PRELIM PROBLEM SOLUTIONS
THE GRAD STUDENTS + KEN

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1. Complex Analysis Practice Problems 2.0

Complex 2.0 #9.2
Let $D$ be a domain which contains in its interior the closed unit disk $|z| \leq 1$. Let $f(z)$ be analytic in $D$ except at a finite number of points $z_1, \ldots, z_k$ on the unit circle $|z| = 1$ where $f(z)$ has first order poles with residues $s_1, \ldots, s_k$. Let the Taylor series of $f(z)$ at the origin be $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Prove that there exists a positive constant $M$ such that $|a_n| \leq M$.

Proof. From the given information, the function $g$ defined by

$$ g(z) = f(z) - \sum_{j=1}^{k} \frac{s_j}{z - z_j} $$

is holomorphic on all of $D$. In particular, its Taylor series $\sum_{n=0}^{\infty} b_n z^n$ converges to $g$ with a radius of convergence $r > 1$. Then

$$ \limsup_{n \to \infty} \sqrt[n]{|b_n|} = \frac{1}{r} < 1, $$

which implies that

$$ \limsup_{n \to \infty} |b_n| < 1, $$

which in turn implies that there exists some constant $B > 0$ such that $|b_n| < B$ for all $n \geq 0$. If we solve for $f(z)$ in the formula for $g(z)$ above, we get

$$ f(z) = g(z) + \sum_{j=1}^{k} \frac{s_j}{z - z_j} = \sum_{n=0}^{\infty} b_n z^n + \sum_{j=1}^{k} -\frac{s_j}{z_j} \frac{1}{1 - \frac{z}{z_j}} $$

$$ = \sum_{n=0}^{\infty} b_n z^n + \sum_{j=1}^{k} -\frac{s_j}{z_j} \sum_{p=0}^{\infty} (\frac{z}{z_j})^{-p} z^p $$

$$ = \sum_{n=0}^{\infty} \left( b_n - \sum_{j=1}^{k} \frac{s_j}{z_j} \frac{1}{z^{n+1}} \right) z^n = \sum_{n=0}^{\infty} a_n z^n. $$
Then
\[
|a_n| = \left| b_n - \sum_{j=1}^{k} \frac{s_j}{z_j^{n+1}} \right|
\leq B + \sum_{j=1}^{k} \frac{|s_j|}{|z_j|^{n+1}}
= B + \sum_{j=1}^{k} |s_j|.
\]

Thus, the conclusion holds with \( M = B + \sum_{j=1}^{k} |s_j| \). \( \square \)

**Complex 2.0 #12.5**

Let \( A \) be a simply-connected open set in \( \mathbb{C} \), and let \( \alpha \) be a closed, Jordan, rectifiable curve in \( A \) with interior \( I(\alpha) \). Suppose that \( f \) is a holomorphic function on \( A \) such that the restriction \( f|_{\alpha} \) is one-to-one. Prove that \( f \) has at most one zero in \( I(\alpha) \).

**Proof.** Note that \( f \) has no poles.

First, suppose that \( f \) has no zero on \( \alpha \).

Then the argument principle implies that
\[
Z = Z - P = \frac{1}{2\pi i} \int_{\alpha} \frac{f'(z)}{f(z)} \, dz = n(f(\alpha), 0),
\]
where \( Z \) is the number of zeroes of \( f \) in \( I_{\alpha} \). \( P \) is the number of poles of \( f \) in \( I_{\alpha} \), and \( n(f(\alpha), 0) \) is the winding number of \( f(\alpha) \) around zero. Since the restriction \( f|_{\alpha} \) is one-to-one, the \( f(\alpha) \) cannot cross itself. Thus the winding number can at most be one (one if 0 is inside \( f(\alpha) \) and zero if 0 is not inside \( f(\alpha) \)). Therefore, \( Z = 0 \) or \( Z = 1 \). Thus \( f \) has at most one zero in \( I_{\alpha} \).

Finally, we consider the case when \( f \) has a zero \( z_0 \) on \( \alpha \). (to be continued...)

**Complex 2.0 #13.7**

Find the set of all possible orientation-preserving conformal maps from \( A = \{ z \in \mathbb{C} : 0 < \Im(z) < \pi \} \) to \( B = \{ z \in \mathbb{C} : \Im(z) > 0 \} \), and prove that no other maps are possible.

