4/9 Calc I: Quiz 18

$$1. \quad \lim_{x \to 3} \frac{x^2 - 9}{x^3 - 3x^2} \overset{0/0}{=} \lim_{x \to 3} \frac{2x}{3x^2 - 6x} = \frac{6}{9} = 2/3$$

$$2. \quad \lim_{x \to 0} \frac{\cos(\boxed{3x}) - 1}{x^2} \overset{0/0 \ LH}{=}$$

$$\lim_{x \to 0} \frac{-\sin(3x) \cdot 3}{2x} \overset{0/0}{=}$$

$$\lim_{x \to 0} \frac{-\cos(3x) \cdot 3 \cdot 3}{2}$$

$$= \lim_{x \to 0} \frac{-9\cos(3x)}{2} = -9/2$$

$$3. \quad L = \lim_{x \to \infty} \left(1 - \frac{2}{x}\right)^x$$

$$\ln L = \lim_{x \to \infty} \ln\left(1 - \frac{2}{x}\right)^x$$

$$= \lim_{x \to \infty} x \ln\left(1 - \frac{2}{x}\right)$$

$$= \lim_{x \to \infty} \frac{\ln\left(\left(1 - \frac{2}{x}\right)\right)}{\frac{1}{x}} \to 0$$

$$\to 0$$

$$= \lim_{x \to \infty} \frac{\frac{\frac{2}{\cancel{x^2}}}{\left(1 - 2/x\right)}}{-\frac{1}{\cancel{x^2}}}$$

$$= \lim_{x \to \infty} \frac{-2}{\left(1 - \frac{2}{x}\right)} = -2$$

$\frac{2}{x} \to 0$

$$-2 = \ln L$$

$$L = e^{-2} = \frac{1}{e^2}$$

Last time: Antiderivatives

$F(x)$ antideriv. of $f(x) \Rightarrow F'(x) = f(x)$

Notation: $\int f(x)\,dx = F(x) + C$

Indefinite integral

Integration constant

Ex1: $\int x^9\,dx = \frac{1}{10}x^{10} + C$

Ex2: A few general examples

$f(x)$	$\int f(x)\,dx$
x^p	$\frac{1}{p+1}x^{p+1} + C$ $(p \neq -1)$

$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + C$
$\cancel{e^x}$ $\quad e^{kx}$	$\frac{1}{k}e^{kx} + C$
$\sin(kx)$	$\frac{-\cos(kx)}{k} + C$
$\cos(kx)$	$\frac{\sin(kx)}{k} + C$
$\sec^2(kx)$	$\frac{1}{k}\tan(kx) + C$
$\frac{1}{k^2+x^2}$	$\frac{1}{k}\arctan\frac{x}{k} + C$

<u>HW</u>: $\frac{d}{dx}\left(\frac{1}{7}\arctan\frac{x}{7} + C\right) =$

$$\frac{1}{7}\,\frac{1}{\left(\frac{x}{7}\right)^2+1}\cdot\frac{1}{7} = \frac{1}{49\left(\frac{x^2}{49}+1\right)} =$$

