4/11/ Calc I

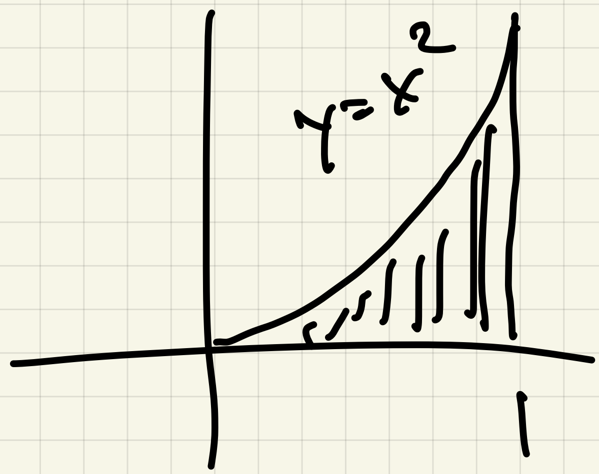
$\boxed{5.1 + 5.2}$

Last time: Area

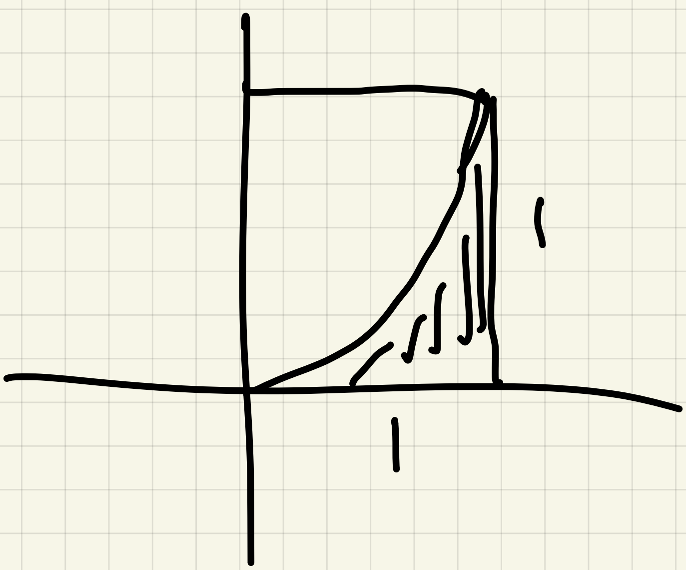
Idea: can compute exact area by

1. making estimate
2. take limit

Ex)

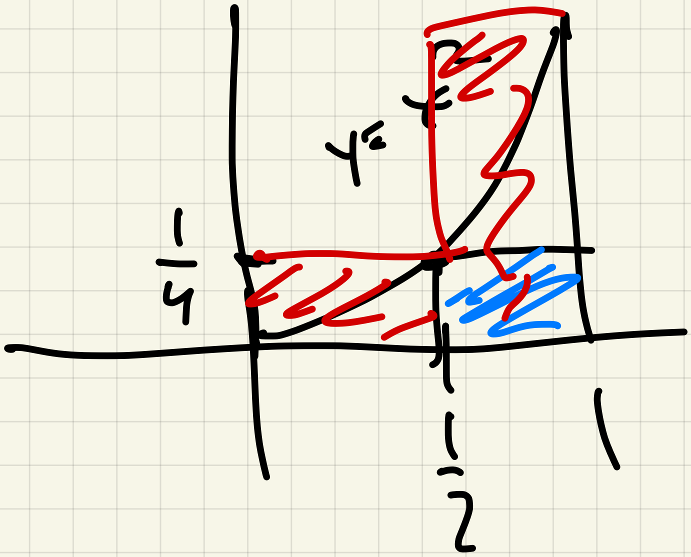


1.



