classes $[\![\psi_t]\!]_X$ of asymptotic $C_0(X)$-morphisms $\{\psi_t\}: A \to B$.

From the discussion in Section 2, it is easy to see that $C_0(X)$-homotopy defines an equivalence relation on the set of asymptotic $C_0(X)$-morphisms from A to B. Furthermore, there is a canonical map

$$[A, B]_X \to [\![A, B]\!]_X.$$

In addition, if asymptotic $C_0(X)$-morphisms are equivalent, then they are actually $C_0(X)$-homotopic via the straight-line homotopy, i.e., there is a canonical map

$$[\![A, B]\!]_X^a \to [\![A, B]\!]_X.$$

LEMMA 3.5. *Let $\{\psi_t\}: A \to B$ be an asymptotic $C_0(X)$-morphism. If $r: [1, \infty) \to [1, \infty)$ is a continuous function such that $\lim_{t \to \infty} r(t) = \infty$, then $\{\psi_{r(t)}\}$ is an asymptotic $C_0(X)$-morphism that is $C_0(X)$-homotopic to $\{\psi_t\}$.*

Proof. It is easy to show that $\{\psi_{r(t)}\}$ is an asymptotic morphism. Let $a \in A$, $f \in C_0(X)$, and $\varepsilon > 0$ be given. By definition, there is a $t_0 > 0$ such that

$$\|\psi_t(f \cdot a) - f \cdot \psi_t(a)\| < \varepsilon$$

for all $t \geqslant t_0$. Now choose $t_1 \geqslant t_0 > 0$ such that $r(t) \geqslant t_0$. Then for all $t \geqslant t_1$ we have that

$$\|\psi_{r(t)}(f \cdot a) - f \cdot \psi_{r(t)}(a)\| < \varepsilon.$$

Thus, $\{\psi_{r(t)}\}: A \to B$ is asymptotically $C_0(X)$-linear. By defining $\{\Psi_t\}: A \to B[0, 1]$ via the formula

$$\Psi_t(a)(s) = \psi_{sr(t) + (1-s)t}(a)$$

for all $a \in A$, $t \in [1, \infty)$ and $s \in [0, 1]$, we obtain a $C_0(X)$-homotopy connecting $\{\psi_t\}$ and $\{\psi_{r(t)}\}$. Q.E.D

Let $\{\psi_t\}: A \to B$ be an asymptotic $C_0(X)$-morphism. If $\phi: B \to C$ is a $C_0(X)$-morphism, then the composition $\{\phi \circ \psi_t\}: A \to C$ is clearly an asymptotic $C_0(X)$-morphism. Similarly, composition on the left with a $C_0(X)$-morphism and $\{\psi_t\}$ produces an asymptotic $C_0(X)$-morphism. As for ordinary asymptotic morphisms, composition is a delicate issue, but we have the following parallel result for composing asymptotic $C_0(X)$-morphisms. The proof is analogous to the one given in [CH89], but with a few extra observations needed for preserving asymptotic $C_0(X)$-linearity. See also the expositions in [Gue99, Bla98].

Recall that a *generating system* $\{A_k\}$ for a separable C^*-algebra A is a countable family of compact subsets A_k of A such that for all k:

- $A_k \subset A_{k+1}$;
- $\bigcup A_k$ is dense in A;
- each of $A_k + A_k$, $A_k A_k$, A_k^* and λA_k for $|\lambda| \leqslant 1$ is contained in A_{k+1}.

Definition 3.6. Let X be a second countable, locally compact space. A $C_0(X)$-*generating* system $(\{A_k\}, \{C_k\})$ for a $C_0(X)$-algebra A is a family of compact subsets $\{A_k\}$ of A and compact subsets $\{C_k\}$ of $C_0(X)$ such that $\{A_k\}$ is a generating system for A, $\{C_k\}$ is a generating system for $C_0(X)$, and $C_k \cdot A_k \subset A_{k+1}$.

The following is then easy to prove, since $C_0(X) \cdot A$ is dense in A.

Lemma 3.7. *Suppose $\{C_k\}$ and $\{A_k\}$ are generating systems for $C_0(X)$ and A, respectively, and let $A'_k = C_k \cdot A_k$. Then $(\{A'_k\}, \{C_k\})$ is a $C_0(X)$-generating system for the $C_0(X)$-algebra A.*

Theorem 3.8 (Technical Theorem on Compositions). *Let X be a second countable, locally compact space. Asymptotic $C_0(X)$-morphisms may be composed at the level of $C_0(X)$-homotopy; that is, there is an associative map*

$$[\![A, B]\!]_X \otimes [\![B, C]\!]_X \to [\![A, C]\!]_X.$$

The composition of $[\![\phi_t]\!]_X$ and $[\![\psi_t]\!]_X$ is the $C_0(X)$-homotopy class of an asymptotic $C_0(X)$-morphism $\{\theta_t\}$ constructed from $\{\phi_t\}$ and $\{\psi_t\}$.

Proof. Let $(\{A_k\}, \{C_k\})$ be a $C_0(X)$-generating system for the $C_0(X)$-algebra A. Let $\{\psi_t\}: A \to B$ and $\{\phi_t\}: B \to C$ be asymptotic $C_0(X)$-morphisms. We may assume by Lemma 3.3 that these asymptotic morphisms are given by equicontinuous families of functions.

Since each A_k is compact and $\{\psi_t\}$ is equicontinuous on A_k, there exists $t_k > 0$ such that for all $t \geqslant t_k$, a, $a' \in A_k$, $f \in C_k$ and $|\lambda| \leqslant 1$:

$$\begin{aligned} \|\psi_t(aa') - \psi_t(a)\,\psi_t(a')\| &\leqslant 1/k \\ \|\psi_t(a+a') - \psi_t(a) - \psi_t(a')\| &\leqslant 1/k \\ \|\psi_t(\lambda a) - \lambda\psi_t(a)\| &\leqslant 1/k \\ \|\psi_t(a^*) - \psi_t(a)^*\| &\leqslant 1/k \\ \|\psi_t(f \cdot a) - f \cdot \psi_t(a)\| &\leqslant 1/k \\ \|\psi_t(a)\| - \|a\| &\leqslant 1/k. \end{aligned}$$

Let $(\{B_k\}, \{C_k\})$ be a $C_0(X)$-generating system such that for all k,

$$\{\psi_t(a) \colon t \leqslant t_k \text{ and } a \in A_{k+10}\} \subset B_k.$$

There then exist $r_k > 0$ such that for all $r \geqslant r_k$, b, $b' \in B_{k+10}$, $f \in C_{k+10}$ and $|\lambda| \leqslant 1$, estimates similar to the ones above hold for $\{\phi_r\}$. Let σ be a continuous increasing function such that $\sigma(t_k) \geqslant r_k$.