**Proof.** First, observe that for any fixed \( y \in (0, \pi) \), the straight horizontal line \( \{ t + iy : t \in \mathbb{R} \} \) gets mapped by \( f(z) = e^z \) to \( \{ e^t e^{iy} : t \in \mathbb{R} \} \), which is an open ray from the origin to \( \infty \) at angle \( y \).

Thus, since the map is \( 1-1 \), \( f \) is an orientation-preserving conformal mapping of \( A \) to \( B \). Observe that the orientation-preserving conformal maps from \( B \) to itself are of the form
\[
h(z) = \frac{az + b}{cz + d},
\]
where \( a, b, c, d \in \mathbb{R} \) and \( ad - bc \) can be assumed to be 1. Thus, any map of the form \( h \circ f \) is an orientation-preserving conformal map from \( A \) to \( B \).

Conversely, given any orientation-preserving conformal map \( \phi : A \to B \), then \( \phi \circ f^{-1} : B \to B \) is an orientation-preserving conformal map from \( B \) to itself, so that \( \phi \circ f^{-1} \) must have the form of such an \( h \) above, but then \( \phi = h \circ f \) in the form we obtained before.

In summary, all orientation-preserving conformal maps from \( A \) to \( B \) are of the form
\[
F(z) = \frac{ae^z + b}{ce^z + d},
\]
with \( a, b, c, d \in \mathbb{R} \) and we may assume \( ad - bc = 1 \). \( \square \)
Remark: If the word “orientation-preserving” is deleted, we would need to include the possibility of the antiholomorphic maps, which have the form

$$G(z) = \frac{-ae^z + b}{-ce^z + d},$$

with $a, b, c, d \in \mathbb{R}$ and we may assume $ad - bc = 1$.

**Complex 2.0 #14.1**

Let

$$f(z) = \int_{-1}^{1} \frac{e^{-u^2}}{u - z} \, du.$$

(a) Show that $f(z)$ is analytic in $\mathbb{C} - [-1, 1]$.

(b) Show that $f(z)$ may be continued analytically across the open segment $(-1, 1)$.

(c) Show that the analytic continuations of $f$ from above $(-1, 1)$ and from below $(-1, 1)$ are different. What is their difference on the cut $(-1, 1)$?

**Proof.** (a) Let $z_0 = x_0 + iy_0 \in \mathbb{C} \setminus [-1, 1]$. Observe that the integrand, as a function of $z = x + iy$, is smooth as a function of $x$ and $y$. Further, its partial derivatives with respect to $x$ and $y$ are bounded and continuous and thus integrable when $z \in \mathbb{C} \setminus [-1, 1]$. Therefore, we may differentiate under the integral sign to evaluate

$$\frac{d}{dz} f(z) = \int_{-1}^{1} \frac{d}{dz} \left( \frac{e^{-u^2}}{u - z} \right) \, du = 0,$$

since $\frac{1}{u - z}$ is holomorphic in $z$. Therefore, $f$ is holomorphic on $\mathbb{C} - [-1, 1]$.

(b) Observe that if $C_0$ is the upper half of the unit circle, oriented counterclockwise and if $C$ denotes the closed oriented curve that is the union of $C_0$ and $[-1, 1]$, then the residue theorem yields

$$\int_{C} \frac{e^{-u^2}}{w - z} \, dw = \begin{cases} 
2\pi i e^{-z^2} & z \in \mathcal{I}(C) \\
0 & z \text{ outside of } C,
\end{cases}$$

where $\mathcal{I}(C)$ denotes the interior of $C$. Thus,

$$f(z) = \int_{-1}^{1} \frac{e^{-u^2}}{u - z} \, du = \begin{cases} 
2\pi i e^{-z^2} - \int_{C_0} \frac{e^{-w^2}}{w - z} \, dw & z \in \mathcal{I}(C) \\
- \int_{C_0} \frac{e^{-w^2}}{w - z} \, dw & z \text{ outside of } C
\end{cases}$$

Note that both the top and the bottom of the right hand side give two different analytic continuations of $f$ to the inside of $C$, and note that in both cases the formula is valid at all points away from $C_0$. The upper formula gives a continuation from the upper half plane to points at or below $[-1, 1]$, and the lower formula gives a continuation from the lower half plane to points at or above $[-1, 1]$.

