2/20/ Calc

Derivatives Rules

c constant

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \left(\text{if } x^n / x^{n-1} \text{ are defined}\right)$$

$$\frac{d}{dx}(c\,u) = c\frac{d}{dx}(u)$$

u function

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

product

$$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\left(\text{note: } \frac{d}{dx}(uv) \neq \underline{\underline{\frac{du}{dx} \cdot \frac{dv}{dx}}}\right)$$

quotient $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

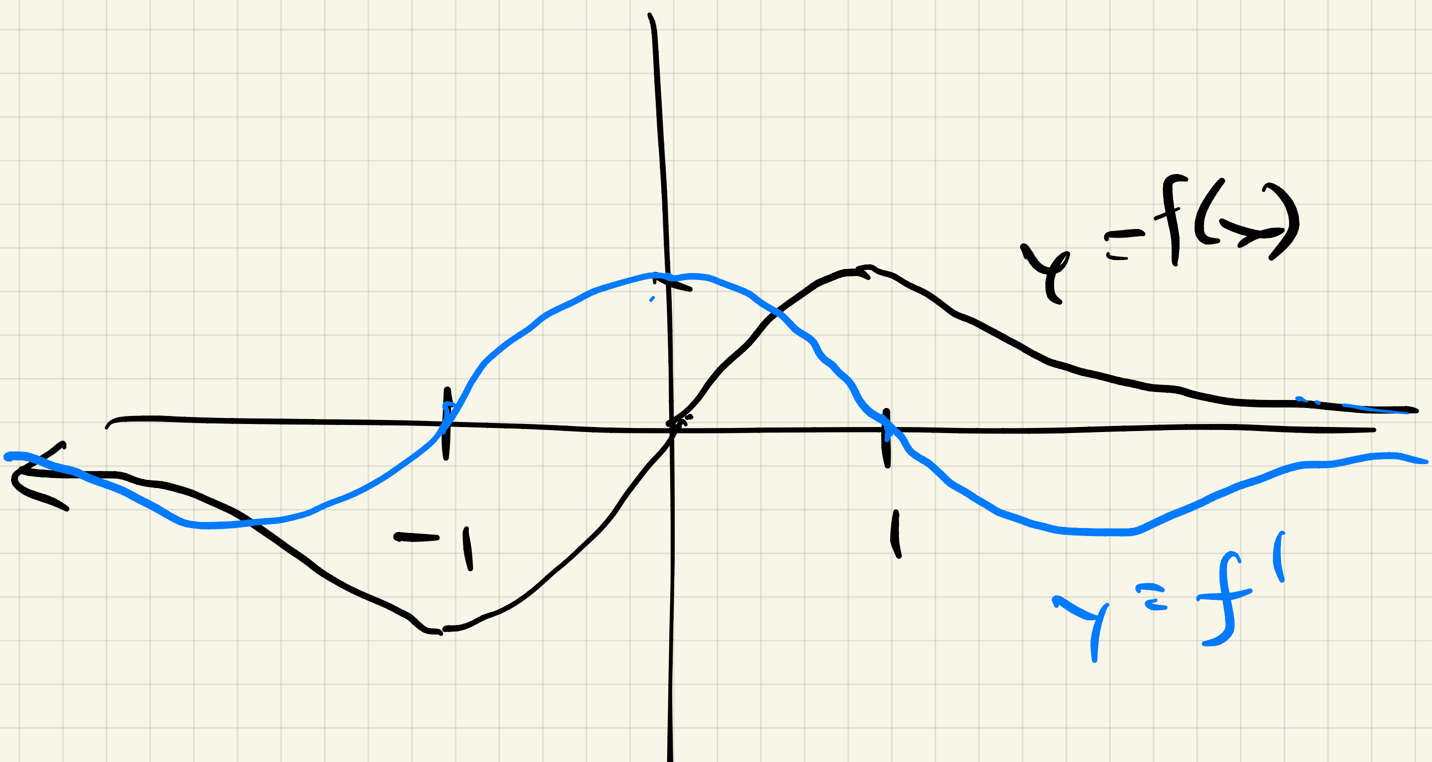
" $\dfrac{lo\ d\,hi - hi\ d\,lo}{lo\,lo}$ "

Ex) $\dfrac{d}{dx}\left(\dfrac{x}{x^2+1}\right) = \dfrac{(x^2+1)(1) - (2x)(x)}{(x^2+1)^2}$