For each continuous increasing function $\rho \geqslant \sigma$, the composition $\theta_t = \phi_{\rho(t)} \circ \psi_t \colon A \to C$ defines an asymptotic $C_0(X)$-morphism. We will call $\{\theta_t\}$ a *composition* of $\{\psi_t\}$ and $\{\phi_t\}$. Different choices of reparametrizations are $C_0(X)$-homotopic through compositions via Lemma 3.5. For different choices of $C_0(X)$-generating systems, it is obviously possible to choose the function σ so that all estimates needed hold for both $C_0(X)$-generating systems. Hence, for all $\rho \geqslant \sigma$ the family $\{\phi_{\rho(t)} \circ \psi_t\}$ is a composition for both $C_0(X)$-generating systems.

Consequently, this construction passes to a well-defined pairing

$$[\![A, B]\!]_X^a \times [\![B, C]\!]_X^a \to [\![A, C]\!]_X.$$

By taking compositions of $C_0(X)$-homotopies, it passes to a well-defined pairing on $C_0(X)$-homotopy classes, as desired. Q.E.D

Define $\mathbf{AM(X)}$ to be the category of separable $C_0(X)$-algebras and $C_0(X)$-homotopy classes of asymptotic $C_0(X)$-morphisms, i.e. $\mathrm{Mor}(A, B) = [\![A, B]\!]_X$. There are obvious functors $\mathbf{SC^*(X)} \to \mathbf{SH(X)} \to \mathbf{AM(X)}$.

Let $\{\psi_t\} \colon A \to B$ be an asymptotic $C_0(X)$-morphism. By Lemma 3.3, we may assume, up to equivalence, that the family $\{\psi_t\}$ is equicontinuous. Let $SA = C_0(\mathbb{R}, A) \cong C_0(\mathbb{R}) \otimes A$ denote the suspension of A, which is a $C_0(X)$-algebra by Lemma 2.5. It follows that the formula

$$g \mapsto \phi_t \circ g$$

then defines an asymptotic morphism $\{S\psi_t\}\colon SA \to SB$ called a *suspension* of $\{\psi_t\}$. It is easily seen to be asymptotically $C_0(X)$-linear. By applying this construction to $C_0(X)$-homotopies we obtain the following result.

PROPOSITION 3.9. *Let A and B be $C_0(X)$-algebras. There are well-defined maps*

$$S\colon [\![A, B]\!]^a_X \to [\![SA, SB]\!]^a_X$$

$$S\colon [\![A, B]\!]_X \to [\![SA, SB]\!]_X$$

which are induced by sending the class of $\{\psi_t\}$ to the class of $\{S\psi_t\}$.

Consider the balanced tensor product $A \otimes_X B$ of A and B over $C_0(X)$. We have the following generalization of the tensor product for regular asymptotic morphisms, which requires an extra bit of proof.

PROPOSITION 3.10. *Let $\{\psi_t\}\colon A \to B$ and $\{\phi_t\}\colon C \to D$ be asymptotic $C_0(X)$-morphisms. The tensor product over X is defined (up to equivalence) by the formula*

$$\{\psi_t \otimes_X \phi_t\}\colon A \otimes_X C \to B \otimes_X D\colon a \otimes_X b \mapsto \psi_t(a) \otimes_X \phi_t(b).$$

It is well-defined on equivalence classes and $C_0(X)$-homotopy classes, i.e., there are well-defined maps

$$[\![A, B]\!]^a_X \otimes [\![C, D]\!]^a_X \to [\![A \otimes_X C, B \otimes_X D]\!]^a_X$$

$$[\![A, B]\!]_X \otimes [\![C, D]\!]_X \to [\![A \otimes_X C, B \otimes_X D]\!]_X.$$

Proof. By Lemma 3.1 of [CH89], the map $a \otimes b \mapsto \psi_t(a) \otimes \phi_t(b)$ determines an asymptotic morphism $\{\psi_t \otimes \phi_t\}\colon A \otimes C \to B \otimes D$ on maximal tensor products which is well-defined on equivalence (and homotopy) classes. We may assume that each of these maps is $*$-linear up to equivalence by Lemma 3.3.

Let I_X and J_X be the balancing ideals of $A \otimes C$ and $B \otimes D$, respectively (see Definition 2.7). Let $q\colon B \otimes D \to B \otimes_X D = (B \otimes D)/J_X$ denote the quotient map. A simple $\varepsilon/3$-argument then shows that for any generator $x = (f \cdot a) \otimes b - a \otimes (f \cdot b)$ of I_X we have

$$\lim_{t \to \infty} \|q((\psi_t \otimes \phi_t)(x))\| = 0.$$

By [Gue99], the family $\{\psi_t \otimes \phi_t\}$ determines an equivalence class of a *relative* asymptotic morphism

$$\{\psi_t \otimes \phi_t\}\colon A \otimes C \rhd I_X \to B \otimes D \rhd J_X.$$

By Lemma 3.8 in [Gue99] there is an induced asymptotic morphism on the quotients

$$\overline{\{\psi_t \otimes \phi_t\}} \colon (A \otimes C)/I_X \to (B \otimes D)/J_X$$

which is well-defined up to equivalence and is given by the formula $a \otimes_X b \mapsto \psi_t(a) \otimes_X \phi_t(b)$. We shall call this asymptotic morphism the *tensor product morphism over* X of $\{\psi_t\}$ and $\{\phi_t\}$ and denote it by $\{\psi_t \otimes_X \phi_t\} \colon A \otimes_X C \to B \otimes_X D$. Q.E.D.