(c) From the formula above, the difference between the analytic continuations from below and above at the point $x \in [-1, 1]$ is $2\pi i e^{-x^2}$. \qedhere

**Complex 2.0 #14.3**

Let

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{zx}}{1 + e^x} \, dx.$$

(a) Determine the set of $z$ for which the integral converges. (b) Show that $F$ can be analytically continued, and find the largest possible domain of its analytic continuation.
2. Real Analysis Practice Problems 2.0

Real 2.0 #2.4
Let $a$ be a positive real number. Define a sequence $(x_n)$ by

$$x_0 = 0, \quad x_{n+1} = a + x_n^2, \quad n \geq 0.$$ 

Find a necessary and sufficient condition on $a$ in order that a finite limit $\lim_{n \to \infty} x_n$ should exist.

Proof. We have $x_0 = 0$, $x_1 = a$, $x_2 = a + a^2$. Clearly each $x_j$ is positive. Note that initially $(x_j)$ is strictly increasing. Also,

$$x_{n+1} - x_n = (a + x_n^2) - (a + x_{n-1}^2) = x_n^2 - x_{n-1}^2 = (x_n - x_{n-1})(x_n + x_{n-1}),$$

which is positive if $(x_n - x_{n-1})$ is positive. By induction $x_{n+1} > x_n$ for all $n$.

Next, suppose the limit of the sequence $L$ exists. Then

$$\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} (a + x_n^2),$$

so that $L^2 - L + a = 0$, so that $L = \frac{1 + \sqrt{1 - 4a}}{2}$, so that a necessary condition on $a$ is that $0 < a \leq \frac{1}{4}$.

Now, if $0 < a \leq \frac{1}{4}$, then observe that $x_0, x_1 \leq \frac{1}{2}$. Then for $n \geq 1$, if $x_n \leq \frac{1}{2}$, then

$$x_{n+1} = a + x_n^2 \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

By induction, $x_n \leq \frac{1}{2}$ for all $n$, and so $(x_n)$ is a bounded increasing sequence and thus converges. Therefore, $0 < a \leq \frac{1}{4}$ is a necessary and sufficient condition. \[\square\]

Real 2.0 #5.3
Show that there exist constants $a$ and $b$ such that, for all integers $N \geq 1$,

$$\left| \sum_{n=1}^{N} \frac{1}{\sqrt{n}} - 2\sqrt{N} - a \right| < \frac{b}{\sqrt{N}}.$$ 

Real 2.0 #5.10
Let $f$ and $g$ be continuous functions on $\mathbb{R}$ such that $f(x + 1) = f(x)$ and $g(x + 1) = g(x)$ for all $x \in \mathbb{R}$. Prove that

$$\lim_{n \to \infty} \int_{0}^{1} f(x) g(nx) \, dx = \int_{0}^{1} f(x) \, dx \int_{0}^{1} g(x) \, dx.$$ 

Proof. Note that

$$L_n = \int_{0}^{1} f(x) g(nx) \, dx = \frac{1}{n} \int_{0}^{n} f\left( \frac{y}{n} \right) g(y) \, dy = \sum_{k=1}^{n} \frac{1}{n} \int_{k-1}^{k} f\left( \frac{y}{n} \right) g(y) \, dy$$

Since $f$ is continuous on $[0, 1]$, it is uniformly continuous, and thus for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ and $x, y \in [0, 1]$ implies that $|f(x) - f(y)| < \varepsilon$. We choose $n \in \mathbb{Z}_{>0}$ such that $\frac{1}{n} < \delta$. Let

