1/25 | Calc : Last time

1.3 $\left\{ \begin{array}{l} \text{Trig functions} \\ \text{Radian measure} \end{array} \right.$

1.5 Exponential functions

For $\boxed{a>0}$, $f(x) = \underline{\underline{a}}^x$ is the exponential function base a

Ex1: Evaluate $f(x) = 2^x$ $(a=2)$

(a) $f(5) = 2^5 = 32$

(b) $f(-3) = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

(c) $f(0) = 1$

(d) $f(3/2) = 2^{3/2} = (2^3)^{1/2} = 8^{1/2} = \sqrt{8}$

(e) $f(-2/7) = 2^{-2/7} =$

$$\frac{1}{2^{2/7}} = \frac{1}{(2^2)^{1/7}} = \frac{1}{4^{1/7}} = \frac{1}{\sqrt[7]{4}}$$

Using: Laws of exponentials

For $a, b > 0$, x, y real.

① $a^x a^y = a^{x+y}$

② $\frac{a^x}{a^y} = a^{x-y}$

③ $(a^x)^y = (a^y)^x = a^{(xy)}$

④ $a^x b^x = (ab)^x$

⑤ $a^x / b^x = (a/b)^x$

(f) $f(\pi) = 2^{\pi}$??

no way to simplify, but

$$\pi = \underbrace{3}.1415926\ldots$$

$$2^3 = 8 < 2^{\pi} < 2^4 = 16$$

$$2^{3.1} = 2^{31/10} = \sqrt[10]{2^{31}} \cong 8.5741$$

$$2^{3.14} = 2^{314/100} = \sqrt[100]{2^{314}} \cong \boxed{8.815}$$

$$2^{3.15} = 2^{315/100} = \sqrt[100]{2^{315}} \cong \boxed{8.87}$$

take a limiting value

as $x \to \pi$, exact value

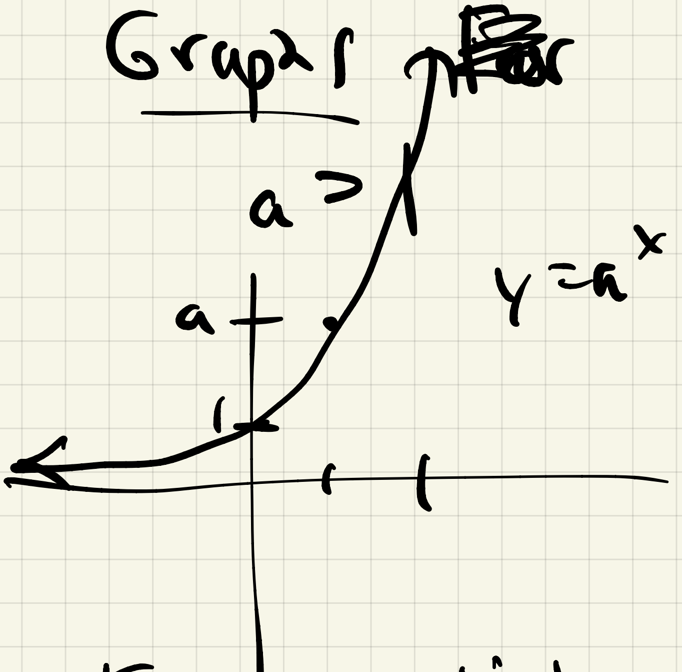
$$2^x \to 2^\pi$$

Ex 2 $\underbrace{27}^{3/4} \cdot 3^{7/4}$

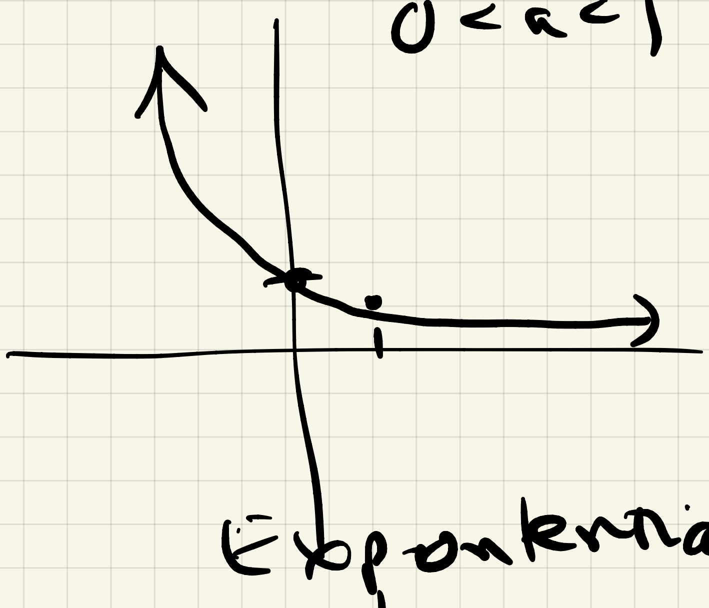
$$\underbrace{(3^3)^{3/4}} \cdot 3^{7/4} =$$

