Homework 7  Algebraic Topology

§71 3. Let $p$ be $S^1 \cap S^2$, $U$ be the union of $S^1$ and a contractible neighborhood of $p \in S^2$ and $V$ be the union of $S^2$ and an arc neighborhood of $p \in S^1$. The edge of the sphere can be retracted to $p$, so $\pi_1(U, p) \cong \pi_1(S^2, p) \cong \mathbb{Z}$. The arc can be retracted to $p$ in $S^1$, so $\pi_1(V, p) \cong \pi_1(S^2, p) = 0$ and $U \cap V$ is contractible to $p$, so $\pi_1(U \cap V, p) = 0$. Van Kampen gives $\pi_1(S^1 \vee S^n, p) \cong \mathbb{Z}$.

§72 1. Adjust proof of 72.1 in text. Step 1 is the same, except replace $B^2$ with $B^n$, so $A$ is a deformation retract of $U$. Step 2 is same except $U \cap V$ is homeomorphic to $B^n - 0$, whose fundamental group is trivial. Therefore $\pi_1(U, b) \to \pi_1(X, b)$ is an isomorphism, as is $\pi_1(A, a) \to \pi_1(X, a)$.

§73 1ac. (a) If $X_n$ is $n$-fold dunce cap and $Y_m$ is $m$-fold dunce cap, then $\pi_1(X \times Y) \cong \mathbb{Z}_n \times \mathbb{Z}_m$. (c) With same notation as part (a), the space $X_n \vee Y_m$ has $\pi_1(X_n \vee Y_m) = \mathbb{Z}_n \ast \mathbb{Z}_m$ by van Kampen.

§79 1. Since $\pi_1(S^n) = 0$ and $\pi_1(\mathbb{R}) = 0$, 79.1 shows that $f$ lifts to $\tilde{f} : S^n \to \mathbb{R}$, where $p : \mathbb{R} \to S^1$ given by $p(x) = e^{2\pi ix}$. Since $\mathbb{R}$ is contractible, $f$ is nulhomotopic (or note that $f_*$ is trivial, so $f$ is nulhomotopic by 55.3).

2. (a) If $f_*(1) = n \in \mathbb{Z}$, then $f_*(0) = f_*(1 + 1) = n + n = 2n = 0 \Rightarrow n = 0$, therefore $f_*$ is trivial. Thus $f$ is nulhomotopic as in previous problem.

(b) Project $S^1 \times S^1$ onto either factor.

3. (⇒). Assume $H_0$ normal and let $e_1, e_2 \in p^{-1}(b_0)$. There are paths $\gamma_1, \gamma_2$ in $E$ from $e_0$ to $e_1, e_2$, let $\alpha_i = p \circ \gamma_i$. Then by 79.3 we have $\alpha_i H_i \alpha_i^{-1} = H_0$ for $i = 1, 2$, so

$$H_1 = \alpha_1^{-1} H_0 \alpha_1 = H_0 = \alpha_2^{-1} H_0 \alpha_2 = H_2$$

the middle two equalities due to $H_0$ being normal. By 79.2 there is a homeomorphism $h : E \to E$ taking $e_1$ to $e_2$.

(⇐). Let $g \in \pi_1(B, b_0)$ be an arbitrary loop at $b_0$. Lift $g$ to a path $\tilde{g}$ in $E$ starting at $e_0$ and ending at $e_1$. Then 79.3 says that $gH_1g^{-1} = H_0$ and the existence of $h : E \to E$ with $h(e_0) = e_1$ implies that $H_0 = H_1$ by 79.2. Therefore $g^{-1} H_0 g = H_1 = H_0$. Since $g$ was arbitrary, $H_0$ is normal.

4. (a) $p : S^1 \times \mathbb{R} \to S^1 \times S^1$ by $p(z, x) = (z^m, e^{2\pi ix})$. (b) $p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ by $p(x, y) = (e^{2\pi ix}, e^{2\pi iy})$. (c) $p : S^1 \times S^1 \to S^1 \times S^1$ by $p(z, w) = (z^m, w^n)$.