$$\frac{1}{4x^2+49}$$

Ex 4

(a) $\int x^7 \, dx = \frac{1}{8} x^8 + C$

$p = \frac{1}{2}$

(b) $\int \sqrt{x} \, dx = \int x^{1/2} \, dx =$

$$\frac{1}{1+\frac{1}{2}} x^{1+\frac{1}{2}} + C =$$

$$\frac{1}{(3/2)} x^{3/2} + C = \frac{2}{3} x^{3/2} + C$$

(c) $\int \frac{1}{x^7} \, dx = \frac{1}{-7+1} x^{-7+1} + C$

$$= \frac{1}{-6} x^{-6} + C =$$

$$= \frac{-1}{6x^6} + C$$

(d) $\int x^{50} + 7e^{5x} + \sec^2(8x) + \frac{9}{x}\,dx$

$$\frac{1}{51}x^{51} + \frac{7}{5}e^{5x} + \frac{1}{8}\tan(8x) +$$

$$9\ln|x| + C$$

(e) $\int x^{-2/3} + \frac{3}{\sqrt{1-x^2}} + \sec x \tan x \, dx$

$$3x^{1/3} + 3\arcsin x + \sec x + C$$

(f) $\int (1+3t^2)\,t^2\,dt$

$$= \int t^2 + 3t^4\,dt$$

$$= \frac{1}{3}t^3 + \frac{3}{5}t^5 + C$$

Note:

$$\int (1+3t^2) \neq \int t^2$$

$$(t+t^3) \cdot \tfrac{1}{3}t^3$$

Ex 5

Is $\int \arcsin x = x\arcsin x + \sqrt{1-x^2} + C$??

$(1-x^2)^{\frac{1}{2}}$

check:

$$\frac{d}{dx}\left(\underline{x\arcsin x} + \sqrt{1-x^2}\right)$$

$$\arcsin x + \cancel{x\frac{1}{\sqrt{1-x^2}}} + \underbrace{\tfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)}_{\cancel{\frac{-x}{\sqrt{1-x^2}}}}$$

YES!