$$3^{9/4} \cdot 3^{7/4} = 3^{9/4+7/4} = 3^4 = 81$$

Graphs for $f(x)=a^x$



Exponential growth



Exponential decay

(population) (radioactive)

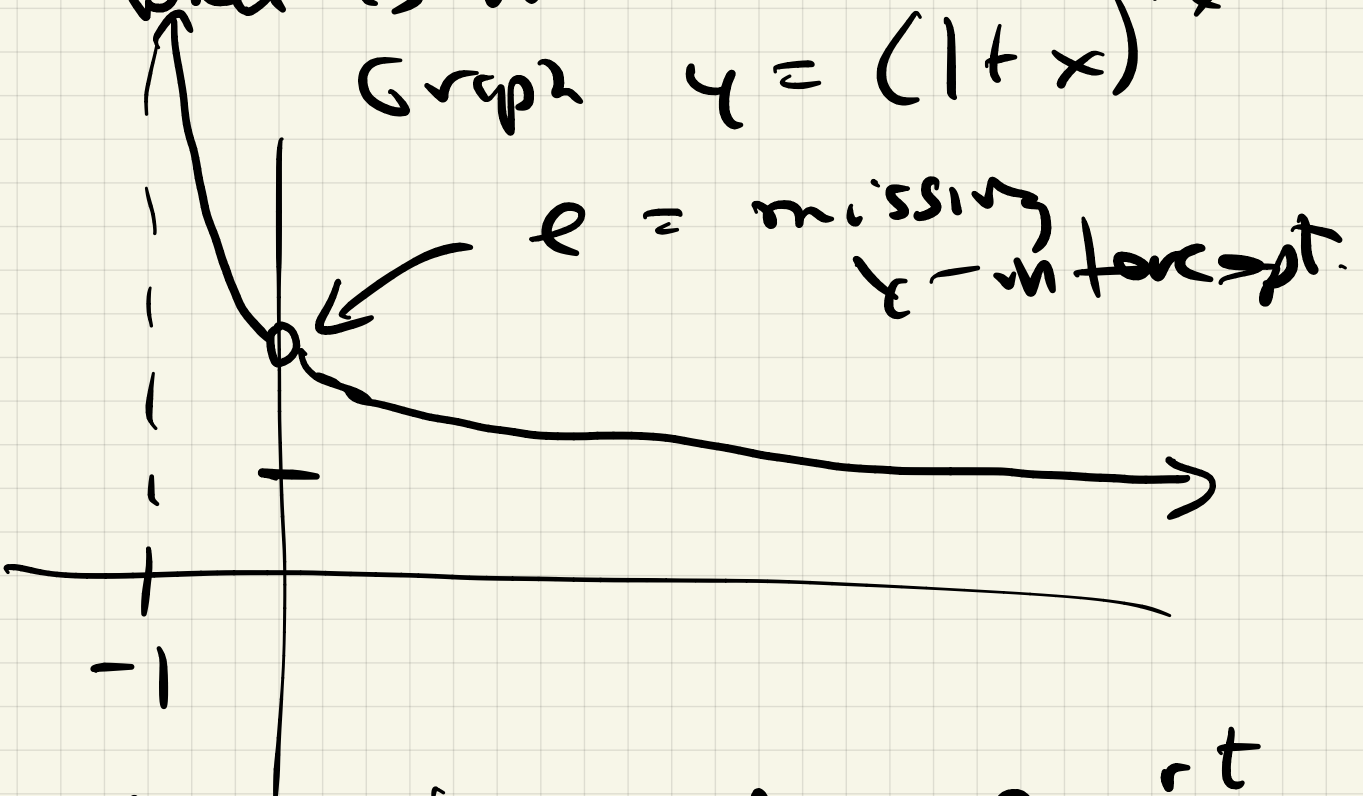
Rmk: In Calc 1-3, Diff Eqns usually use base $a = e$

↗ Euler's number

$e \approx 2.71828\ldots$

What is it?