Let I be an ideal in a separable $C_0(X)$-algebra A. Then I has a canonical $C_0(X)$-algebra structure inherited from A. Let $\{u_t\}$ be a quasicentral approximate unit for $I \lhd A$ [Arv77]. Let $s\colon A/I \to A$ be any set-theoretic section of the $C_0(X)$-linear projection $p\colon A \to A/I$. Connes and Higson [CH89] construct an asymptotic morphism

$$\{\alpha_t\} \colon A/I(0,1) \to I,$$

which on elementary tensors is induced by the formula

$$\alpha_t \colon g \otimes \bar{a} \mapsto g(u_t)\, s(\bar{a}) \tag{1}$$

for any $g \in C_0(0,1)$ and $\bar{a} = p(a) \in A/I$. The asymptotic equivalence class $[\![\alpha_t]\!]_a$ is independent of the choice of the section s. In addition, the (full) homotopy class $[\![\alpha_t]\!]$ is independent of the choice of the quasicentral approximate unit [CH89, Bla98].

Lemma 3.11. *The asymptotic morphism $\{\alpha_t\} \colon A/I(0,1) \to I$ is asymptotically $C_0(X)$-linear, and the $C_0(X)$-homotopy class is independent of the choice of quasicentral approximate unit.*

Proof. Take $\bar{a} \in A/I$ and $f \in C_0(X)$. By Lemma 2.5, $A/I(0,1) = C_0(0,1) \otimes A/I$ has a $C_0(X)$-algebra structure induced from that of A/I. For any $\bar{a} \in A/I$ we have $f \cdot s(\bar{a}) - s(f \cdot \bar{a}) \in I$, since the projection p is a $C_0(X)$-morphism. Therefore, for any $g \in C_0(0,1)$, $f \in C_0(X)$ and $\bar{a} \in A/I$ we compute that

$$\lim_{t \to \infty} \|\alpha_t(f \cdot (g \otimes \bar{a})) - f \cdot \alpha(g \otimes \bar{a})\|$$

$$= \lim_{t \to \infty} \|g(u_t)(f \cdot s(\bar{a}) - s(f \cdot \bar{a}))\| = 0,$$

since $\{u_t\}$ is quasicentral for $I \lhd A$. It follows that the associated $*$-homomorphism $\alpha\colon A/I(0,1) \to I_\infty$ is $C_0(X)$-linear.

Let $\{u_t\}$ and $\{u'_t\}$ be two quasicentral approximate units for $I \lhd A$. Then the convex combination $u_t^\lambda = \lambda u_t + (1-\lambda)\, u'_t$ defines a continuous family for $\lambda \in [0, 1]$ of quasicentral approximate units connecting $\{u_t\}$ and $\{u'_t\}$. Replacing $\{u_t\}$ in formula 1 with $\{u_t^\lambda\}$, we obtain a $C_0(X)$-homotopy by the above argument. Q.E.D

Let $\psi: A \to B$ be a $C_0(X)$-morphism. The *mapping cone* of ψ is the C^*-algebra C_ψ of $A \oplus B[0, 1)$ consisting of all elements (a, G) such that $\psi(a) = G(0)$. It has an induced $C_0(X)$-algebra structure given by $f \cdot (a, G) = (f \cdot a, f \cdot G)$, and there is a commutative diagram of $C_0(X)$-algebras

$$\begin{array}{ccc} C_\psi & \xrightarrow{q} & B[0,1) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ A & \xrightarrow[\psi]{} & B. \end{array}$$

The $C_0(X)$-morphisms p and q are the projections onto the respective factors. There is also a $C_0(X)$-inclusion j: $SB \cong B(0, 1) \to C_\psi$ given by $j: F \mapsto (0, F)$.

The proofs of the following two results are identical to the regular E-theoretic case; one need only see that the relevant constructions are all asymptotically $C_0(X)$-linear.

LEMMA 3.12. *Let $\psi: A \to B$ be a $C_0(X)$-morphism. Then for any $C_0(X)$-algebra D, the following sequence is exact*:

$$[\![D, C_\psi]\!]_X \xrightarrow{p_*} [\![D, A]\!]_X \xrightarrow{\psi_*} [\![D, B]\!]_X.$$

Proof. Same proof as Proposition 5 of [Dad94] and Proposition 3.7 of [Ros82]. Q.E.D

LEMMA 3.13. *Let $\psi: A \to B$ be a $C_0(X)$-morphism. For any $C_0(X)$-algebra D, consider the following diagram*:

$$\begin{array}{ccc} & & [\![B, D]\!]_X & \xrightarrow{\psi^*} & [\![A, D]\!]_X \\ & & \downarrow{\scriptstyle S} & & \\ [\![C_\psi, SD]\!]_X & \xrightarrow{j^*} & [\![SB, SD]\!]_X & & \end{array}$$

Then $S(\mathrm{Ker}(\psi^*)) \subset \mathrm{Range}(j^*)$.

Proof. Same proof as Proposition 8 of [Dad94] and Proposition 3.10 of [Ros82]. Q.E.D

Let $0 \to J \to A \xrightarrow{\psi} B \to 0$ be a short exact sequence of $C_0(X)$-algebras, and consider the $C_0(X)$-morphism $i\colon J \to C_\psi$ given by $i\colon a \mapsto (a, 0)$. There is also a $C_0(X)$-extension

$$0 \to J(0,1) \to A[0,1) \to C_\psi \to 0,$$

where the first map is the obvious inclusion and the map $A[0,1) \to C_\psi$ is given by the formula $F \mapsto (F(0), \psi \circ F)$.

PROPOSITION 3.14. *The suspension $Si\colon SJ \to SC_\psi$ is an isomorphism in the category* **AM(X)**. *The inverse is the $C_0(X)$-homotopy class of the asymptotic $C_0(X)$-morphism $\{\alpha_t\}$ associated to the $C_0(X)$-extension*

$$0 \to J(0,1) \to A[0,1) \to C_\psi \to 0$$

from Lemma 3.11. *That is,* $[\![Si \circ \alpha_t]\!]_X = [\![\mathrm{id}_{SC_\psi}]\!]_X$ *and* $[\![\alpha_t \circ Si]\!]_X = [\![\mathrm{id}_{SJ}]\!]_X$.

Proof. Same as the proof of Theorem 13 in [Dad94]. Q.E.D

THEOREM 3.15. *Let A and B be $C_0(X)$-algebras.*

(i) *If $B \cong B \otimes \mathcal{K}$ is $C_0(X)$-stable, then $[\![A, B]\!]_X$ is an abelian monoid with addition given by direct sum*

$$[\![\psi_t]\!]_X + [\![\phi_t]\!]_X = [\![\psi_t \oplus \phi_t]\!]_X,$$

using fixed $C_0(X)$-isomorphisms $B \cong B \otimes M_2 \cong M_2(B)$. The class of the zero morphism serves as the identity.