$$M_n = \sum_{k=1}^{n} \frac{1}{n} \int_{k-1}^{k} f\left( \frac{k}{n} \right) g(y) \, dy.$$
Then
\[|L_n - M_n| = \left| \sum_{k=1}^{n} \frac{1}{n} \int_{k-1}^{k} \left( f \left( \frac{y}{n} \right) - f \left( \frac{k}{n} \right) \right) g(y) \, dy \right|
\]
\[\leq \sum_{k=1}^{n} \frac{1}{n} \int_{k-1}^{k} \left| f \left( \frac{y}{n} \right) - f \left( \frac{k}{n} \right) \right| |g(y)| \, dy \leq \sum_{k=1}^{n} \frac{1}{n} \int_{k-1}^{k} \varepsilon |g(y)| \, dy
\]
\[\leq \varepsilon \int_{0}^{1} |g(y)| \, dy.
\]
This can be made arbitrarily small by choosing \( n \) large enough. On the other hand,
\[M_n = \sum_{k=1}^{n} \frac{1}{n} \int_{k-1}^{k} f \left( \frac{k}{n} \right) g(y) \, dy
\]
\[= \sum_{k=1}^{n} \frac{1}{n} f \left( \frac{k}{n} \right) \int_{k-1}^{k} g(y) \, dy = \left( \sum_{k=1}^{n} \frac{1}{n} f \left( \frac{k}{n} \right) \right) \int_{0}^{1} g(y) \, dy.
\]
But
\[\sum_{k=1}^{n} \frac{1}{n} f \left( \frac{k}{n} \right) \rightarrow \int_{0}^{1} f(x) \, dx
\]
as \( n \rightarrow \infty \) (a right Riemann sum), so that \( \lim_{n \rightarrow \infty} M_n = \int_{0}^{1} f(x) \, dx \int_{0}^{1} g(x) \, dx \). Then, by the above, \( \lim_{n \rightarrow \infty} L_n \) also exists and
\[\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} M_n = \int_{0}^{1} f(x) \, dx \int_{0}^{1} g(x) \, dx.
\]
\[\square
\]
**Real 2.0 #6.6**

Let
\[F(g)(t) := \int_{0}^{\infty} \exp(-tx)g(x) \, dx.
\]

(a) Prove that if \( g \) is continuous and bounded on \((0, \infty)\), then \( F(g) \) is continuous and bounded on \((1, \infty)\).

(b) Prove that if \( g \) is continuous and bounded on \((0, \infty)\), then it is not necessarily true that \( F(g) \) is continuous and bounded on \((0, \infty)\).

(c) Prove that if \( g \) is continuous, bounded and improper Riemann integrable on \([0, \infty)\), then \( F(g) \) is continuous, bounded and improper Riemann integrable on \([1, \infty)\).

(d) Prove or disprove the converses of (a) and (c).

(e) Show that even if \( g \) is continuous, bounded and improper Riemann integrable on \([0, \infty)\), then \( F(g) \) is not necessarily Riemann integrable on \([0, \infty)\).

**Real 2.0 #11.1**

Discuss the number of solutions in \((x, y)\) to
\[u = x + y^2
\]
\[v = y + xy
\]
for \((u, v)\) sufficiently close to \((0, 0)\).
Proof. Let \( f(x, y) = (x + y^2, y + xy) \). Suppose that \( f(x, y) \) is very close to \((0, 0)\), so there exists a very small \( \varepsilon > 0, \varepsilon < 0.01 \) such that \( |x + y^2| < \varepsilon \) and \( |y + xy| < \varepsilon \). In particular, this means that \((x, y)\) is between the parabolas \( x = -y^2 - \varepsilon \) and \( x = -y^2 + \varepsilon \) and is within the hyperbolic boundaries of \( |y| |1 + x| = \varepsilon \). In particular, this means that \((x, y)\) is within a \(2\varepsilon\)-ball of one of the three intersection points of the parabola \( x = -y^2 \) and the double line \( x(y + 1) = 0\), i.e. the points in \( f^{-1}\{0, 0\} = \{(-1, 1), (-1, -1), (0, 0)\}\). (To see the computation: if \( f(x, y) = (0, 0)\), then \( x + y^2 = 0, y + xy = 0\). Then \( y = 0\) or \( x = -1\). In the first case, \( x = 0\). In the second case, \( y = \pm 1\).) We will choose \( \varepsilon \) to be very small, very soon. What we know now is that if \( f(x, y) = (u, v) \) is within \( \varepsilon \) of \((0, 0)\), then \((x, y)\) is within \(2\varepsilon\) of one of the three points.

