March 24 Algebraic Topology

I hope these notes find you in good health and safe from covid 19! It will be difficult to teach algebraic topology without a board, but I’ll do my best: when I need a picture I will describe things carefully and I can take a picture with my phone and find a way to get it to you. This is not ideal and if it really doesn’t work after a few tries we can try to meet via Zoom.

Scott

This will be the last week on fundamental groups and covers, with homology next week. Home work for this week from Munkres:

§80, #1, §81, #1 − 4, §82, #1.

HW Comments: (a) §81#1 is much like Example 3 in §81. Part (a) is OK, but I was less sure about part (b), hence it is for extra credit. You can use the internet, but then give a clear explanation as to why your answer works. (b) §81#3(b) you will need Theorem 22.2 in Munkres. There’s still something to check to apply it, but it’s much easier than proving the theorem.

Topics from our last meeting on March 5:

1. General lifting theorem for covers $p : (E, e_0) \to (B, b_0)$: If $Y$ is connected, then a continuous function $f : (Y, y_0) \to (B, b_0)$ lifts to a function $\tilde{f} : Y \to E$ with $\tilde{f}(y_0) = e_0$ if and only if $f_*\pi_1(Y, y_0) \subset p_*\pi_1(E, e_0)$.

2. Group of covering transformations (or deck transformations)

3. Equivalence of covers

4. There is an equivalence $h : E \to E'$ with $h(e_0) = e'_0$ if and only if $H = H'$, where $H = p_*\pi_1(E, e_0) \subset \pi_1(B, b_0)$ and analogously for $H'$.

5. There is an equivalence $h : E \to E'$ if and only if $H$ is conjugate to $H'$.

6. Previous follows from a fact we proved, that changing the base point corresponds to forming a conjugate subgroup: i.e. if $\gamma : I \to E$ is a path from $e_0$ to $e_1$ and $\alpha = p \circ \gamma$, then $\alpha p_*\pi_1(E, e_1)\alpha^{-1} = \pi_1(E, e_0)$.

7. We observed that if $E \to B$ is a simply connected cover, then there is a neighborhood $b_0 \in U \subset B$ such that $\pi_1(U, b_0) \to \pi_1(B, b_0)$ is trivial.
Example 1. Item 7 can be an issue: Let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ centered at $(1/n, 0)$ and $B = \bigcup_n C_n$ with base point $b_0 = (0, 0)$. This is the famous Hawaiian earring, with infinitely many loops of smaller radius passing through $b_0$. No matter how small a neighborhood $U$ you choose about $b_0$, one of those loops will be contained in $U$ and $\pi_1(U, b_0) \to \pi_1(B, b_0)$ will be nontrivial, so $B$ has no simply connected covering space (§80 Example 1).

Definition 1. A space $B$ is semi-locally simply connected if for each $b \in B$, there is a neighborhood $b \in U \subset B$ for which $\pi_1(U, b_0) \to \pi_1(B, b_0)$ is trivial.

That might be the longest definition in algebraic topology! With Definition 1, we can form a covering space corresponding to any subgroup:

Theorem 1. Assume $B$ is path connected, locally path connected and semi-locally simply connected. If $H \subset \pi_1(B, b_0)$ is any subgroup, then $B$ has a cover $p : E \to B$ with $p_* \pi_1(E, e_0) = H$.

Construction of $p : E \to B$ (see Munkres, proof of Theorem 82.1). Take set $\mathcal{P}$ of all paths in $B$ starting at $b_0$ and take the quotient by $\sim$, where $\alpha \sim \beta$ if $\alpha(1) = \beta(1)$ and $\alpha \ast \beta \in H$. This is an equivalence relation (easy) and if $\alpha^\#$ is the equivalence class of $\alpha$ it’s obvious that the map $p(\alpha^\#) = \alpha(1)$ is
well-defined and onto. But then one has to put a topology on the space \( E \) of all equivalence classes and show that \( p \) is a covering, this takes some effort.

Consequence: we know that all covers exist under the hypothesis on \( B \). When \( H \) is trivial (consisting only of the identity), the the corresponding cover \( p : E \to B \) is called the the universal cover. It is unique up to isomorphism by item 4 or 5 above. The next result (80.2) will be useful in the homework:

**Lemma 2.** Let \( q : X \to Y \) and \( r : Y \to Z \) be continuous maps with \( p = r \circ q \).

(a) If \( p \) and \( r \) are covering maps, then so is \( q \).

(b) If \( p \) and \( q \) are covering maps, then so is \( r \).

I won’t prove this here, but you should read the proof of part (a) in the text.

**Remark 3.** If \( r : Y \to B \) is any cover, then there is a lift \( \tilde{p} : E \to Y \) with \( r \circ \tilde{p} = p \) by the general lifting theorem (the trivial subgroup is contained in every subgroup) and by Lemma 2 the lifting \( \tilde{p} \) is a covering map! In other words, the universal cover of \( B \) covers every cover of \( B \)!

**Example 2.** We have already seen several universal covers:

(a) Take \( B = S^1 \) and \( E = \mathbb{R} \) with \( p(x) = e^{2\pi ix} \).