Initial value problems

Ex 7

Find a function $F(x)$ such that

(a) $F'(x) = 2$, $\quad F(0) = 3$

$$F(x) = \int 2\,dx$$

$$F(x) = 2x + C$$

$$3 = F(0) = C \qquad C = 3$$

$$\therefore F(x) = 2x + 3$$

(b) $F'(x) = 2\sin 3x \qquad F(0) = 5$

$$F(x) = \int 2\sin 3x\,dx$$

$$F(x) = -\frac{2}{3}\cos(3x) + C$$

$$5 = F(0) = -\frac{2}{3} + C$$

$$C = 5 + 2/3 = 17/3$$

so $$F(x) = -\frac{2}{3}\cos(3x) + \frac{17}{3}$$

(c) $$F'(x) = \frac{5}{x^2+49} \qquad F(7) = 2$$

$$F = \int \frac{5}{x^2+49}\,dx =$$

$$F = \underbrace{\frac{5}{7}\arctan\frac{x}{7} + C}$$

$$2 = F(7) = \frac{5}{7}\underbrace{\arctan\frac{7}{7}} + C$$

$$2 = \frac{5}{7}\cdot\frac{\pi}{4} + C$$

$$C = 2 - \frac{5\pi}{28}$$

(d) ~~F~~ $F''(x) = 3x,\quad F'(0) = -10$
$F(0) = 17$

Step 1

$$F''(x) = 3x \Rightarrow F'(x) = \frac{3}{2}x^2 + C$$

$$F'(0) = -10 \Rightarrow C = -10$$

so $F' = {}^{3}/_{2}\, x^2 - 10$

$$\Downarrow$$

Step 2

$$F = \frac{1}{2}x^3 - 10x + D$$

$$F(0) = 17 \Rightarrow D = 17$$

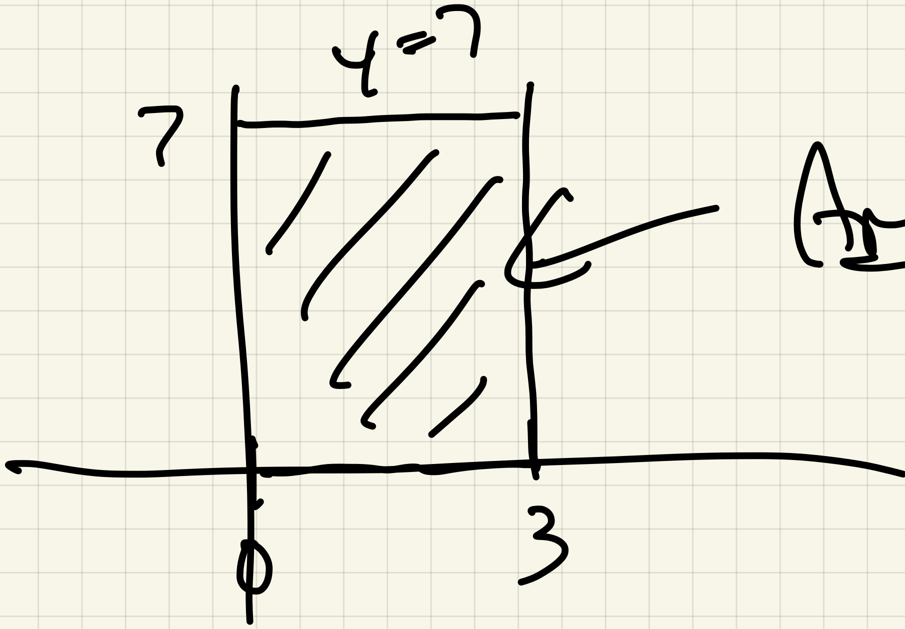
so $F = \frac{1}{2}x^3 - 10x + 17$

Chapter 5

§5.1 & 5.2 $\boxed{\text{Area}}$

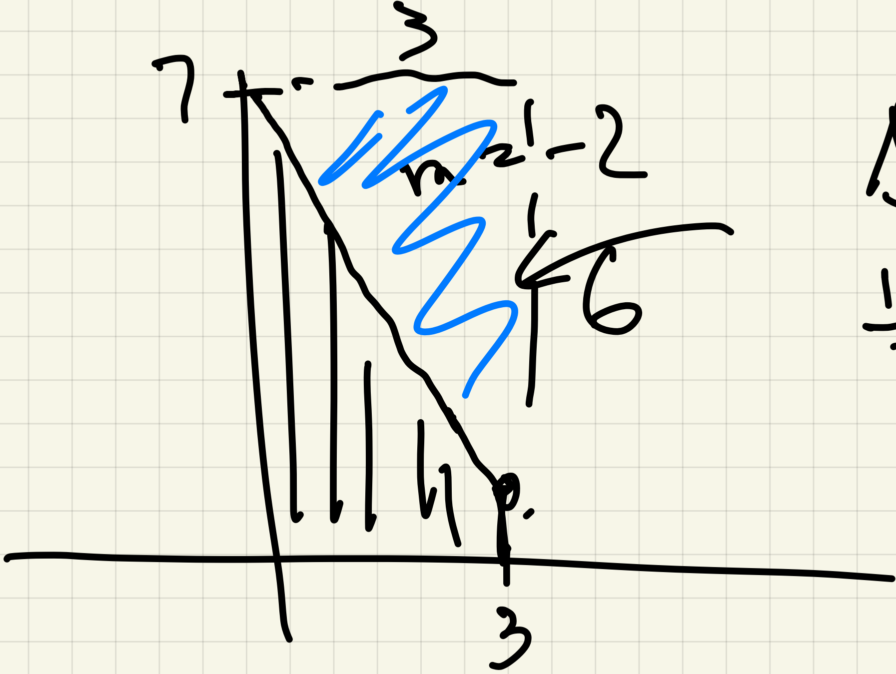
Ex1 Find the area bounded by the function $y = f(x)$ and $x = 0$ and $x = 3$ and x-axis

(a) $y = f(x) = 7$:



Area: $bh = 3 \cdot 7 = 21$

(b) $y = 7 - 2x$



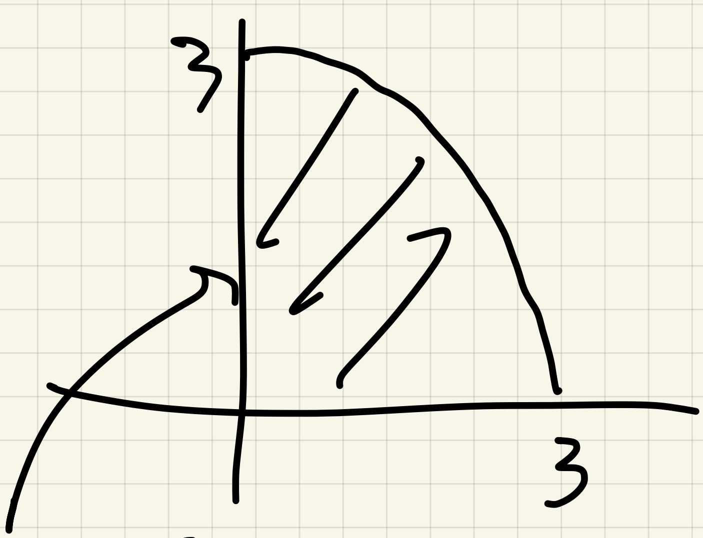
$\triangle$ has area

$\frac{1}{2} b h =$

$\frac{1}{2} \cdot 3 \cdot 6 = \frac{18}{2} = 9$

So Area $= 21 - 9 = 12$

(c) $y = f(x) = \sqrt{9 - x^2}$



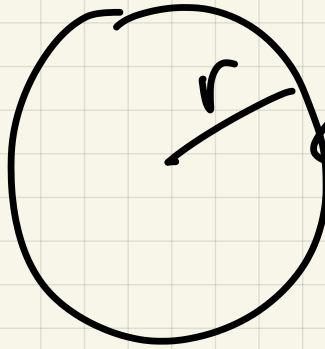
$y = \sqrt{9 - x^2}$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$

$\frac{\pi \cdot 9}{4} = \frac{9\pi}{4}.$

Uses : fact

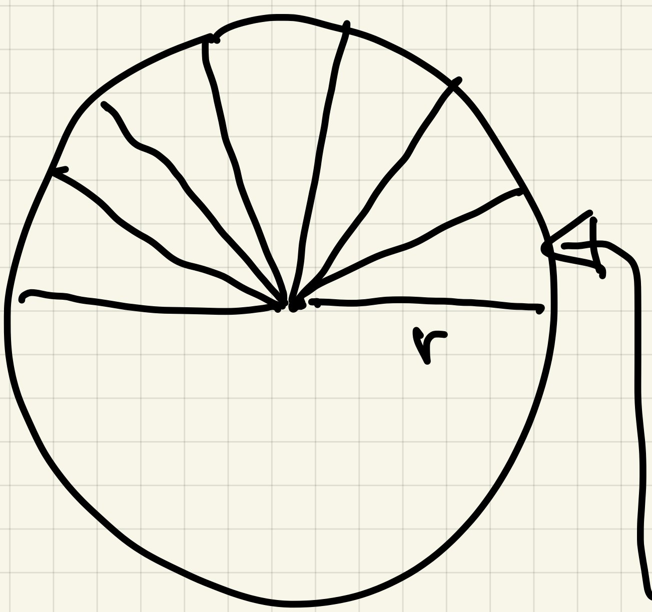


← Area = πr^2

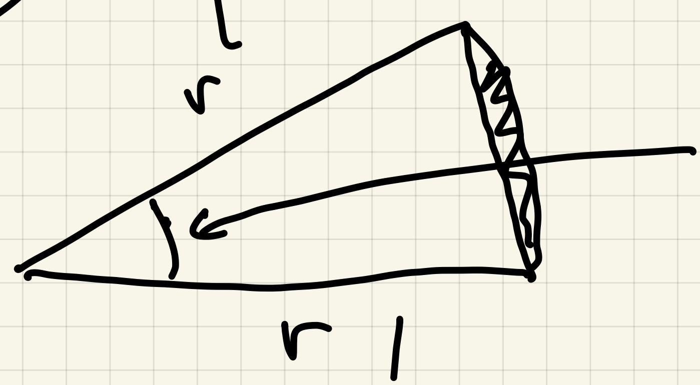
Archimedes, 250 BC

$A = \pi r^2$?? How ?