Observe that \( f \) is \( C^\infty \). We compute
\[
f'(x, y) = \begin{pmatrix} 1 & 2y \\ y & x + 1 \end{pmatrix},
\]
and
\[
f'(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f'(-1, 1) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad f'(-1, -1) = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix},
\]
all of which have nonzero determinants and so are invertible. By the inverse function theorem, there exists a neighborhood \( U_1 \) of \((0, 0)\) and a neighborhood \( V_1 \) of \((0, 0)\) such that \( f : U_1 \to V_1 \) is a homeomorphism and that \((f|_U)^{-1} : V_1 \to U_1\) is differentiable and continuous. Similarly, there exists a neighborhood \( U_2 \) of \((-1, 1)\) and a neighborhood \( V_2 \) of \((0, 0)\) such that \( f : U_2 \to V_2 \) is a homeomorphism and that \((f|_U)^{-1} : V_2 \to U_2\) is differentiable and continuous, and there exists a neighborhood \( U_3 \) of \((-1, -1)\) and a neighborhood \( V_3 \) of \((0, 0)\) such that \( f : U_3 \to V_3 \) is a homeomorphism and that \((f|_U)^{-1} : V_3 \to U_3\) is differentiable and continuous.

Next, choose \( \varepsilon > 0 \) such that the \(2\varepsilon\)-balls around \( f^{-1}\{0, 0\}\) are contained inside \( f^{-1}\{U_1 \cap V_2 \cap V_3\}\). Then we have shown that if \((u, v)\) is in the ball of radius \( \varepsilon \) of \((0, 0)\), there exist exactly 3 points in \( f^{-1}\{(u, v)\}\). \(\square\)

Real 2.0 #11.4
Let \( A \subseteq \mathbb{R}^3 \) be the set defined by \( x^3y + y^3z^2 - 2xz^4 = 2 \). Prove or disprove that there exists \( \delta > 0 \) and a curve \( \alpha : (-1 - \delta, -1 + \delta) \to A \) such that \( \alpha(t) = (t, g(t), h(t)) \) with \( g \) and \( h \) differentiable.

Real 2.0 #13.3

The function \( h \) is periodic of period 4 and satisfies
\[
h(t) = \begin{cases} 
1, & 0 \leq t < 1 \\
0, & 1 \leq t < 4
\end{cases}
\]

(a) Find the corresponding Fourier series for \( h \).
(b) Prove or disprove that the Fourier series converges uniformly to \( h \) on \([1, 2]\).
(c) Prove or disprove that the Fourier series converges uniformly to \( h \) on \([2, 3]\).

Proof. (a) The Fourier series of \( h \) is
\[
H(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi}{2} t \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi}{2} t \right),
\]
and the coefficients satisfy
\[
a_0 = \frac{1}{4} \int_0^4 h(t) \, dt = \frac{1}{4};
\]
\[
a_n = \frac{1}{2} \int_0^4 h(t) \cos\left(\frac{n\pi}{2} t\right) \, dt
= \frac{1}{2} \int_0^1 \cos\left(\frac{n\pi}{2} t\right) \, dt
= \begin{cases} 
\frac{(-1)^{k}}{n\pi} & n = 2k + 1 \text{ for } k \in \mathbb{Z}_{\geq 0}; \\
0 & n = 2\ell \text{ for } \ell \in \mathbb{Z}_{\geq 1}
\end{cases};
\]
\[
b_n = \frac{1}{2} \int_0^4 h(t) \sin\left(\frac{n\pi}{2} t\right) \, dt
= \begin{cases} 
\frac{1}{n\pi} & n = 2k + 1 \text{ for } k \in \mathbb{Z}_{\geq 0} \\
\frac{2}{n\pi} & n = 2(2\ell + 1) \text{ for } \ell \in \mathbb{Z}_{\geq 0} \\
0 & n = 2(2m) \text{ for } m \in \mathbb{Z}_{\geq 1}
\end{cases}.
\]

(b) Observe that if we let
\[
S_N(t) = a_0 + \sum_{n=1}^N a_n \cos\left(\frac{n\pi}{2} t\right) + \sum_{n=1}^N b_n \sin\left(\frac{n\pi}{2} t\right),
\]
Then this function evaluated at \( x = 1 \) is \( S_N(1) = \frac{1}{2} \) (the average of the one sided limits). If \( S_N \) did converge uniformly to \( h \), then the limit function would be continuous, and then \( S_N(1) \) would have to converge to 0, a contradiction. Thus \( S_N \) does not converge to \( h \) uniformly on \([1, 2] \).

