A Remark on Connectedness in Hilbert Schemes

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Dedicated to Robin Hartshorne with affection and gratitude

Let $H_{d,g}$ denote the Hilbert scheme of locally Cohen-Macaulay curves in $\mathbb{P}^3$. For any $d > 4$ and $g \leq \binom{d-3}{2}$, $H_{d,g}$ has two well-understood irreducible families: There is a component $E \subset H_{d,g}$ corresponding to extremal curves (see [5]; these are the curves with maximal Rao function) and $S$, the family of subextremal curves (see [7]; these have the next largest Rao function). In this short note we show that $S \cap E \neq \emptyset$ in $H_{d,g}$ by constructing an explicit specialization (Prop. 1). Our construction also works for ACM curves of genus $g = \binom{d-3}{2} + 1$ (Remark 2) and hence $H_{d,g}$ is connected for $g > \binom{d-3}{2}$ (Corollary 3).

**Proposition 1** For each $d \geq 4$ and $g \leq \binom{d-3}{2}$ there exist extremal curves in $H_{d,g}$ which lie in the closure of the family of subextremal curves.

**Proof:** Fixing $g$ as in the statement, let $(x, y^{d-2})$ be the ideal of a planar multiple line of degree $d - 2$ with support the line $L$ given by \{x = y = 0\}. We define the map

$$(x, y^{d-2}) \mapsto S_L(-1)$$

by $x \mapsto 1, y^{d-2} \mapsto z^{d-3}$. This map is surjective and the kernel

$I_V = (x^2, xy, y^{d-1}, xz^{d-3} - y^{d-2}) = (x^2, xy, xz^{d-3} - y^{d-2})$

is the total ideal of an ACM curve $V$ of degree $d - 1$ and arithmetic genus $\binom{d-3}{2}$ supported on $L$. 

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Now we construct a multiplicity $d$-line with support $L$ as follows. We define the map $I_V \xrightarrow{\psi} S_L(b)$ by $x^2 \mapsto 0, xy \mapsto z^{b+2}, xz^{d-3} - y^{d-2} \mapsto w^b + d - 2$, where $b = \binom{d-3}{2} - 1 - g$ (it is easy to check that the kernel of the surjection $S(-d + 2) \oplus S(-2)^2 \to I_V$ is $S(-d + 1) \oplus S(-3)$ and maps to zero under $\psi$). Although $\psi$ is not surjective (unless $b = -2$; see Remark 2), its sheafification is. The kernel

$$I_W = (x^2, xy^2, y^{d-1} - xz^{d-3}, xyw^{b+d-2} - z^{b+2}(y^{d-2} - xz^{d-3})).$$

is the total ideal of a multiple $d$-line $W$ of genus $g$ as we can see from the exact sequence

$$0 \to I_W \to I_V \to \mathcal{O}_L(b) \to 0.$$ 

This sequence further shows that $H_1^*(I_W) \cong S/(x, y, z^{b+2}, w^{b+d-2})$ and hence the Rao function achieves the upper bound given in ([7], Thm. 2.11): thus $W$ is a subextremal curve.

Now we deform $W$ by considering the ideal

$$I_t = (x^2, xy^2, ty^{d-1} - xz^{d-3}, xyw^{b+d-2} - tz^{b+2}(ty^{d-2} - xz^{d-3}))$$

parametrized by $t \in \mathbb{A}^1$. Flattening over $\mathbb{A}^1$, we add to this ideal polynomials $p$ such that $pt \in I_t$ (see [3], III, Example 9.8.4). If $A, B, C$ are the last three generators appearing in $I_t$, we add

$$D = (w^{b+d-2}B + z^{d-3}C)/t = w^{b+d-2}y^{d-1} - z^{b+d-1}(ty^{d-2} - xz^{d-3})$$

and

$$E = (z^{d-3}A + yB)/t = y^d.$$ 

Letting $t \to 0$ we obtain the limit ideal

$$I_0 = (x^2, xy^2, xz^{d-3}, xyw^{b+d-2}, y^d, w^{b+d-2}y^{d-1} - z^{2d+b-4}x).$$

The saturation $I_0$ contains $(x^2, xy, y^d, w^{b+d-2}y^{d-1} - z^{2d+b-4}x)$, which is the saturated ideal of an extremal curve of degree $d$ and genus $g$ ([5], Prop. 0.6), completing the proof.

**Remark 2.** The deformation used in the proof above works if we take $b = -2$ (i.e. $g = \binom{d-3}{2} + 1$), but in this case the map $I_V \to S_L(b)$ is surjective and hence $H_1^*(I_W) = 0$. Thus $W$ is an ACM curve and we have produced extremal curves in the closure of the ACM curves of genus $g = \binom{d-3}{2} + 1$. 

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Corollary 3 The Hilbert scheme $H_{d,g}$ of locally Cohen-Macaulay curves in $\mathbb{P}^3$ is connected for $d \geq 4$ and $g > \binom{d-3}{2}$.

Proof: For $g > \binom{d-3}{2} + 1$ this is easy because $H_{d,g}$ is irreducible (the curves are either extremal or ACM; see [7] Lemma 2.5) or empty. In the remaining case $g = \binom{d-3}{2} + 1$ there are two irreducible components given by the ACM curves and the extremal curves. By Remark 2, we conclude that $H_{d,g}$ is connected.

Remark 4 In his thesis [1], Aït-Amrane proves Proposition 1 above using the theory of triades of families of curves [4], although he doesn’t recover the extension of Remark 2. His main result is that $H_{d,g}$ is connected for $d \geq 4, g = \binom{d-3}{2}$. This case is more complicated, as there is another irreducible component to deal with.

References