Graph $y = (1+x)^{1/x}$



Application: $A = P_0 e^{rt}$

A: amt investin

P_0: principal

r = int rate

t: time

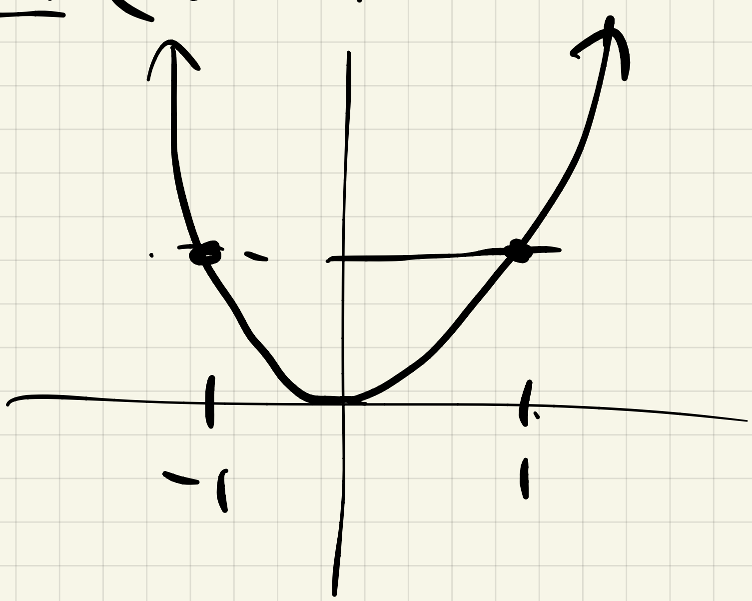
1.6 Inverse functions

Defn: A function $f: \overset{\text{domain}}{D} \to \underset{\text{target}}{\underset{\|}{Y}}$

is **one-to-one** (**1-1**)

if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

Ex1 (a) $f(x) = x^2$ $D = \mathbb{R}$



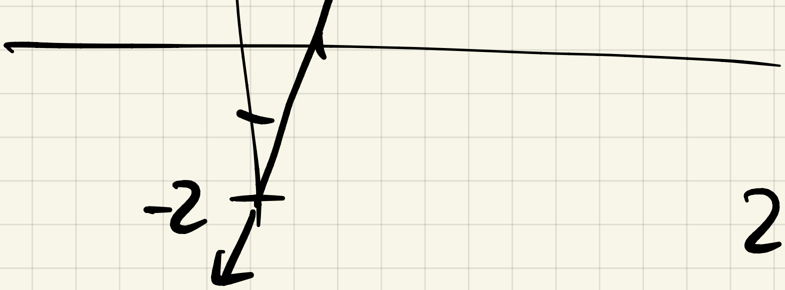
f is **not** 1-1, b/c

$1 \neq -1$

but

$f(1) = f(-1)$

(b) $f(x) = 2x-2$



f is 1-1:

$x_1 \neq x_2$

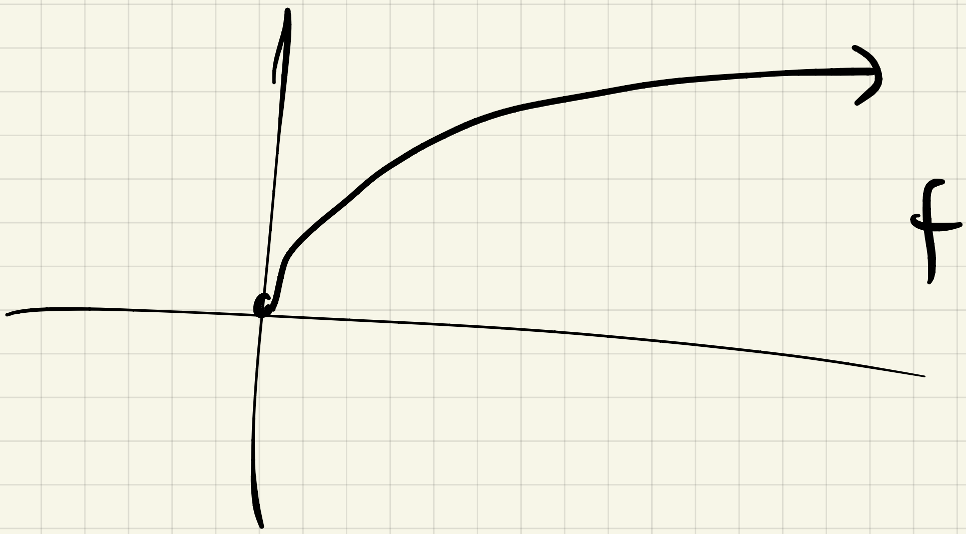
$\Downarrow$

$2x_1 \neq 2x_2$

$\Downarrow$

$2x_1-2 \neq 2x_2-2$

(c) $f(x) = \sqrt{x}$ $D = [0, \infty)$



f is 1-1

Notice: f is 1-1 $\Longleftrightarrow$

Every horizontal line meets graph of f in at most one point

(Horizontal line test)

Defn: If $f : D \to Y$ is 1-1,

and $R = \{f(d) \mid d \text{ in } D\}$ is

range f, ~~then~~

Inverse $f^{-1} : R \to D$

is defined

$$f^{-1}(b)=a \iff f(a)=b$$

Ex2 (Ex1 revisited):

(a) $f(x)=x^2$ has no inverse b/c f not 1-1.

(b) $f(x)=2x-2$

Range is $\mathbb{R}$.

f^{-1} exists:

How to find it?