(c) Because \( h \) is continuously differentiable on the interval \([2, 3] \), the convergence is uniform there. \( \square \)

**Additional Problem**

Let \( f : \mathbb{R} \to \mathbb{R} \) satisfy

1. \( f \) is continuous on \([0, \infty) \).
2. \( f' \) (x) exists for all \( x \geq 0 \).
3. \( f (0) = 0 \).
4. \( f' \) is increasing.

For \( x > 0 \), define \( g(x) = \frac{f(x)}{x} \). Prove that \( g \) is increasing.

**Proof.** Observe that \( g' \) exists on \((0, \infty) \), and
\[
g'(x) = \frac{f'(x)x - f(x)}{x^2}.
\]
So it suffices to show that \( f'(x)x - f(x) \geq 0 \). Note that by the FTC \( f(x) = \int_0^x f'(t) \, dt \leq xf'(x) \) since \( f' \) is increasing. Thus,
\[
f'(x)x - f(x) \geq f'(x)x - xf'(x) = 0.
\]

Alternately, we could use the mean value theorem: For any \( x > 0 \), there exists \( c \) such that \( 0 < c < x \) and
\[
f'(x) \geq f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.
\]
Then since \( x > 0 \), we have \( xf'(x) \geq f(x) \), so \( f'(x)x - f(x) \geq 0 \). \( \square \)

**Additional Problem**

Prove that
\[
\lim_{t \to \infty} \int_1^2 \frac{\sin(tx)}{x^2\sqrt{x - 1}} \, dx = 0.
\]
Proof. The integral is
\[ \int_{1}^{2} \frac{\sin(tx)}{x^2 \sqrt{x-1}} \, dx \]
\[ \square \]

Additional Problem
Let \( f : [0, 1] \to \mathbb{R} \). Suppose that

1. \( f \in C^\infty [0, 1] \)
2. \(|f(1)| \geq |f(0)|\)

Prove that either there exists \( x \in (0, 1) \) such that \( f(x) \) and \( f'(x) \) have the same sign, or \( f \) is constant.

Proof. Observe that the given \( f(x) \) satisfies \( g(x) = f(x)^2 \in C^\infty [0, 1] \), \( g(1) \geq g(0) \). We see by the mean value theorem that there exists \( c \in (0, 1) \) such that \( g'(c) = g(1) - g(0) \). If \( g(1) > g(0) \), we are done, because then \( g'(c) = 2f'(c) f(c) > 0 \), and so \( f \) and \( f' \) have the same sign at \( c \). If \( g(1) = g(0) \), then \( g'(c) = 0 \). If there is no point \( x \in (0, 1) \) at which \( g'(x) > 0 \), then \( g'(x) \leq 0 \) for all \( x \in (0, 1) \). If there exists \( x \in (0, 1) \) such that \( g'(x) < 0 \), then in fact this is true for an interval around \( x \), so that
\[ g(1) - g(0) = \int_{0}^{1} g'(x) < 0, \]
a contradiction, so in this case we must have \( g'(x) = 0 \) for all \( x \). Therefore there exists \( c \in (0, 1) \) such that \( g'(c) = f(c) f'(c) > 0 \), or else \( g(x) \) is constant (and thus \( f(x) \) is constant).

\[ \square \]

Additional Problem
Does there exist a continuous function \( f \) such that

1. \( f : (0, 1) \to [0, 1] \) is onto?
2. \( f : [0, 1] \to (0, 1) \) is onto?
3. \( f : (0, 1) \to [0, 1] \) is 1-1 and onto?
4. \( f : [0, 1] \to (0, 1) \) is 1-1 and onto?

Proof. (1) yes. eg. let \( f(x) = \frac{1}{2} + \frac{1}{2} \sin(2\pi x) \).
(2) no. Since the continuous image of a compact set is compact, \( f([0,1]) \) must be compact, but \((0,1)\) is not compact.
(3) no. Suppose you have such and \( f \). Then there exist \( a, b \in (0, 1) \) such that \( f(a) = 0, f(b) = 1 \). Without loss of generality, suppose \( a < b \). By the intermediate value theorem, for all \( x \in (0, 1) \), there exists \( y \in (a, b) \) such that \( f(y) = x \). Thus, \( f : (a, b) \to (0, 1) \) is onto. Thus, \( f \) cannot be 1-1, since for any \( z \in (0, a), f(z) = f(x) \) for some \( x \in [a,b] \).
(4) no. Same reason as (2).

\[ \square \]

3. Algebra Practice Problems 2.0

Algebra 2.0 #2.2
Prove that any group \( G \) of order 6 is isomorphic to \( Z_6 \) or \( S_3 \).