(ii) *$[\![A, SB]\!]_X$ is a group under* (*not necessarily abelian*) "*loop composition*": *given $\{\psi_t\}$, $\{\phi_t\}\colon A \to SB$, define $\{\psi_t \cdot \phi_t\}\colon A \to SB$ via*

$$(\psi_t \cdot \phi_t)(a)(s) = \begin{cases} \psi_t(a)(2s), & \text{if } 0 \leqslant s \leqslant 1/2 \\ \phi_t(a)(2s-1), & \text{if } 1/2 \leqslant s \leqslant 1. \end{cases}$$

(iii) *$[\![A, S(B \otimes \mathcal{K})]\!]_X$ is an abelian group and the two operations described in* (i) *and* (ii) *coincide.*

DEFINITION 3.16. Let A and B be $C_0(X)$-algebras. The *representable E-theory group* $\mathcal{RE}(X; A, B)$ is defined as the direct limit

$$\mathcal{RE}(X; A, B) = \varinjlim_{n \in \mathbb{N}} [\![S^n(A \otimes \mathcal{K}), S^n(B \otimes \mathcal{K})]\!]_X,$$

where the direct limit is taken over the suspension operation of Proposition 3.9.

We now assume that X is a second countable locally compact space.

THEOREM 3.17. *Let A and B be $C_0(X)$-algebras. The representable E-theory group $\mathscr{R}E(X; A, B)$ is a $C_0(X)$-stable, $C_0(X)$-homotopy invariant and half-exact bifunctor from* $\mathbf{SC}^*(\mathbf{X})$ *to the category of abelian groups that satisfies the following properties*:

(i) *There exists a group homomorphism*

$$\mathscr{R}E(X;\ A,\ B) \otimes \mathscr{R}E(X;\ B,\ C) \to \mathscr{R}E(X;\ A,\ C);$$

(ii) *There exists a group homomorphism*

$$\mathscr{R}E(X; A, B) \otimes \mathscr{R}E(X; C, D) \to \mathscr{R}E(X; A \otimes_X C, B \otimes_X D);$$

(iii) *$\mathscr{R}E(X; A, B)$ is an $\mathscr{R}E(X; C_0(X), C_0(X))$-module;*

(iv) *$\mathscr{R}E(X; A, B) \cong \mathscr{R}E(X; SA, SB)$;*

(v) *A continuous map $p: Y \to X$ induces a group homomorphism*

$$p^*: \mathscr{R}E(X; A, B) \to \mathscr{R}E(Y; p^*A, p^*B).$$

Proof. $\mathscr{R}E(X; A, B)$ is $C_0(X)$-stable and $C_0(X)$-homotopy invariant by definition. Half-exactness follows from Lemmas 3.11, 3.12, 3.13 and Proposition 3.14. Property (i) follows from Theorem 3.8. Property (ii) follows from Theorem 3.10. The third property follows from (ii) and the natural isomorphisms $A \otimes_X C_0(X) \cong A$ and $B \otimes_X C_0(X) \cong B$. Property (iv) follows by definition. The last property follows from Definition 2.9 of the pullback $p^*A = C_0(Y) \otimes_X A$ and the proof of Proposition 3.10. Q.E.D

4. CATEGORICAL ASPECTS OF $\mathscr{R}KK$-THEORY AND $\mathscr{R}E$-THEORY

In this section, we prove universality properties of $\mathscr{R}KK$-theory and $\mathscr{R}E$-theory; our proofs are largely based on the work of [Hig87] and [Bla98]. We begin by recalling the definition of $\mathscr{R}KK$-theory:

DEFINITION 4.1. Let A and B be $C_0(X)$ algebras. A *cycle* in $\mathscr{R}KK(X; A, B)$ is a triple $(\mathscr{E}, \phi, F)$, where

- $\mathscr{E}$ is a countably generated $\mathbb{Z}_2$-graded Hilbert B-module;
- $\phi: A \to \mathscr{B}(\mathscr{E})$ is a $*$-homomorphism;
- $F \in \mathscr{B}(\mathscr{E})$ has degree one, and $[F, \phi(a)]$, $(F - F^*)\,\phi(a)$, and $(F^2 - 1)\,\phi(a)$ are all in $\mathscr{K}(\varepsilon)$ for all a in A;
- $\phi(fa)\, eb = \phi(a)\, e(fb)$ for all $a \in A$, $b \in B$, $e \in \mathscr{E}$, and $f \in C_0(X)$.

The abelian group $\mathscr{R}KK(X; A, B)$ is formed by identifying these cycles under the equivalence relations of unitary equivalence and homotopy.

By the same arguments that are used in the "quasihomomorphism picture" of KK-theory ([Bla98], Section 17.6), we may represent elements of $\mathscr{R}KK(X; A, B)$ by pairs (ϕ^+, ϕ^-) of homomorphisms from A to $M(B \otimes \mathscr{K})$ such that

- $\phi^{\pm}(fa)(b \otimes k) = \phi^{\pm}(a)(fb \otimes k)$ for all $a \in A$, $b \in B$, $k \in \mathscr{K}$, and $f \in C_0(X)$;
- $\phi^+(a) - \phi^-(a)$ is in $B \otimes \mathscr{K}$ for each a in A.

We shall call such pairs $C_0(X)$-*quasihomomorphisms*. A $C_0(X)$-quasihomomorphism is *degenerate* if ϕ^+ and ϕ^- are equal, and the appropriate equivalence relations on $C_0(X)$-quasihomomorphisms (ϕ^+, ϕ^-) are addition of degenerates pairs, conjugation of ϕ^+ and ϕ^- by the same unitary, and homotopies given by paths (ϕ_t^+, ϕ_t^-), where (ϕ_t^+, ϕ_t^-) is a $C_0(X)$-quasihomomorphism for each t in $[0, 1]$ and $a \mapsto \phi_t^{\pm}(a)$ is continuous for each $a \in A$.