Area
"
$0 < A \leq 1$

②

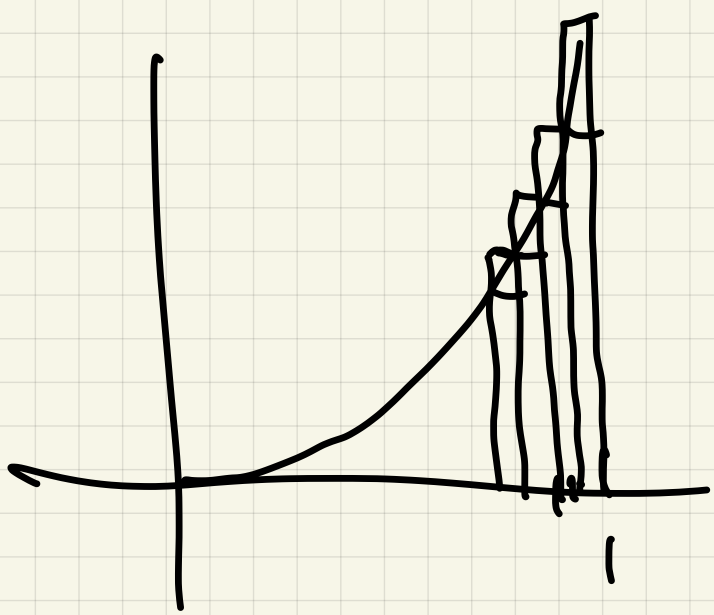


Area of blue box

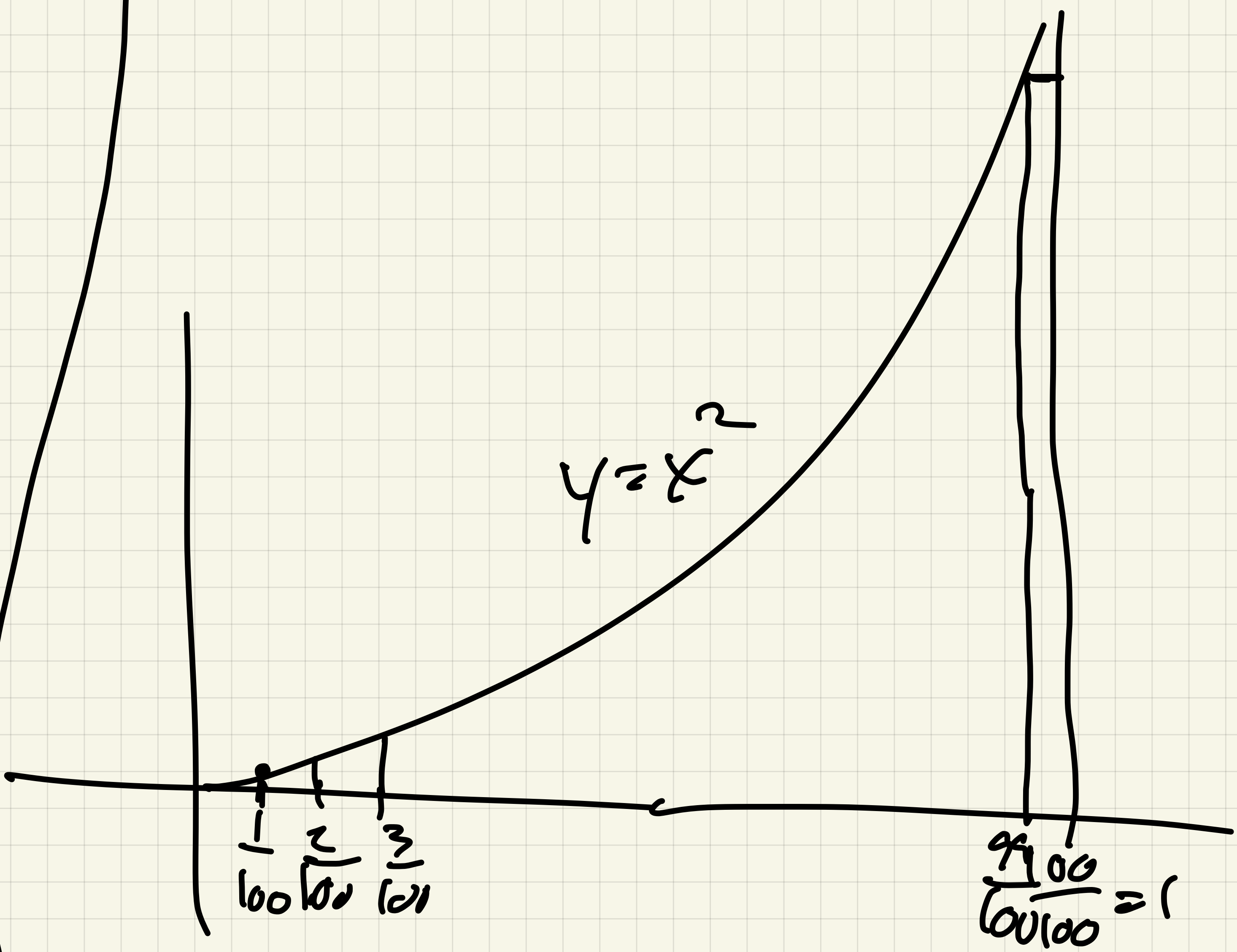
$\frac{1}{8} < A < \frac{5}{8}$

③ Try $n = 100$ rectangles

Break $[0,1]$ into 100 equal pieces



$\boxed{\text{sum area of smaller}} < A <$ sum area of bigger boxes



$$\frac{1}{100}(0) + \frac{1}{100}\left(\frac{1}{100}\right)^2 + \frac{1}{100}\left(\frac{2}{100}\right)^2 + \cdots + \frac{1}{100}\left(\frac{99}{100}\right)^2$$

smaller □s

$$\wedge$$

$$A$$

$$\wedge$$

$$\frac{1}{100}\left(\frac{1}{100}\right)^2 + \frac{1}{100}\left(\frac{2}{100}\right)^2 + \dots + \frac{1}{100}\left(\frac{100}{100}\right)^2$$

bigger $\square$s

$$.328350 < A < .338350$$

Calculator

Much better

error $< .01$

(4) Use n equal intervals, do same thing; looks like

small Δs