Proof. (Case 1) If \( G \) is abelian, then by the FTFTGAG, \( G \cong Z_6 \) (since \( 6 = 2 \cdot 3 \) and \( \gcd(2,3) = 1 \), so \( Z_6 \cong Z_2 \times Z_3 \).
(Case 1') If \( G \) has an element of order 6, then it is cyclic and is therefore isomorphic to \( Z_6 \).
(Case 2) Otherwise, \( G \) must have elements of order 1 (the identity), so every other element has order 2 or 3, by Lagrange’s theorem. By Cauchy’s Theorem, there exist elements of order two and three. Let \( H \) be the subgroup of generated by an element \( g \) of order three. This subgroup is normal since \([G:H] = \frac{|G|}{|H|} = 2\). Let \( a \in G \) such that \( a^2 = e \), so \( a \notin H \) since all elements of \( H \) have order
1 or 3. Then $aH \neq H$. Since $G = H \cup aH$, the elements of $G$ are \{e, g, g^2, a, ag, ag^2\}. The only multiplications that are not yet determined are $aga$, $agag$, $agag^2$, $ag^2a$, $ag^2ag$, $ag^2ag^2$, etc.

$aga$ can’t be $e$ because $ga$ can’t be $a = a^{-1}$.

Cases: (A) If $aga = g$, then $ga = ag$, and the group is abelian. (contradiction to Case 2 assumption).

(B) so we must have $aga = g^2$. Then $agag = e$, $agag^2 = g$, $ag^2a = agag = g^2g^2 = g$, $ag^2ag = g^2$, $ag^2ag^2 = g^2g = e$.

Also $ga = aaga = agg = a$, $gag^2 = ag$, $g^2a = g(ag^2) = ag$, $g^2ag = ag^2$, $g^2ag^2 = a$.

To prove $G \cong S_3$, we determine a map $\phi : G \rightarrow S_3$ by $\phi(g) = (1, 2, 3)$, and $\phi(a) = (1, 2)$. We check that

\[
\phi(aga) = \phi(a)\phi(g)\phi(a) = (1, 2)(1, 2, 3)(1, 2) = (1, 3, 2) = (1, 2, 3)^2 = \phi(g)^2 = \phi(g^2).
\]

Therefore, $\phi$ is a homomorphism. Since $S_3$ is generated by $(1, 2)$ and $(1, 2, 3)$, $\phi$ is onto, and since $|G| = |S_3|$, $\phi$ is bijective and thus and isomorphism.

\[\square\]

Algebra 2.0 #6.2

Let $R$ be a ring with identity, and let $u$ be an element of $R$ with a right inverse. Prove that the following conditions on $u$ are equivalent:

1. $u$ has more than one right inverse;
2. $u$ is a left zero divisor;
3. $u$ is not a unit.

Proof. (1)$\Rightarrow$(2). If $x$ and $y$ are right inverses of $u$, and if 1 is the multiplicative identity. If $ux = uy = 1$ then $ux - uy = 0$, so $u(x - y) = 0$. So if $x \neq y$ then $u$ is a left zero divisor.

(2)$\Rightarrow$(1). Let $ux = 1$, and suppose that $uz = 0$ for some $z \neq 0$. Then $1 = ux + uz = u(x + z)$, so $u$ has more than one right inverse.

(2)$\Rightarrow$(3). Let $u$ be a left zero divisor, so there exists $z \neq 0$ such that $uz = 0$. So if there exists $v$ such that $vw = vu = 1$, then $v(uz) = (vu)z = 1z = z$ but $v(uz) = v(0) = 0$, a contradiction. So $u$ is not a unit.

(3)$\Rightarrow$(2). Suppose that $u$ is not a unit but has a right inverse $x$. Then $ux = 1$, so $uxu = u$, or $uxu - u = 0$. Then $u(xu - 1) = 0$. But $xu - 1$ is nonzero because otherwise $x$ would be the inverse of $u$. Thus, $u$ is a left zero divisor. \[\square\]