Let

$$0 \to B \otimes \mathscr{K} \xrightarrow{j} D \underset{s}{\overset{q}{\rightleftarrows}} A \to 0$$

be a split exact sequence of $C_0(X)$-algebras, and let $\imath$ be the canonical inclusion of D into $M(B \otimes \mathscr{K})$. Then $\imath$ is a $C_0(X)$-homomorphism, and

$$\left(H_B \oplus H_B^{op}, \imath \oplus (\imath \circ s \circ q), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

determines an element π_s of $\mathscr{R}KK(X; A, B)$ that is called a *splitting morphism*. Furthermore, every element of $\mathscr{R}KK(X; A, B)$ is the pullback of a splitting morphism via a $C_0(X)$-homomorphism:

PROPOSITION 4.2 (Compare Proposition 17.8.3, [Bla98]). *Let (ϕ^+, ϕ^-) be a $C_0(X)$-quasihomomorphism from A to B. Then there exists a split exact sequence*

$$0 \to B \otimes \mathscr{K} \xrightarrow{j} D \underset{s}{\overset{q}{\rightleftarrows}} A \to 0$$

of $C_0(X)$-algebras and a $C_0(X)$-homomorphism $f: A \to D$ such that $f^*(\pi_s) = [\phi^+, \phi^-]$.

Proof. Let $D = \{(a, \phi^+(a) + \beta) : a \in A, \beta \in B \otimes \mathscr{K}\}$, and let q be projection onto the first factor. The kernel of q is the ideal $\{(0, \beta) \colon \beta \in B \otimes \mathscr{K}\}$, which is of course isomorphic to $B \otimes \mathscr{K}$. The map $s \colon A \to D$ defined by $s(a) = (a, \phi^-(a))$ is obviously a $C_0(X)$-splitting for q, and thus we have π_s as an element of $\mathscr{R}KK(X; D, B)$. Define $f \colon A \to D$ as $f(a) = (a, \phi^+(a))$; an easy computation shows that $f^*(\pi_s) = [\phi^+, \phi^-]$ in $\mathscr{R}KK(X; A, B)$. Q.E.D

THEOREM 4.3. *Let* **RKK** *denote the category of separable $C_0(X)$-algebras, with* **RKK** *classes serving as morphisms, and let C be the obvious functor from* **SC*(X)** *to* **RKK**. *Let* **A** *be an additive category and let F:* **SC*(X)** $\to$ **A** *be a covariant functor satisfying the following*:

- *F is a $C_0(X)$-homotopy functor*;
- *F is $C_0(X)$-stable; i.e., if e is a rank one projection, then the map $b \mapsto e \otimes b$ induces an invertible morphism $Fe \colon F(B) \to F(B \otimes \mathscr{K})$*;
- *if $0 \to J \xrightarrow{j} D \underset{s}{\overset{q}{\rightleftarrows}} A \to 0$ is a spit short exact sequence of $C_0(X)$-algebras, then $F(D)$ is isomorphic to the direct sum of $F(J)$ and $F(A)$ via the morphisms j_* and s_*.*

Then there is a unique functor $\hat{F}$: **RKK** $\to$ **A** *such that $\hat{F} \circ C = F$.*

Proof. Since **RKK** and **SC*(X)** consist of precisely the same objects, the only question is how $\hat{F}$ acts on morphisms. If x is a morphism arising from a $C_0(X)$-homomorphism, then $\hat{F}(x) = F(x)$. On the other hand, suppose that x comes from a splitting morphism represented by the split short exact sequence $0 \to B \otimes \mathscr{K} \xrightarrow{j} D \underset{s}{\overset{q}{\rightleftarrows}} A \to 0$. Then $\hat{F}(x) \in \mathbf{A}(F(D), F(B))$ is given by the following composition:

$$F(D) \xrightarrow{(Fj,\, Fs)^{-1}} F(B \otimes \mathscr{K}) \oplus F(A) \xrightarrow{p_1} F(B \otimes \mathscr{K}) \xrightarrow{Fe^{-1}} F(B).$$

The fact that $F(x)$ is unique follows from the fact that both (F_j, Fs) and Fe are invertible, and the proof that $\hat{F}$ is well-defined is the same as that in [Bla98], Theorem 22.2.1 Q.E.D

Next we consider the universality property of $\mathscr{R}E$-theory. The proofs of this are almost exactly the same as the ones given for E-theory in Section 25.6 of [Bla98], so we shall only outline the argument here. Let $\{\psi_t\}_{t \in [1, \infty)} \colon A \to B$ be an asymptotic $C_0(X)$-morphism, let $\Psi \colon A \to B_\infty$ be the map defined in Proposition 3.2, and let q be the quotient map from $C_b([1, \infty), B)$ onto B_∞. Set $D = (q^{-1}(\Psi(A))$. Then we have a short exact sequence $0 \to C_0([1, \infty), B) \to D \xrightarrow{\pi} A \to 0$. Since $C_0([1, \infty), B)$ is contractible, π induces an isomorphism $[\![\pi]\!]_X^a$ in $\mathscr{R}E(D, A)$.

LEMMA 4.4 (Compare Proposition 25.6.2, [Bla98]). *$[\![\mathrm{ev}_1]\!]^a_X \circ ([\![\pi]\!]^a_X)^{-1} = [\![\psi]\!]^a_X$ in $\mathscr{RE}(X; A, B)$.*

Proof. For each $t \geqslant 1$, the asymptotic $C_0(X)$-morphism $\{\psi_t \circ \pi\}_{t \in [1, \infty)}$ from D to B is equivalent to $\{\mathrm{ev}_t \circ \pi\}_{t \in [1, \infty)}$, which in turn is homotopic to the constant morphism $\{\mathrm{ev}_1 \circ \pi\}_{t \in [1, \infty)}$. Therefore $[\![\psi]\!]^a_X \circ [\![\pi]\!]^a_X = [\![\mathrm{ev}_1]\!]^a_X$ in $\mathscr{RE}(X; A, B)$, whence the desired result follows. Q.E.D

We also need the following lemma, whose proof is easily adapted from the proof of the "classical" result:

LEMMA 4.5 ([J.C84]). *Let* **A** *be an additive category and let* F: **SC*(X)** $\to$ **A** *be a covariant functor satisfying the following*:

- *F is a $C_0(X)$-homotopy functor*;
- *F is $C_0(X)$-stable*;
- *F is half-exact.*

Then F satisfies Bott periodicity.