f: $y=2x-2$

$\therefore f^{-1}$: $x=2y-2$

$$\Downarrow$$

$$x+2=2y \Rightarrow y=\frac{x+2}{2}$$

$$=\frac{x}{2}+1$$

(c) $f(x)=\sqrt{x}$

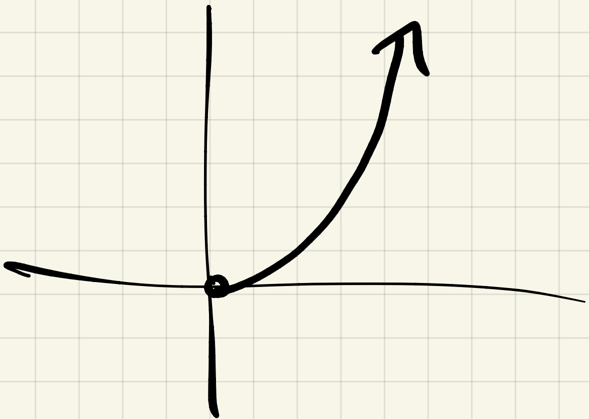
$D=[0,\infty)$

$R = [0, \infty)$

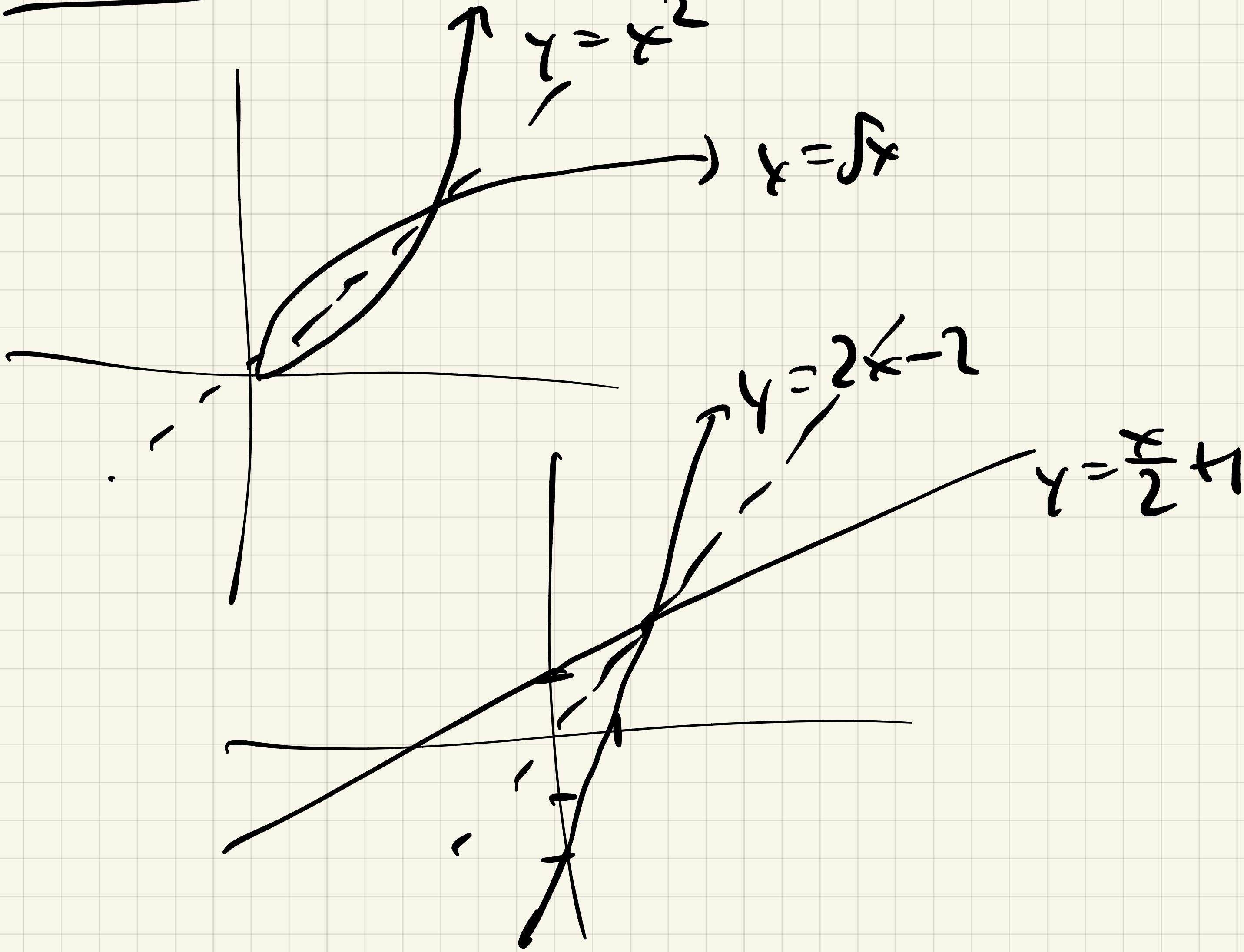
f[illegible](: $y = \sqrt{x}$

$x = \sqrt{y} \Rightarrow y = x^2$

$x \geq 0$



Sketch graph of last 2 examples:



In general: graph of f^{-1} = graph of f reflected through line $y=x$

Dom f = Range f^{-1}

Dom f^{-1} = Range f

b/c (x,y) on graph of f

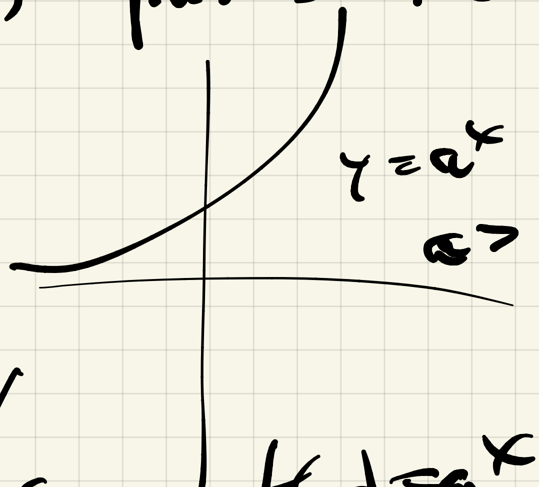
$$\Updownarrow$$

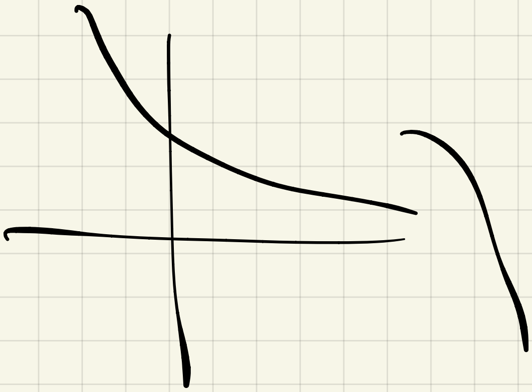
(y,x) on graph of f^{-1}

Ex2 (Very important)