Idea: Estimate area and use limits to find exact answer

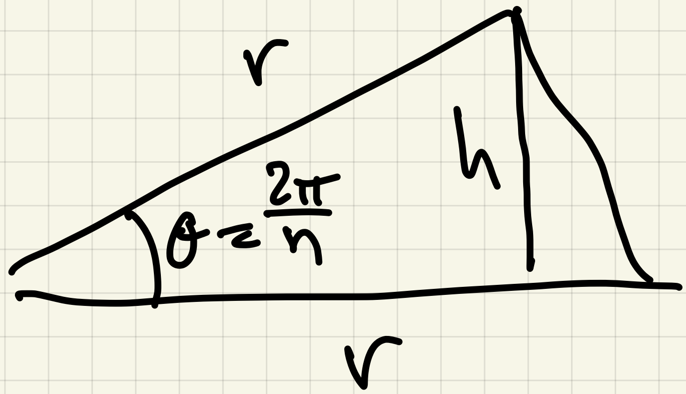


break circle into n equal wedges



$\theta = \frac{2\pi}{n}$

Area wedge $\approx$ Area of triangle



$$\frac{h}{r} = \sin\frac{2\pi}{n} \Rightarrow$$

$$h = r\sin\frac{2\pi}{n}$$

$$\therefore \text{Area of } \Delta = \frac{1}{2}\cdot r \cdot \underbrace{r\sin\left(\frac{2\pi}{n}\right)}_{h}$$

Putting n wedges together,

$$A \approx n \cdot \frac{1}{2} r^2 \sin\left(\frac{2\pi}{n}\right)$$

For exact area

$$A = \lim_{n\to\infty} \underbrace{n}_{\infty} \cdot \underbrace{\tfrac{1}{2} r^2 \sin\left(\tfrac{2\pi}{n}\right)}_{0}$$

$$= \lim_{n\to\infty} \frac{\tfrac{1}{2} r^2 \sin\left(\tfrac{2\pi}{n}\right)}{\tfrac{1}{n}}$$