THEOREM 4.6 (Compare Theorem 2.6.1, [Bla98]). *Let* **RE** *denote the category of separable $C_0(X)$-algebras, with* **RE** *classes serving as morphisms, and let C be the obvious functor from* **SC*(X)** *to* **RE**. *Let A be an additive category and let F:* **SC*(X)** $\to$ **A** *be a covariant functor satisfying the following*:

- *F is a $C_0(X)$-homotopy functor*;
- *F is $C_0(X)$-stable*;
- *F is half-exact.*

Then there is a unique functor $\hat{F}$: **RE** $\to$ **A** *such that $\hat{F} \circ D = F$.*

Proof. As in the analogous theorem for $\mathscr{RKK}$, the only question is how $\hat{F}$ acts on morphisms. Let $\Psi = \{\psi_t\}_{t \in [1, \infty)}: A \to B$ be an asymptotic $C_0(X)$-morphism, and let $0 \to C_0([1, \infty), B) \to D \xrightarrow{\pi} A \to 0$ be the corresponding short exact sequence described above. Since $C_0([1, \infty), B)$ is contractible, it follows from the long exact sequence for F that $F\pi$: $F(D) \to F(A)$ is an isomorphism. Set $\hat{F}(\Psi) = F(\mathrm{ev}_1) \circ (F(\pi))^{-1}$. It is straightforward to show that this map is well-defined, so $\hat{F}(\Psi)$ is a morphism from $F(A)$ to $F(B)$. Next, since F satisfies Bott periodicity, we have isomorphisms among $F(A)$, $F(S^2A)$, $F(A \otimes \mathscr{K})$ and $F(S^2A \otimes \mathscr{K})$ for all A in **SC*(X)**, whence each element of $\mathscr{RE}(X; A, B)$ determines a morphism from $F(A)$ to $F(B)$. Q.E.D

THEOREM 4.7 (Compare Theorem 25.6.3, [Bla98]). *Let A be a separable $C_0(X)$-algebra for which $\mathscr{RKK}(X; A, B)$ is half-exact. Then $\mathscr{RE}(X; A, B)$ is naturally isomorphic to $\mathscr{RKK}(X; A, B)$ for every separable $C_0(X)$-algebra B. In* *particular, if A is $\mathscr{R}KK(X)$-nuclear in the sense of* [Bau98], *then $\mathscr{R}E(X; A, B) \cong \mathscr{R}KK(X; A, B)$ for every separable $C_0(X)$-algebra B.*

Proof. From Theorem 4.3, we have a homomorphism α from $\mathscr{R}KK(X; A, B)$ to $\mathscr{R}E(X; A, B)$. Theorem 4.6 implies that we have a map $\mathscr{R}KK(X; A, A) \times \mathscr{R}E(X; A, B) \to \mathscr{R}KK(X; A, B)$, and so pairing with the element $\mathbf{1}_A \in \mathscr{R}KK(X; A, A)$ defines a homomorphism β: $\mathscr{R}E(X; A, B) \to \mathscr{R}KK(X; A, B)$ that is the inverse to α. Q.E.D

5. EXAMPLES AND APPLICATIONS

In this section, we consider examples and applications of the preceding theory. First, we consider $\mathscr{R}E$-elements associated to unbounded $\mathscr{R}KK$-elements, which can also arise from families of elliptic differential operators parametrized by X. We also construct "fundamental classes" for unital $C_0(X)$-algebras, and define invariants of central bimodules in noncommutative geometry.

5.1. *Unbounded $\mathscr{R}KK$-elements.* Let X be a locally compact space, and let A and B be separable $C_0(X)$-algebras. Given any Hilbert B-module $\mathscr{E}$ with B-valued inner product $\langle\ ,\ \rangle$, let $\mathscr{B}(\mathscr{E})$ denote the C^*-algebra of bounded B-linear operators on $\mathscr{E}$ that possess an adjoint. The ideal $\mathscr{K}(\mathscr{E})$ of compact operators on $\mathscr{E}$ has a natural $C_0(X)$-action [Kas88]; the structural homomorphism Φ: $C_0(X) \to M(\mathscr{K}(\mathscr{E})) = \mathscr{B}(\mathscr{E})$ is defined as follows: For all $e \in \mathscr{E}$ and $f \in C_0(X)$,

$$\Phi(f)\, e = \lim_{\lambda} e \cdot (f \cdot b_\lambda),$$

where $\{b_\lambda\}$ is any approximate unit for B. Thus, $\mathscr{B}(E)$ is a central Banach $C_0(X)$-module.

Definition 5.1. An unbounded $\mathscr{R}KK(X; A, B)$-cycle is a triple $(\mathscr{E}, \phi, D)$ where

- $\mathscr{E}$ is a countably generated $\mathbb{Z}_2$-graded Hilbert B-module;
- ψ: $A \to \mathscr{B}(\mathscr{E})$ is a $C_0(X)$-morphism;
- D is a self-adjoint regular operator on $\mathscr{E}$ of grading degree one satisfying
  (i) $(D \pm i)^{-1}\, \psi(a) \in \mathscr{K}(\mathscr{E})$ for all $a \in A$;
  (ii) $\{a \in A : [D, \psi(a)]$ is densely defined and extends to $\mathscr{B}(\mathscr{E})\}$ is dense in A.

We define $\Psi(X; A, B)$ to be the collection of all unbounded $\mathscr{R}KK(X; A, B)$-cycles.

Let ε be the grading operator of $\mathscr{E}$. For any $x \in \mathbb{R}$ and $t \geqslant 1$, the operator $x\varepsilon + t^{-1}D$ is a self-adjoint regular operator [Lan95] on $\mathscr{E}$ that satisfies

$$(x\varepsilon + t^{-1}D)^2 = x^2 1_{\mathscr{E}} + t^{-2}D^2 \geqslant x^2 1_{\mathscr{E}}.$$

Define a one-parameter family of maps $C_0(\mathbb{R}) \odot A \to C_0(\mathbb{R}) \otimes \mathscr{K}(\mathscr{E})$ on elementary tensors $g \otimes a$ by the formula

$$g \otimes a \mapsto g(x\varepsilon + t^{-1}D) \circ \psi(a)$$

and extend linearly, where $g(x\varepsilon + t^{-1}D)$ is computed using the functional calculus for self-adjoint regular operators [Lan95, BJ83].

Proposition 5.2. *Let* $(\mathscr{E}, \psi, D) \in \Psi(X; A, B)$. *The above formula extends to an asymptotic* $C_0(X)$*-morphism*

$$\{\psi_t^D\}\colon C_0(\mathbb{R}) \otimes A \to C_0(\mathbb{R}) \otimes \mathscr{K}(\mathscr{E}),$$

which determines a well-defined map $\Psi(X; A, B) \to \mathscr{R}E(X; A, B)$.