(b) Take \( B = S^1 \times S^1 \) and \( E = \mathbb{R} \times \mathbb{R} \).

(c) Take \( B \) to be the figure 8 space and \( E \) to be that infinite antenna space.
Finally we will study the group of deck transformations of a cover. Part of the next theorem will look familiar to you.

**Theorem 4.** Let \( p : (E, e_0) \to (B, b_0) \) be a cover with \( E \) path connected, \( B \) path connected and locally path connected and set \( H = p_* \pi_1(E, e_0) \). Then

(a) For each \( x_1, x_2 \in p^{-1}(b_0) \), there is a deck transformation \( h : E \to E \) with \( h(x_1) = x_2 \) if and only if \( H \) is a normal subgroup.

(b) The group of deck transformations \( G_{E/B} \) is isomorphic to \( N(H)/H \).

**Remark 5.** (a) should look familiar because it is what you proved in this week’s home work! When \( H \) is normal, one says that the covering \( p : E \to B \) is normal (unfortunate, as the word “normal” is greatly overused).

(b) The normalizer of \( H \subset G \) is \( N(H) = \{g \in G : g^{-1}Hg = H\} \). It is the largest subgroup of \( G \) containing \( H \) as a normal subgroup (so that \( N(H) = G \iff H \) is normal).

**Proof.** We prove Theorem 4. Part (a) was on this week’s home work. For part (b), we have two maps to the fiber (here \( H_0 = p_* \pi_1(E, e_0) \) as usual):

\[
\pi_1(B, b_0)/H_0 \xrightarrow{\Phi} p^{-1}(b_0) \xleftarrow{\Psi} G_{E/B}
\]

namely the lifting correspondence \( \Phi(\alpha) = \tilde{\alpha}(1) \) and \( \Psi(h) = h(e_0) \). Note that \( \Phi \) is a bijection by a theorem we proved a while ago (Munkres 54.6) and \( h \) is injective because of uniqueness in the general lifting theorem.

**Claim 1:** The image of \( \Psi \) is equal to the image of \( N(H_0)/H_0 \) under \( \Phi \).

Given a loop \( \alpha \) at \( b_0 \), \( \tilde{\alpha} \) the lift with \( \tilde{\alpha}(0) = e_0 \) and \( e_1 = \tilde{\alpha}(1) \), we need to see that there is an equivalence \( h : E \to E \) with \( h(e_0) = e_1 \iff [\alpha] \in N(H_0) \).

But this isn’t bad: There is such an equivalence \( h \iff H_0 = H_1 \) and \([\alpha]H_1[\alpha]^{-1} = H_0 \) by a formula from before the break (item 6), so \( h \) exists if and only if \( \alpha H_0 \alpha^{-1} = H_0 \iff \alpha \in N(H_0) \).

From this we see a bijection \( \Phi^{-1} \circ \Psi : G_{E/B} \to N(H_0)/H_0 \) and we claim this map is a group homomorphism. This is Theorem 81.2 in Munkres.

Let me highlight two special cases:

**Corollary 6.** If \( H \) is normal, then \( G_{E/B} \cong \pi_1(B, b_0)/H \).

**Corollary 7.** If \( E \) is the universal covering space, then \( G_{E/B} \cong \pi_1(B, b_0) \).
Example 3. Concrete examples of Theorem 4.
(a) If \( B = S^1 \) and \( E = \mathbb{R} \), then \( G_{E/B} \cong \mathbb{Z} \): the deck transformations are exactly translations of \( \mathbb{R} \) by an integer, as I mentioned before.
(b) If \( B = S^1 \times S^1 \) and \( E = \mathbb{R} \times \mathbb{R} \), then \( G_{E/B} \cong \mathbb{Z} \times \mathbb{Z} \) is given by a pair of translations (which amounts to one translation of \( \mathbb{R}^2 \) by a vector with integer coordinates).
(c) If \( B \) is the figure 8 space and \( E \) is the antenna space, then \( G_{E/B} \cong \langle a \rangle \ast \langle b \rangle \) the free group. Looking at the picture, you can see how this works, since a given word in the free group tells you where to slide \( e_0 \).
(d) If \( B = E \) is the trivial cover, then \( H_0 = \pi_1(B, b_0) \) and \( G_{E/B} \) is trivial.

Example 4. Let \( B \) be the figure 8 space with loops \( a \) and \( b \). Let \( E \) be the covering space that looks like a circumscribed triangle: place a triangle with corners touching the circle, label the edges of the triangle with \( b \) and the three circle arcs with \( a \). This gives a three sheet covering space of \( B \). By computing the loops in \( E \) we find that \( H = p_* \pi_1(E, e_0) = \langle a^3, b^3, ab, ba \rangle \subset \langle a, b \rangle \). What are the deck transformations of \( E \)? Hopefully you can see exactly 3 of them, by rotating the triangle, so the group is \( \mathbb{Z}_3 \). On the other hand, the subgroup \( H \) is normal and the relation \( ab \) really says that \( b = a^{-1} \), so the quotient group is \( \langle a \rangle / \langle a^3 \rangle \cong \mathbb{Z}_3 \).