$$\frac{1}{n}(0)+\frac{1}{n}\left(\frac{1}{n}\right)^2+\frac{1}{n}\left(\frac{2}{n}\right)^2+\dots+\frac{1}{n}\left(\frac{n-1}{n}\right)^2$$

$$\wedge$$

$$A$$

$$\wedge$$

big Δs

$$\frac{1}{n}\left(\frac{1}{n}\right)^2+\dots \quad \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$\frac{1}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\dots+\frac{(n-1)^2}{n^3}$$

$$\|$$

$$\boxed{\frac{1}{n^3}\left(1^2+2^2+3^2+\dots+(n-1)^2\right)}$$

$$\wedge$$

$$A$$

$$\wedge$$

$$\frac{1}{n^3}\left(1^2+2^2+3^2+\dots+n^2\right)$$

Formula $1^2+2^2+3^2+\dots+k^2 = \dfrac{k(k+1)(2k+1)}{6}$

$k=1$ $\quad 1 = 1$ ✓

$k=2$ $\quad 1^2+2^2 = 5 = \dfrac{2(3)(5)}{6} = 5$ ✓

$k=3$ $\quad 1^2+2^2+3^2 = 14$

$$\frac{3(4)(7)}{6} = 14 \checkmark$$

So

$$\frac{1}{n^3}\left(\frac{(n-1)(n)(2n-1)}{6}\right) < A < \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$k = n-1$ $\qquad\qquad k = n$