Proof. That this defines an asymptotic morphism is well-known [CH89, HKT98]; we need only show that it is asymptotically $C_0(X)$-linear. By an approximation argument, we only need to check on elementary tensors $g \otimes a$. Since ψ is $C_0(X)$-linear, we have for each $f \in C_0(X)$ that

$$g(x\varepsilon + t^{-1}D) \circ \psi(f \cdot a) = \Phi(f)\ g(x\varepsilon + t^{-1}D) \circ \psi(a),$$

since $\Phi(f) \in ZM(\mathscr{K}(\mathscr{E})) = Z(\mathscr{B}(\mathscr{E}))$. The result now easily follows. Q.E.D

As in [BJ83] we have a map $\Psi(X; A, B) \to \mathscr{R}KK(X; A, B)$ given by the formula

$$(\mathscr{E}, \psi, D) \mapsto (\mathscr{E}, \psi, F(D)),$$

where $F(D) = D(D^2+1)^{-1/2}$. Take $[(\mathscr{E}, \psi, T)] \in \mathscr{R}KK(X; A, B)$ (Definition 4.1). We may and do assume that $T = T^*$ and $\|T\| \leqslant 1$. Then $(\mathscr{E}, \psi, G(T))$, where $G(x) = x(1-x^2)^{-1/2}$, is an unbounded $\mathscr{R}KK(X; A, B)$-cycle which maps to $(\mathscr{E}, \psi, T)$, since $(F \circ G)(x) = x$. Therefore the map $\Psi(X; A, B) \to \mathscr{R}KK(X; A, B)$ is surjective.

Proposition 5.3. *Let* $\mathscr{R}KK(X; A, B) \to \mathscr{R}E(X; A, B)$ *be the natural transformation from Theorem* 4.7. *The following diagram commutes*:

$$\begin{tikzcd} & \Psi(X; A, B) \arrow[dl] \arrow[dr] & \\ \mathscr{R}KK(X; A, B) \arrow[rr] & & \mathscr{R}E(X; A, B). \end{tikzcd}$$

5.2. *Elliptic differential $C_0(X)$-operators.* For the material and definitions in this subsection, we refer the reader to [MF80] and [Tro99].

Let B be a separable $C_0(X)$-algebra with unit and let M be a smooth compact oriented Riemannian manifold. Let $E \to M$ be a smooth vector B-bundle, i.e., a smooth locally trivial fiber bundle with fibers $E_p \cong \mathscr{E}$, a finite projective (right) B-module. Equip the fibers E_p with smoothly-varying B-valued metrics

$$\langle \cdot, \cdot \rangle_p : E_p \times E_p \to B;$$

this is always possible since $\mathscr{E}$ admits partitions of unity [Bla98]. Let $C^\infty(E)$ denote the module of smooth sections of E. We let $\mathscr{H}_E = L^2(E)$ denote the Hilbert B-module completion of $C^\infty(E)$ with respect to the B-valued Hermitian metric

$$\langle s, s' \rangle = \int_M \langle s(p), s'(p) \rangle_p \, \mathrm{dvol}_M(p),$$

where $s, s' \in C^\infty(E)$ and dvol_M is the Riemannian volume measure.

Let $D: C^\infty(E) \to C^\infty(E)$ be a self-adjoint elliptic partial differential B-operator. For simplicity, we assume that D has order one and that E splits as a direct sum $E = E_0 \oplus E_1$ with respect to a grading bundle automorphism ε. Also, assume that D is of degree one with respect to this grading, i.e.,

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

where $(D^-)^* = D^+$.

In the special case where X is compact and $B = C(X)$, then we can view $D = \{D_x\}_{x \in X}$ as a family of elliptic differential operators on M parametrized by X; this is accomplished by letting $D_x = D \otimes_{\mathrm{ev}_x} 1$ on the complex vector bundle $E^x = E \otimes_{\mathrm{ev}_x} \mathbb{C}$, where $\mathrm{ev}_x : C(X) \to \mathbb{C}$ denotes evaluation at x (note that $C^\infty(E^x) = C^\infty(E) \otimes_{\mathrm{ev}_x} \mathbb{C}$). Thus, for a general unital $C_0(X)$-algebra B, we think of D as a generalized family of elliptic operators parametrized by X.

Since B is a $C_0(X)$-algebra and $\mathscr{H}_E$ is a Hilbert B-module, the C^*-algebra of compact operators $\mathscr{K}(\mathscr{H}_E)$ is also a $C_0(X)$-algebra. For each $x \in \mathbb{R}$ and $t \geqslant 1$, the operator $x\varepsilon + t^{-1}D$ is a self-adjoint regular operator on $\mathscr{H}_E$ with compact resolvents $(x\varepsilon + t^{-1}D \pm i)^{-1}$. There is a continuous family of $*$-homomorphisms [Tro99]

$$\{\phi_t^D\} : C_0(\mathbb{R}) \to C_0(\mathbb{R}) \otimes \mathscr{K}(\mathscr{H}_E) : g \mapsto g(x\varepsilon + t^{-1}D)$$

which is defined by the functional calculus for self-adjoint regular operators [Lan95]. Using the universal property of the maximal tensor product, we obtain an asymptotic $C_0(X)$-morphism

$$\{\hat{\phi}_t^D\}: C_0(X) \otimes C_0(\mathbb{R}) \to C_0(\mathbb{R}) \otimes \mathscr{K}(\mathscr{H}_E)$$

by mapping $f \otimes g \mapsto f \cdot g(x\varepsilon + t^{-1}D)$.

DEFINITION 5.4. We define $[\![D]\!]_X = [\![\hat{\phi}_t^D]\!]_X \in \mathscr{R}E(X; C_0(X), \mathscr{K}(\mathscr{H}_E))$.

Let ℓ_B^2 denote the standard Hilbert B-module. Using the isomorphism $\mathscr{H}_E \oplus \ell_B^2 \cong \ell_B^2$ that comes from the Kasparov Stabilization Theorem, we have an inclusion $i: \mathscr{K}(\mathscr{H}_E) \hookrightarrow \mathscr{K}(\ell_B^2) \cong \mathscr{K} \otimes B$, where $\mathscr{K}$ is the algebra of compact operators on separable Hilbert space. Thus, we can push forward the $\mathscr{R}E$-theory class of D to obtain $i_* [\![D]\!]_X \in \mathscr{R}E(X; C_0(X), B)$.