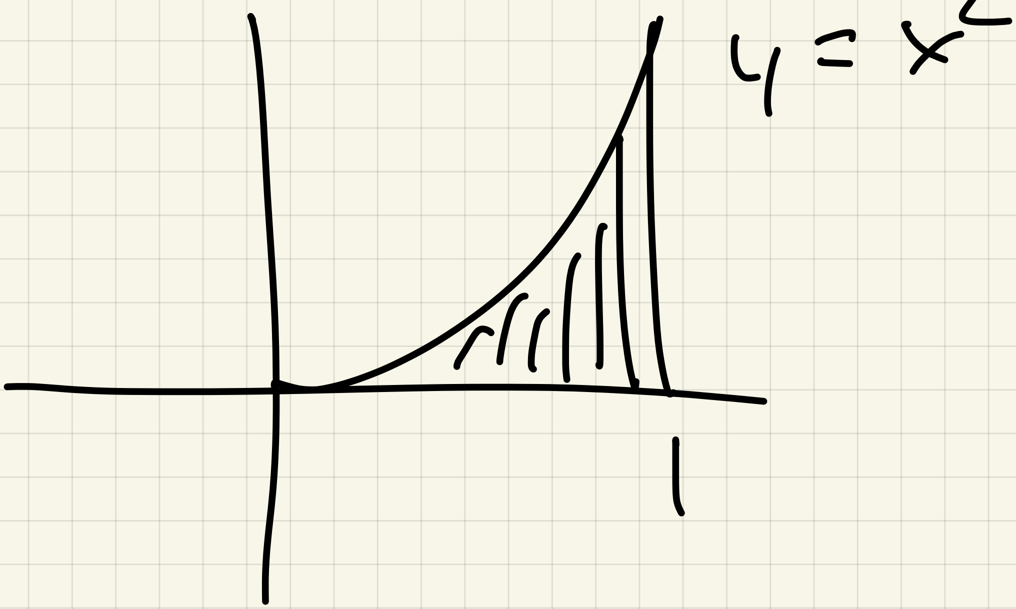
LH

$$\lim_{n\to\infty} \frac{\tfrac{1}{2} r^2 \cos\left(\tfrac{2\pi}{n}\right) \cdot \cancel{-}\tfrac{2\pi}{\cancel{n^2}}}{\cancel{-}\tfrac{1}{\cancel{n^2}}}$$

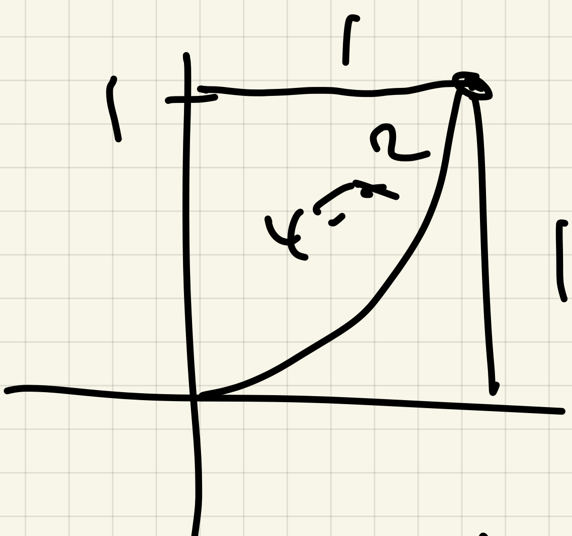
$$= \lim_{n\to\infty} \tfrac{1}{2} r^2 \underbrace{\cos\left(\tfrac{2\pi}{n}\right)}_{\to 0} \cdot 2\pi$$

$$\tfrac{1}{2} r^2 \cdot 2\pi = \pi r^2$$

Ex 2: Use this idea to estimate area under curve using rectangles



① Use $n=1$ rectangle:

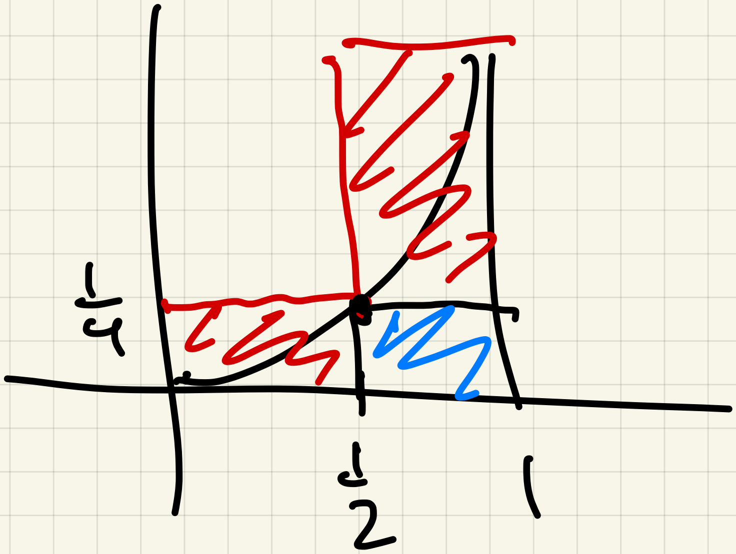


"big square"

$$0 < A = \text{Area} < 1$$

(2) Can do better:

$n = 2$ rectangles



Area of small box $<$ A $<$ area of 2 big rectangles

$$\frac{1}{2} \cdot \frac{1}{4} < A < \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 1$$

$$\frac{1}{8} < A < \frac{5}{8}$$