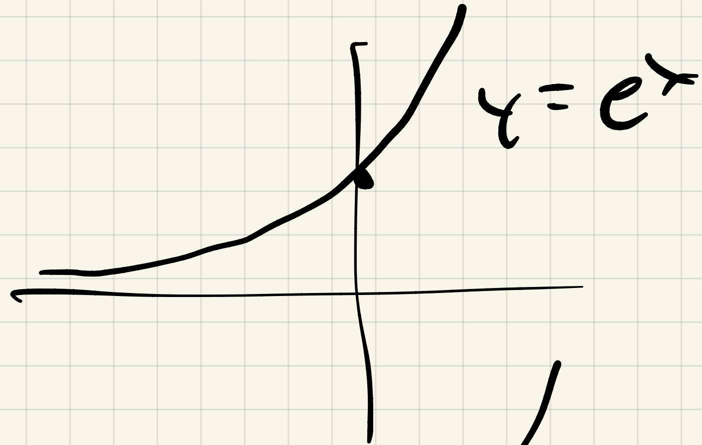
"f"

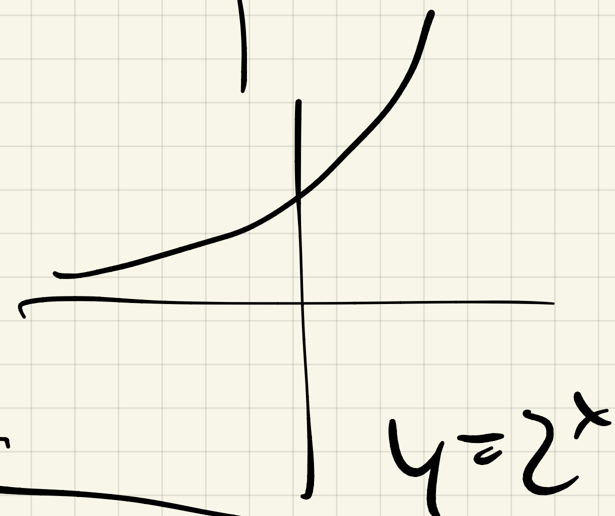
x^2+1-2x^2

$= \dfrac{1-x^2}{(x^2+1)^2}$



$$\boxed{\frac{d}{dx}(e^x) = e^x}$$





Why??

Recall $e =$ missing y int of $y = (1+x)^{1/x}$

$$= \lim_{x \to 0} \boxed{(1+x)^{1/x}}$$

Now $\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{\boxed{e^{x+h}} - e^x}{h} =$

$$\lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$

$$e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

For h small, $e \approx (1+h)^{1/h}$

$$\frac{e^h - 1}{h} \approx \frac{\left((1+h)^{1/h}\right)^h - 1}{h} = 1$$

$\therefore$

$$e^x \cdot 1 = e^x.$$

Ex 1

(a) $$\frac{d}{dx}(7e^x) = 7\frac{d}{dx}(e^x)$$
$$= 7e^x$$

(b) $$\frac{d}{dx}\left(3e^x + 7x^4 + \frac{8}{x^2}\right)$$
$$\frac{d}{dx}\left(3e^x + \underline{7x^4} + \underline{8x^{-2}}\right)$$
$$= 3e^x + 28x^3 - 16x^{-3}$$

(c) $$\frac{d}{dx}\left(\underbrace{(x^3+x+6)}_{\text{LHS}}\underline{e^x}\right)$$

$$(3x^2+1)(e^x) + (x^3+x+1)(e^x)$$

$$(x^3+3x^2+x+2)e^x$$

(d) $\frac{d}{dx}\left(\frac{e^x}{x+1}\right) = \frac{(x+1)e^x - (1)(e^x)}{(x+1)^2} =$

$$= \frac{xe^x}{(x+1)^2}$$

(e) $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x) =$

e^{x+x}

$$e^x \cdot e^x + e^x e^x = 2e^x e^x$$

$$= \boxed{2e^{2x}}$$

(f) $\frac{d}{dx}(e^{3x}) = \frac{d}{dx}(e^{2x+x}) =$

$$\frac{d}{dx}(\underbrace{e^{2x}} \cdot e^x)$$

$$2\underbrace{e^{2x}\cdot e^{x}} + \underbrace{e^{2x}\cdot e^{x}}_{e^{3x}}$$

$$= 2e^{3x} + e^{3x} = 3e^{3x}$$

Guess $\frac{d}{dx}(e^{kx}) = ke^{kx}$??