If $f(x)=a^x$ is exponential function here $a \neq 1$,

f passes horizontal line test





so $f(x)=a^x$ has an inverse

function called logarithm function base a

$y = f^{-1}(x) = \log_a x$

$y = \log_a x$

Dom = $(0, \infty)$

Range = $\mathbb{R}$

Special ~~if~~ case $a = e \approx 2.71828\ldots$

$\log_e x = \ln x$ = natural log function

Inverse properties:

$\left\{ a^{\log_a x} = x \qquad x > 0 \right.$

$\log_a a^x = x$ all x

$e^{\ln x} = x$ $x > 0$

$\ln e^x = x$ all x

- **Algebraic properties** $x, y > 0$

① $\log xy = \log x + \log y$

② $\boxed{\log \frac{x}{y}} = \boxed{\log x - \log y}$

③ $\log \frac{1}{x} = -\log x$

④ $\log x^p = p \log x$

Calculator: $\log_a x$ is not on calculator, but ok, because

$$\log_a x = \frac{\ln x}{\ln a}$$

Why $y = \log_a x \iff \underset{\Downarrow}{a^y} = x$

take ln x

$\ln a^y = \ln x$

$\parallel$

$y \ln a$

$y = \frac{\ln x}{\ln a}$ ✓

<u>**Usage**</u>: Use inverse property to solve equations:

(a) $2^x = 8 \Rightarrow x = 3$

(b) $2^x = 7 \Rightarrow$

$\log_2 2^x = \log_2 7$

$\parallel$

$x \log_2 2$ //

$x \cdot 1$

$\boxed{x = \log_2 7}$

(c) $e^{x^2+2x} = 20$

take ln of both sides

$x^2 + 2x = \ln 20$

$$x^2+2x-\ln 20 = 0$$

$$x = \frac{-2 \pm \sqrt{4-4(-\ln 20)}}{2\cdot 1}$$

$$= \frac{-2 \pm \sqrt{4+4\ln 20}}{2}$$

$$= \frac{-1 \pm \sqrt{1+\ln 20}}{2}$$

d) $$\underbrace{\ln(x-1) - \ln(x+1)} = 3$$

$$\ln\left(\frac{x-1}{x+1}\right) = 3$$

exponentiale

$$\frac{x-1}{x+1} = e^3$$

$$x-1 = (x+1)e^3 = xe^3+e^3$$

$$\Downarrow$$

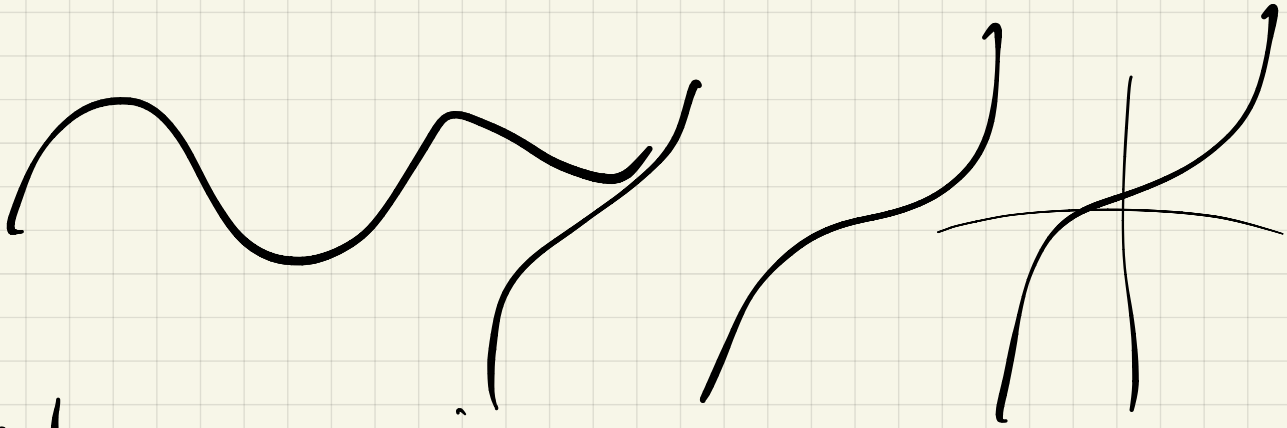
$$-e^3-1 = xe^3-x = x(e^3-1)$$

✓ $$\boxed{x = \frac{-e^3-1}{e^3-1}}$$

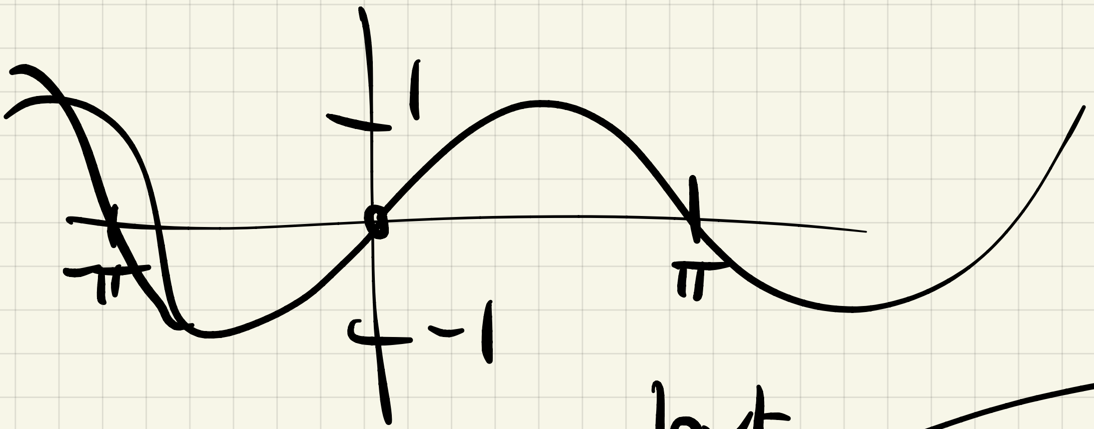
$\ln(\underset{\sim}{x-1})$ so $\underline{\text{undefined}}$

$x-1<0$ $\boxed{\text{no solution}}$

<u>Trig functions</u> are not 1-1



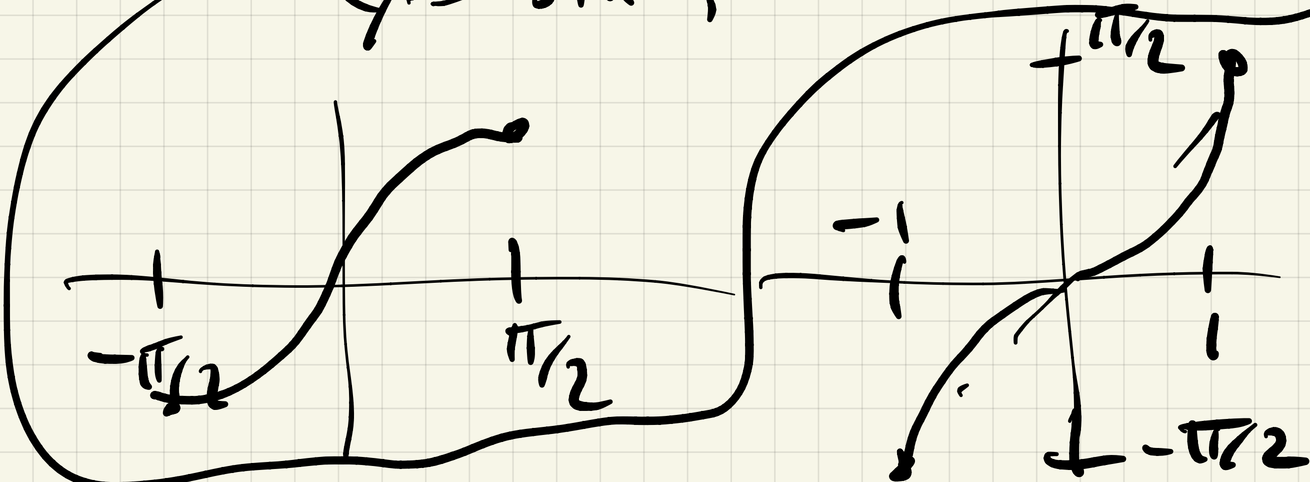
<u>Ex</u> $y=\sin x$



fails H.L.T

but

$y=\sin x, \quad -\pi/2 \le x \le \pi/2$



Inverse function

$$\frac{1}{\sin x} \neq \overset{\substack{\arcsin x \\ \|}}{\boxed{\sin^{-1} x}}$$