$$\frac{2n^3-3n^2+n}{6n^3} < A < \frac{2n^3+3n^2+n}{6n^3}$$

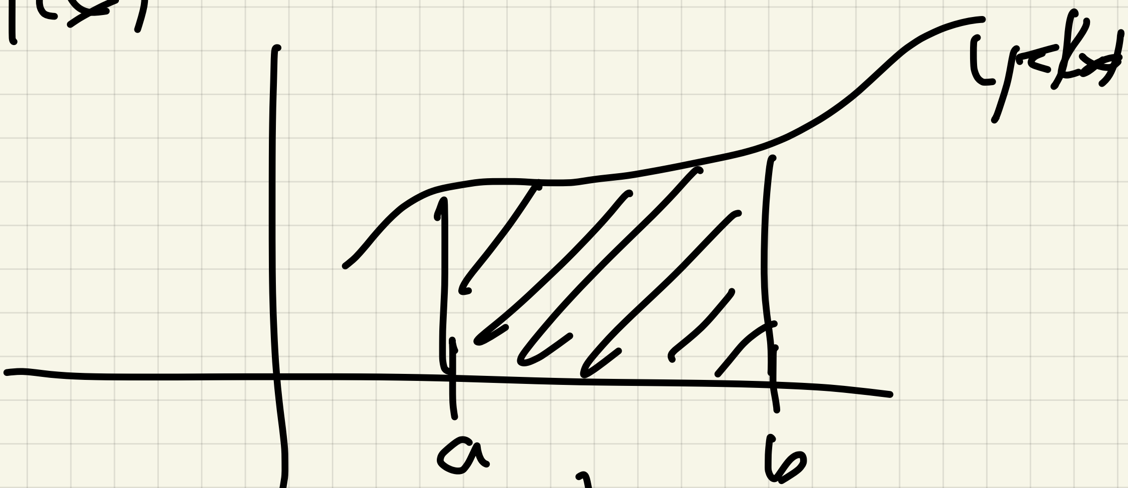
(Take limit as $n \to \infty$)

L

$$2/6 \le A \le 2/6$$

$$\therefore A = 2/6 = 1/3.$$

This gives a method to find exact area under graph of $y = f(x)$



① Estimate area with n ~~fee~~ boxes

② Take limit $n \to \infty$

Notation:

If $a_1, \ldots a_n$ are numbers,

then

$$a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

$n \leftarrow$ upper lim

a_i — terms

i = index

1 — lower

<u>Ex 1</u> If $a_i = i^2$

def

(a) $\sum_{i=1}^{4} a_i = \sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

(b) $\sum_{i=1}^{5} (i+2) = 3+4+5+6+7 = 25$

(c) $\sum_{i=-2}^{2} (i+1) = \underset{-2}{-1} + \underset{-1}{0} + \underset{0}{1} + \underset{1}{2} + \underset{2}{3} = 5$

(d) $\sum_{i=1}^{100} 4 = \underbrace{4+4+4+\ldots+4}_{100} = 400$

(e) $\sum_{i=1}^{100} i = 1+2+3+\cdots + 100$

$$1+2+3+\cdots+100 = S$$
$$100+99+98+\cdots+1 = S$$
$$\underbrace{101+101+101 \cdots 101}_{100} = 2S$$

$$10{,}100 = 2S$$
$$S = \frac{10{,}100}{2} = 5{,}050$$

Summation rules/formulas

c constant

$$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = 1+2+\dots+n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Ex: Cal $\sum_{i=1}^{20} i^2 - 3i$

$= -2 + -2 + 0 + 4 + \dots$

$$\sum_{i=1}^{20} i^2 + \sum_{i=1}^{20} -3i$$

$$\boxed{\sum_{i=1}^{20} i^2} - 3\boxed{\sum_{i=1}^{20} i}$$

$$\underline{\frac{20(21)(41)}{6}} - 3\,\frac{20(21)}{2}$$

$$2870 - 630 = 2240$$

(b) $\displaystyle\sum_{i=1}^{10} (2i-1)^2 = 1^2 + 3^2 + 5^2 + \cdots$

$$\parallel$$

$$\sum_{i=1}^{10} 4i^2 - 4i + 1$$

$$4$$

$$4\sum_{i=1}^{10} i^2 - 4\sum_{i=1}^{10} i + \underbrace{\sum_{i=1}^{10} 1}_{10}$$