<u>Higher derivatives</u>:

can keep differentiating:

<u>Notation</u>:

$$\frac{d}{dx}\left(\frac{d}{dx}(f(x))\right) = f''(x) = y''(x)$$

$$\frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}$$

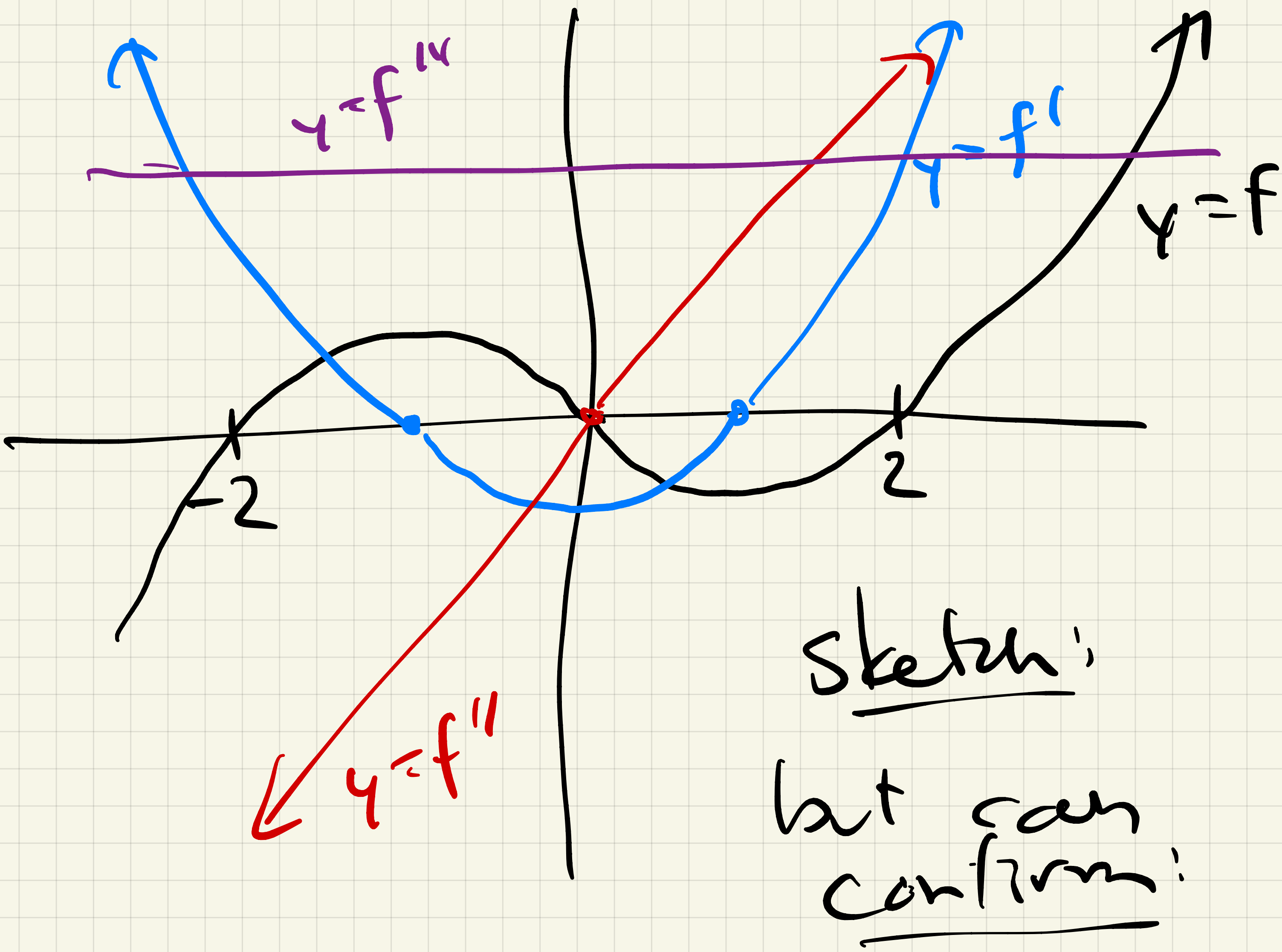
$$\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}f\right)\right) = f'''(x),\ y'''(x)$$

$$\frac{d^3y}{dx^3}$$

d d

$$\frac{d^4y}{dx^4} = f^{(4)}(x) = f^{(iv)}(x)$$

Ex 1 Find all higher derivatives of $y = x^3 - 4x$



sketch:

but can confirm:

$$y = x^3 - 4x$$

$$y' = 3x^2 - 4$$

$$y'' = 6x$$

$$y''' = 6$$

$$y^{(4)} = 0 \qquad y^{(5)} = 0 \ldots$$

<u>Ex2</u>

$$y = e^{2x}$$

$$y' = 2e^{2x}$$

$$y'' = 2 \cdot 2e^{2x}$$

$$y''' = 2 \cdot 2 \cdot 2e^{2x} = 2^3 e^{2x}$$

$$y^{(4)} = 2^4 e^{2x}$$

$$y^{(n)} = 2^n \cdot e^{2x}$$

<u>Ex3</u>:

$$y = \frac{1}{x} = x^{-1}$$

$$y' = -1x^{-2}$$

$$y'' \quad 2x^{-3}$$

$$y''' = -6x^{-4}$$

$$y^{(4)} = +24x^{-5}$$

$$\vdots$$

Pattern:

$$y^{(n)} = (-1)^n n!\, x^{-(n+1)}$$

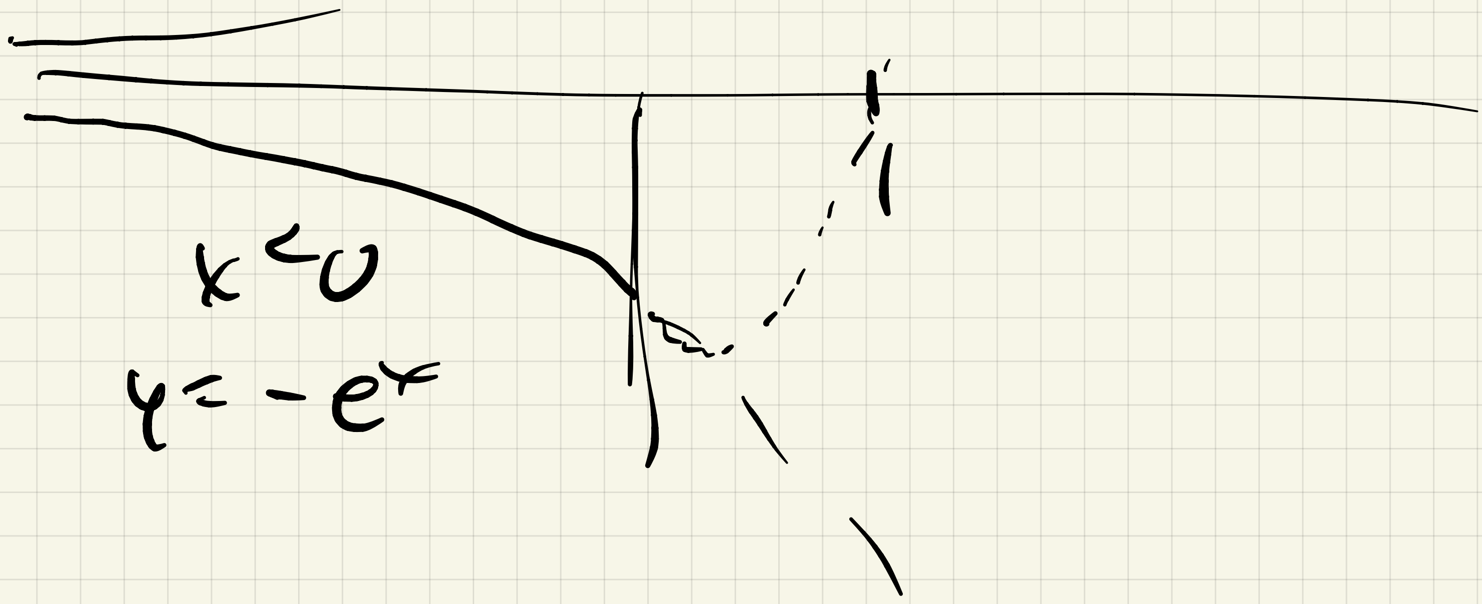
$$n! = n(n-1)(n-2)\cdots(1)$$

Ex 4 For what a, b, c is the function

$$f(x) = \begin{cases} -e^x & x < 0 \\ ax^2 + bx + c & x \geq 0 \end{cases}$$

$\boxed{\text{differentiable}}$ for all x ??

and x-intercept is 1



<u>Need 3 things</u>

① x-int = 0 ∴ $\boxed{a+b+c=0}$

② Continuity at $x=0$

$$\lim_{x\to 0^-} f = \lim_{x\to 0^-} -e^x = -1$$

$$\lim_{x\to 0^+} f = \lim_{x\to 0^+} ax^2+bx+c = c$$

$$-1 = c$$

③ $\boxed{\text{derivative from right}} = \boxed{\text{derivative from left}}$

$$\frac{d}{dx}(-e^{x}) = -e^{x}\Big|_{x=0} = -1$$

$$\frac{d}{dx}(ax^2+bx+c) = 2ax+b\Big|_{x=0} = b$$

so $\boxed{b=-1}$

so $b=-1, \; c=-1$

$a+b+c=0 \Rightarrow a=2$

so the parabola is

$y = 2x^2 - x - 1$

<u>Skip</u> §3.4

<u>§3.5</u> <u>Trig functions</u>

<u>Recall</u>:

$$\begin{cases} \lim_{x \to 0} \dfrac{\sin x}{x} = 1 \\ \lim_{x \to 0} \dfrac{\cos x - 1}{x} = 0 \end{cases}$$