As an example, let M be a smooth compact even-dimensional spin manifold with Dirac operator $\not{D}: C^\infty(S) \to C^\infty(S)$, where $S = S^+ \oplus S^- \to M$ is the (graded) spinor bundle. Let $F \to M$ be a smooth vector B-bundle with connection $\nabla^F: C^\infty(F) \to C^\infty(T^*M \otimes F)$. By twisting the Dirac operator $\not{D}$ with the connection ∇^F, we obtain an elliptic differential B-operator $\not{D}_F: C^\infty(S \otimes F) \to C^\infty(S \otimes F)$ associated to the vector B-bundle $E = S \otimes F \to M$ as above.

5.3. *Fundamental classes of unital $C_0(X)$-algebras.* Let A be a unital separable $C_0(X)$-algebra. Since A is unital, $ZM(A) = Z(A) \subset A$, and so we can consider the structural homomorphism $\Phi_A: C_0(X) \to ZM(A) \hookrightarrow A$ as a canonically given $C_0(X)$-morphism.

DEFINITION 5.5. Let A be a unital separable $C_0(X)$-algebra. We define the *fundamental class of A* to be the $\mathscr{R}E$-theory element

$$[\![A]\!]_X = [\![\Phi_A]\!]_X \in \mathscr{R}E(X; C_0(X), A)$$

determined by the structural homomorphism.

PROPOSITION 5.6. *Let A and B be unital $C_0(X)$-algebras. Under the $C_0(X)$-tensor product operation*

$$\mathscr{R}E(X; C_0(X), A) \otimes \mathscr{R}E(X; C_0(X), B) \to \mathscr{R}E(X; C_0(X), A \otimes_X B)$$

we have $[\![A]\!]_X \otimes [\![B]\!]_X = [\![A \otimes_X B]\!]_X$.

Proof. This follows from Definition 2.7 and the isomorphism $C_0(X) \otimes_X C_0(X) \cong C_0(X)$. Q.E.D

PROPOSITION 5.7. *Let $p\colon Y \to X$ be a continuous map of compact spaces. Let A be a unital $C(X)$-algebra. Under the pullback transformation*

$$p^*\colon \mathscr{R}E(X; C(X), A) \to \mathscr{R}E(Y; C(Y), p^*A)$$

we have $p^*[\![A]\!]_X = [\![p^*A]\!]_Y$.

Proof. Since X and Y are compact and A is unital, $p^*A = C(Y) \otimes_X A$ is also unital. Let $\mathrm{id}_Y\colon C(Y) \to C(Y)$ denote the identity map. From the definition of the pullback,

$$p^*(\Phi_A) = \mathrm{id}_Y \otimes_X \Phi_A.$$

Take $g \in C(Y)$ and $f \in C(X)$. On elementary tensors $h \otimes_X a \in p^*A$ we have

$$\begin{aligned}(\mathrm{id}_Y \otimes_X \Phi_1)(g \otimes_X f)(h \otimes_X a) &= gh \otimes_X fa = g(f \circ p)\, h \otimes_X a \\ &= \Phi_{p^*A}(g(f \circ p))(h \otimes_X a).\end{aligned}$$

Since $p^*C(X) \cong C(Y)$, we see that $p^*(\Phi_A) = \Phi_{p^*A}$, as desired. Q.E.D

5.4. *Central bimodules.* Let A be a separable C^*-algebra with unit and center $Z(A)$ (we restrict to unital C^*-algebras so that we are guaranteed that A has a nonempty center).

DEFINITION 5.8. A *central bimodule* over A is an A-A-bimodule $\mathscr{E}$ such that $z \cdot e = e \cdot z$ for all $e \in \mathscr{E}$ and $z \in Z(A)$.

Central bimodules have found several applications in noncommutative geometry and noncommutative physics [DVM96, DHLS96, Mas96]. This is due to the fact that the module $\mathscr{E} = \Gamma(E)$ of continuous sections of a complex vector bundle $E \to X$ on a compact space X naturally defines a bimodule over the C^*-algebra $A = C(X)$ for which the left and right actions are compatible. Allowing A to be noncommutative, we see that the notion of a central bimodule is a slight generalization of the notion of a complex vector bundle to the noncommutative setting.

Let $X = \widehat{Z(A)}$ denote the spectrum of the center of A, i.e., $Z(A) \cong C(X)$ (note that X is compact since $Z(A)$ is unital). This gives A the structure of a $C(X)$-algebra. Thus, if $(\mathscr{E}, \psi, T)$ is a (unbounded) $\mathscr{R}KK(X; A, A)$-cycle, then $\mathscr{E}$ is naturally endowed with the structure of a central bimodule over A.

Dubois-Violette and Michor [DVM96] have asked how to define a suitable notion of K-theory for A which is appropriate for defining invariants for central bimodules. The previous observations lead us to propose the following definition.

DEFINITION 5.9. Let A be a C^*-algebra with unit. We define the *central K-theory* $ZK(A)$ of A to be

$$ZK(A) = \mathscr{R}E(\widehat{Z(A)}; A, A).$$

If A is $\mathscr{R}KK(X)$-nuclear (see Theorem 4.7), then $ZK(A) \cong \mathscr{R}KK(\widehat{Z(A)}; A, A)$.

The relationship between K-theory and central K-theory is contained in the following proposition, which gives the two extremes for the center of A.

LEMMA 5.10. *Let A be a C^*-algebra with unit. Then $ZK(A)$ is a ring with multiplication given by the composition product. If A is commutative, then $ZK(A) \cong K_0(A)$. If $Z(A) = \mathbb{C}$ and A is K-nuclear, then $ZK(A) \cong KK(A, A)$.*

Proof. That $ZK(A)$ is a ring with multiplication given by the composition product follows from Theorem 3.17. If $A \cong C(X)$ then $ZK(A) = \mathscr{R}E(X; C(X), C(X)) \cong \mathscr{R}KK(X; C(X), C(X)) \cong K^0(X) \cong K_0(A)$ by Proposition 2.20 in [Kas88]. If $Z(A) = \mathbb{C}$ then $X = \widehat{Z(A)} = \bullet$ and $ZK(A) = \mathscr{R}E(\bullet, A, A) = E(A, A)$ and the result follows. Q.E.D

Note that the ring structure incorporates the fact that if $\mathscr{E}$ and $\mathscr{F}$ are central bimodules over A then $\mathscr{E} \otimes_A \mathscr{F}$ is also a central bimodule over A.

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