$$1540 - 220$$

$$\parallel$$

$$1330$$

Write i ~~as~~ a using $\sum$ notation

Ex (a) $1+3+5+\dots+11 = \sum_{i=1}^{6}(2i-1)$

$2i$

$\parallel\ 2+4+6 \dots + 12$

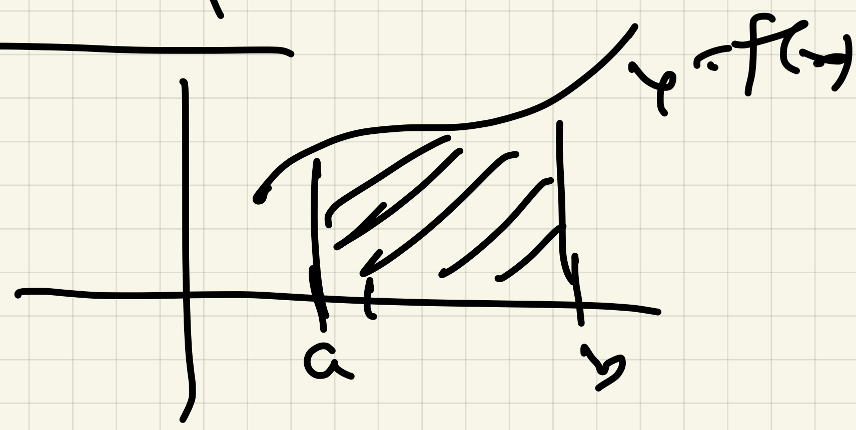
$$\sum_{i=0}^{5}(2i+1)$$

(b) $0+2+8+18+32+50+\dots+20000$

$$2(0+1+4+9+16+25+36+\dots 100^2)$$

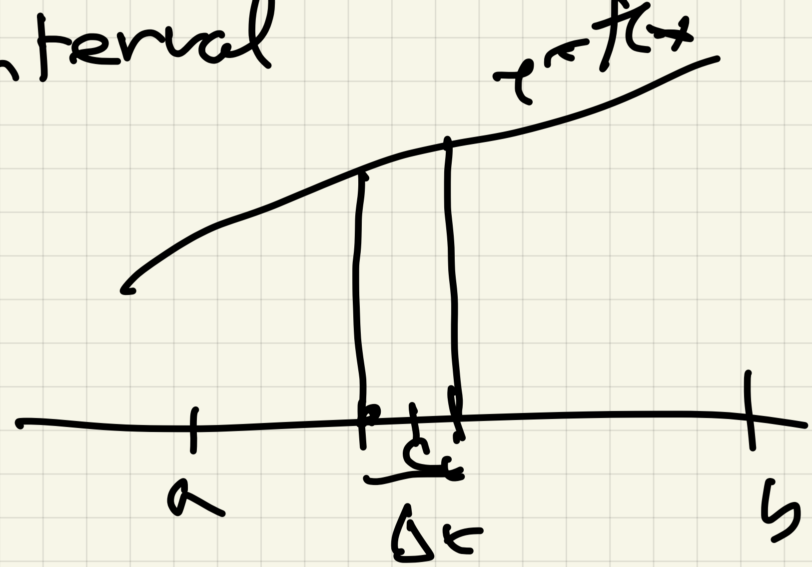
$$2\sum_{i=0}^{100} i^2 = 676700$$

Recipe for area



Step1 : break interval $[a,b]$ into n equal pieces of width $\Delta x = \frac{b-a}{n}$

Step2 : Choose a point c_i in i^{th} subinterval



Step3 Estimate for area

$$\sum_{i=1}^{n} \underbrace{f(c_i)}_{\text{height}} \underbrace{\Delta x}_{\text{width}}$$

Definition of area:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x$$