<u>Usage</u>:

<u>Ex)</u> If $f(x) = \sin x$, what is $f'(x)$?

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$\lim_{h \to 0} \frac{\boxed{\sin x \cosh + \cos x \sin h} - \sin x}{h}$$

$$\lim_{h \to 0} \sin x \left(\frac{\cosh - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right)$$

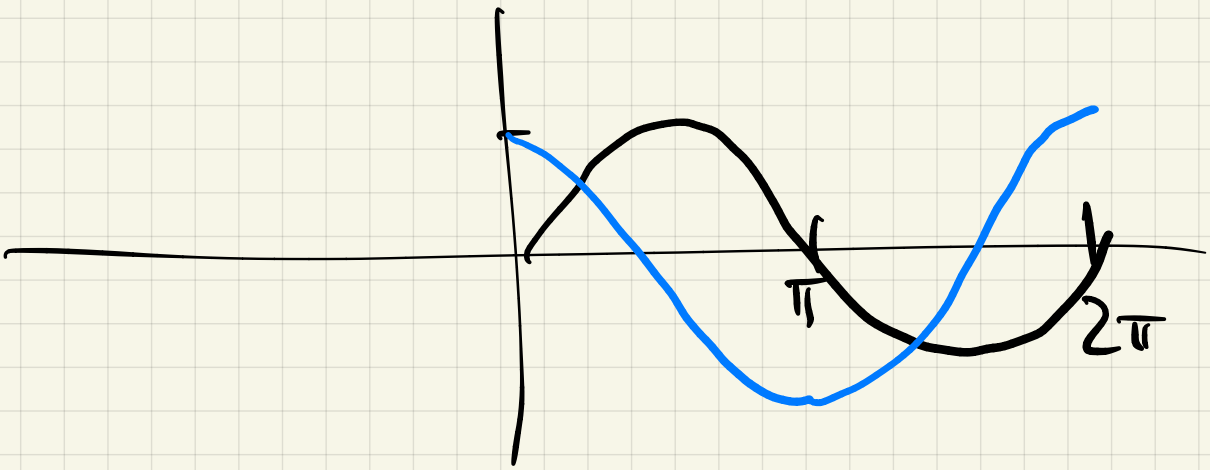
$$\frac{\cosh - 1}{h} \downarrow 0 \qquad \frac{\sin h}{h} \downarrow 1$$

So

$$\sin x \boxed{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{=0} + \cos x \boxed{\lim_{h\to 0} \frac{\sin h}{h}}_{=1}$$

$$= 0 + \cos x.$$

So $\frac{d}{dx}(\sin x) = \cos x$



Similarly $\frac{d}{dx}(\cos x) = -\sin x$

Ex2 Differentiate:

(a) $y = x^8 + 3\cos x + \frac{1}{x}$

$$y' = 8x^7 - 3\sin x + \frac{-1}{x^2}$$

(b) $$y = x^5 + \underbrace{\overbrace{x^2}^{u}\,\overbrace{\sin x}^{v}}$$

$$y' = 5x^4 + 2x\sin x + x^2\cos x$$

(c) $$y = \frac{\sin x}{x}$$

$$y' = \frac{x(\cos x) - 1(\sin x)}{x^2}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

(d) $$y = \underbrace{x^2}_{u}\,\underbrace{e^x\cos x}_{v}$$

dv/dx

$$2x e^x\cos x + x^2\left(e^x\cos x + e^x(-\sin x)\right)$$

$$\shortparallel$$

$$2xe^x\cos x + x^2e^x\cos x - x^2e^x\sin x$$

1 $(x^2 e^x)\cos x$

$y' = (2xe^x + x^2 e^x)\cos x$

$+ \; x^2 e^x(-\sin x)$

$\shortparallel$

$2xe^x \cos x + x^2 e^x \cos x - x^2 e^x \sin x$

ex Find all derivative of

$y = \sin x$

$y' = \cos x$

$y'' = -\sin x$

$y''' = -\cos x$

$y'''' = \sin x$